AN INTRODUCTION TO THE

INFINITESIMAL CALCULUS

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AN INTRODUCTION TO THE INFINITESIMAL CALCULUS

WITH APPLICATIONS TO MECHANICS AND PHYSICS

BY

G. W. CAUNT, M.A., M.Sc.

LECTURER IN MATHEMATICS AT ARMSTRONG COLLEGE

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PREFACE

In this book I have endeavoured to present the fundamental principles and applications of the Differential and Integral Calculus in as simple a form as possible consistent with an adequate comprehension of the ideas upon which they are based. Although the work is intended primarily for students who wish to obtain a sound working knowledge of the subject and its application, whether to Mechanics, Physics, Chemistry, Engineering, or any other science, I hope that it will also prove useful to those who are studying for pass degrees in Mathematics and to those who are working for Mathematical Scholarships.

In most of the old text-books on the Calculus, and in some modern ones, all the different standard forms and methods of differentiation are discussed before any applications of them are considered, and similarly with integration. It takes the ordinary student a long time to master all these methods, especially in the Integral Calculus, and if he sets out to learn them all before he knows the object of them, and what use he is to make of differential coefficients and integrals when he has obtained them, he is apt to get discouraged and take no interest in the subject; if he succeeds in learning them, he is apt to look upon the processes of differentiation and integration as a kind of mathematical juggling with symbols without any real comprehension of their meanings. In order to avoid these dangers, and in the hope of arousing the interest of the student at the outset, I have introduced easy applications at an early stage. After treating of the differentiation of quite simple algebraical and trigonometrical functions, I have considered their applications to properties of curves, to maxima and minima, and to mechanics. Similarly, after obtaining the integrals of a few simple types of functions, I have considered their applications to areas and volumes and to mechanics. All this is done before dealing at all with the inverse circular functions, or with exponential, logarithmic, and

hyperbolic functions. Afterwards I have treated of these latter functions and of more difficult methods and applications.

There is really no reason why the first part, just described, and which is comprised in the first nine chapters, should not (with the omission, perhaps, of parts of Chapter II) form part of the mathematical course of the best form of a good school. The amount of mathematical knowledge required before beginning the calculus is, I think, less than is often supposed. A sound knowledge of Elementary Geometry and of comparatively elementary algebraical and trigonometrical processes is the one essential and absolutely necessary requisite.

No attempt has been made to treat the bookwork with the precision and rigour required in the light of modern mathematical investigations. This would be quite out of place in a book intended for those who wish to acquire a knowledge of the calculus as a tool to work with, rather than for those who are training to be mathematicians, and in any case it is not suitable for a first course on the subject. At the same time, I have attempted not to ignore or conceal points of difficulty, and in several places where I have considered it advisable to assume theorems, of which the proof seemed to me beyond the scope of the book, I have not hesitated to do so, at the same time expressly stating that they are assumptions. Considerable space has been devoted to explanations and illustrations of the meanings of 'limits' and 'continuous functions', for I am convinced that, unless the student has clear ideas on these points, it is impossible for him to grasp the true meaning of a differential coefficient, although he may be able to acquire a certain amount of facility in the use of it. In connexion with limits, I think that the recent introduction of the symbol $x \to a$, in place of $x = a$, is a most valuable improvement. This is used throughout the book. There is no doubt that the older symbol is calculated to cause confusion and lead to erroneous ideas, since in most cases x can *not* be taken equal to a, but only as near to it as we please without actual coincidence.

In order to bring home the meaning of a formula or a theorem to the beginner, I have frequently introduced numerical examples and appeals to the geometrical intuitions of the student. These geometrical 'proofs' are generally much more interesting and

indeed much more convincing to the ordinary student than an analytical proof which, however rigorous, probably conveys no very definite meaning to him, and they are quite sufficient for many purposes.

I have assumed that the student is familiar with the theory of graphs as treated in text-books on Elementary Algebra, and in the first chapter I have given a short discussion on the method of sketching a graph from its equation (a most valuable exercise for mathematical students) in some rather more difficult cases, including the conic sections. For the benefit of students who have not done Analytical Conics, I have appended to this chapter a short discussion of the simplest forms of the equations of these curves, to which frequent references are made in the sequel.

Before proceeding to the differentiation and integration of exponential and logarithmic functions, &c., I have briefly recapitulated the chief properties of these functions, together with as much of the theory of convergency of series as seemed necessary. Many students of the calculus have but an imperfect knowledge of these important functions, and in any case it is hoped that this chapter may serve as a useful revision. Many examples, which illustrate the application of the exponential function e^x and show how it continually occurs in all branches of science, will be found in the last part of Chapter XVIII, which deals with the Compound Interest Law.

In the chapter on Methods of Integration, some of the well-known general processes, including the general discussion of resolution into Partial Fractions, are omitted; but I think that the methods given are sufficient to enable the student to integrate most of the expressions he is likely to meet with in practical applications of the subject, and at the end of the chapter [as also at the end of Chapter IV after the chief methods of differentiation have been considered] a long collection of miscellaneous exercises is given in which the student does not know beforehand which particular method he has to employ, as he does with the exercises in the body of the chapter. It is not, of course, expected that the student will work straight through all the examples or even all the articles in this chapter on a first reading, but a selection should be made, and he can return to them again and again as occasion arises for revision.

The same remark applies to the contents of Chapters XVIII and XIX. The applications to Centres of Gravity, Centres of Pressure, Moments of Inertia, Electricity, Potential and Attractions, Dynamics, &c., given in these chapters, are included in the hope of making the book useful to students in different branches of science and of varying interests; but the student of the calculus who is not also a student of Hydrostatics will naturally omit the section that deals with Centres of Pressure, and so on. In Chapter XIX a considerable amount of Particle Dynamics is included, for while this subject depends for the most part upon quite elementary principles of mechanics, it affords excellent illustrations of the application of the principles of the calculus.

It is obviously impossible in a book of this type to give any adequate discussion of the subject of Differential Equations, but the simplest and most useful types of equations of the first and second order are collected in Chapter XXI (although differential equations have been solved in the earlier chapters which deal with Physics and Mechanics). These are sufficient to enable the student to solve most of the equations he is likely to meet in elementary applications.

The chapters on Taylor's Theorem and Partial Differentiation are placed last, because they are not needed in the development of the subject along the lines I have adopted, but the treatment is such that the student who requires these particular subjects (for instance, the student of Thermodynamics and Chemistry who has to deal chiefly with functions of more than one variable and will therefore need Chapter XXIII) can take them at a much earlier stage. Chapter XXII on Taylor's Theorem can be taken, if desired, immediately after Chapter XIII on the Mean-Value Theorem, and the greater part of Chapter XXIII can be taken as soon as the student has finished the ordinary differentiation.

It is essential that students of the calculus should have a liberal supply of examples for practice, and sets of exercises are inserted at short intervals. The examples are plentiful in number and carefully graded, and I have endeavoured to include problems and applications from different sources of as varied, instructive, and interesting a nature as possible. With

such a large number of examples it cannot be hoped that the answers will be free from mistakes, but I hope that there are not many errors, and I shall be glad to receive corrections or hear of cases where answers are found to be wrong.

A collection of numerical tables is added at the end of the book, and it is hoped that this will prove very useful. It is important that, where possible, students should be able to work examples fully out and obtain definite numerical answers. This part of the work is often neglected. Some of the tables required for this purpose are not usually given in text-books on the Calculus and are not always easily accessible to the ordinary student.

In preparing this book I have frequently consulted many of the existing text-books on the subject, including those of Williamson, Lamb, Gibson, Osgood, and others, and I wish to make acknowledgement of my indebtedness to these works. I wish also to express my obligations to Professor Jessop of Armstrong College for his encouragement and for much valuable advice in connexion with the work. My sincere thanks are also due to Mr. J. W. Bullerwell of Armstrong College for his kindness in reading through the proofs and for the time and care that he gave to them, which led to the detection of many errors.

G. W. CAUNT.

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CHAPTER I

FUNCTIONS AND THEIR GRAPHS

1. Constants and variables.

In any equation or any investigation the quantities which occur are of two kinds: (i) those which retain the same value throughout the particular equation or investigation which is under consideration; these are called *constants*, and are generally denoted by the earlier letters of the alphabet, a, b, c, l, m, n, &c.; and (ii) those which take different values; these are called *variables*, and are generally denoted by the later letters of the alphabet, u, v, x, y, z. For instance, one of the commonest forms of the equation of a straight line is $y = mx + c$. Here x and y are variables; they are the coordinates of any point whatever on the straight line, and can take values from $-\infty$ to $+\infty$; m and c are constants, and have fixed values for any particular straight line, m being the tangent of the angle which the straight line makes with the positive direction of the axis of x, and c the intercept on the axis of y; but they have different values for different straight lines.

Again, the equation of a circle of radius a, taking its centre as the origin, is $x^2 + y^2 = a^2$. Here x and y are variables and a is a constant; x and y are the coordinates of any point on the circle, and therefore each may take any value from $-a$ to $+a$; a is the same for all points on any particular circle, but will of course have different values for circles of different sizes.

In mechanics, the distance s travelled in time t by a point moving in a straight line with constant acceleration a is given by the formula $s = ut + \frac{1}{2}at^2$. In this case s and t are variables and u and a are constants, u being the initial velocity of the moving point, and a its constant acceleration; s changes as t changes, but u and a remain the same during the particular motion which is under consideration.

As an example of an equation which contains three variables, we have in pneumatics the equation $p = k\rho(1 + \alpha t)$; in this case p, ρ,

and t are variables, being respectively the pressure, density, and temperature of a given mass of gas; k and α are constants.

2. Functions.

If two variables x *and* y *are so related that one or more values of* y *can be determined when the value of* x *is assigned, then* y *is said to be a function of* x.

In the first two of the cases just mentioned, the value of y can be calculated when the value of x is assigned (the values of the constants being supposed known); in the first case we obtain *one* value of y, in the second case *two* values, from a given value of x; y is said to be a function of x. In the third case, s can be calculated when the value of t is assigned (u and a being known); s is said to be a function of t. In the last case, the value of p can be found when the values of ρ and t are given (k and α being known); p is said to be a function of ρ and t.

Similarly, in the first two cases, we can, if values of y be assigned, calculate the corresponding values of x; in the third case, given the value of s, we can calculate values for t (two values of t for each value of s, since the equation is a quadratic for t in terms of s); and in the last case, given the values of p and t, we can calculate the value of ρ; i.e. we may regard x as a function of y, t as a function of s, and ρ as a function of p and t.

A magnitude may be a function of any number of variables.

Further examples are the following: the volume and superficial area of a sphere are functions of one variable, the radius of the sphere; the volume and superficial area of a cone or a cylinder are functions of two variables, the height and the radius of the base; the volume and superficial area of a rectangular block are functions of three variables, the length, breadth, and thickness.

In this book we deal chiefly with functions of a single variable.

The expressions x^3; $\sqrt{x}$; $\sqrt{(16+x^2)}$; $\sin x$; $\tan x$; $\log x$; 2^x; $\sin^{-1} x$ are all functions of x; their values can be calculated when the value of x is given. In these, and in the cases mentioned above, the relation between the variables can be expressed by a formula, but this is not always the case. For example, the height of the barometer at any given place is a function of the time; at any particular instant the barometer has a definite height, but there is no mathematical formula connecting the height with the time, although the relation between them can be represented graphically,

and instruments are used which draw the graph, and thereby exhibit to the eye the height as a function of the time.

The symbol $f(x)$ is used to denote a function of x in general; sometimes the symbols $F(x)$, $\phi(x)$, &c. are used, so that $y=f(x)$ or $y=\phi(x)$ is merely a symbolic way of expressing the fact that y is a function of x. x and y are often referred to as the *independent variable* and the *dependent variable* respectively, implying that any value may be assigned at will to x, and the corresponding value of y then calculated from it. Sometimes y is said to be a function of the *argument* x.

If $y=f(x)$, then $f(a)$ denotes the value of y when a is substituted for x; e.g. if $f(x)=x^2+4x+5$, then $f(a)=a^2+4a+5$; $f(3)=9+12+5=26$; $f(0)=5$; $f(-2)=4-8+5=1$; and so on.

In some cases, real values of y are obtained for every real value of x; this is so in the example just mentioned, but it is not always the case; e.g. if $y=\sqrt{(1-x^2)}$, only values of x from -1 to $+1$ inclusive give real values for y; if x is numerically >1, y is imaginary. Again, if $y=x/(x-1)$, we get a definite real value of y for every real value of x except $x=1$. If $x=1$, the function takes the form $1/0$, which has no definite value. In this case, y is said to be *defined* for all values of x except $x=1$. In the preceding example, y is defined only for values of x from -1 to $+1$ inclusive.

3. Single-valued and many-valued functions.

If to each value of x there corresponds one and only one value of y, then y is said to be a one-valued or single-valued function of x; e.g. $y=(ax+b)/(cx+d)$; $y=\sin x$; $y=(1+x^2)^2$ are one-valued functions of x.

If to each value of x there correspond more than one value of y, then y is said to be a many- or multiple-valued function of x; e.g. in the second example of Art. 1, $y^2=a^2-x^2$ and $y=\pm\sqrt{(a^2-x^2)}$. To each value of x correspond two values of y equal in magnitude and opposite in sign; therefore y is a two-valued function of x. In this case, if only real values of y are to be considered, x must not be numerically $>a$. If $x^2>a^2$, y will be imaginary.

If $y^3-6y^2+11y=x$, to each value of x correspond three values of y obtained by solving this equation of the third degree; therefore y is a three-valued function of x.

For example, if $x=6$, we have

$$y^3-6y^2+11y-6=0, \text{ whence } y=1 \text{ or } 2 \text{ or } 3.$$

3. If $f(x)=(x^2-1)(x^2-4)$, prove that $f(1)$, $f(-1)$, $f(2)$. $f(-2)$ are all zero.

4. If $f(x)=ax^2+bx+c$, find $f(x+1)$, $f(x-1)$, $f(x+h)-f(x)$.

5. If $f(x)=a^x$, prove that $f(m)\times f(n)=f(m+n)$,
$$f(m)\div f(n)=f(m-n).$$

6. If $f(x)=\log x$, prove that
$$\text{(i) } f(abc)=f(a)+f(b)+f(c);$$
$$\text{(ii) } f(a/b)=f(a)-f(b);$$
$$\text{(iii) } f(a^n)=nf(a).$$

7. If $f(x)=\tan x$, prove that $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)\cdot f(y)}$.

8. Classify the following functions as 'even' or 'odd' functions of x:
$\cot x$, $\sec x$, $\operatorname{cosec} x$, $(x-1)^2+(x+1)^2$, $(x^3+1)^2-(x^3-1)^2$, $x/(1+x^2)$, $\sin 2x$, $\sin^2 x$, $\cos 2x$, $x(x-2)(x+2)$, a^x+a^{-x}, $x\sin x$, $x\cos x$.

9. Express y explicitly in terms of x in the following cases:
(i) $x^3+y^3=a^3$, (ii) $x^2y^2=a^4+b^4$,
(iii) $\log y+\log x=\log a$, (iv) $y^2+2ay-x^2=0$,
(v) $a\sin y+b=cx$, (vi) $axy+bx+cy+d=0$.

10. Transform the following into implicit relations between x and y, free from fractions and radical signs:
(i) $y=(3x-2)/(2x-1)$, (ii) $y=(a^n-x^n)^{1/n}$,
(iii) $y=(a+x)/\sqrt{x}$, (iv) $y=\log_a\{x/(1+x^2)^{1/2}\}$,
(v) $y=x\pm\sqrt{(1-x^2)}$, (vi) $y=\sin^{-1}(x/a)$.

11. Given the following functions, find in each case the corresponding inverse function:
(i) $y=x^4$, (ii) $y=1+\sqrt{x}$, (iii) $y=\cos^{-1}2x$,
(iv) $y=a^x$, (v) $y=a\tan^2 x$, (vi) $y=\sqrt[n]{(a^n-x^n)}$,
(vii) $y=\sqrt{(5-x^2)}$, (viii) $y=\sqrt{(2x-x^2)}$, (ix) $y=4x/(x-1)$,
(x) $y=\frac{1}{4}\log_a(x+1)$.

12. In each of the first nine examples of Question 11, state how many values of y correspond to each value of x, and for what values of x y is defined; also, in the inverse functions, how many values of x correspond to each value of y; and for what values of y x is defined.

8. Graphs.

A general survey of the relation between the variable x and the function y can be obtained by drawing the graph of the function. If we suppose that x increases through a given range of values, and calculate the values of y corresponding to different values of x within this range, we shall, by plotting on squared paper the points which have these corresponding values of x and y as coordinates, obtain a series of points; the locus of these points is called the *graph* of the function. For an account of the theory of continuous number and a discussion of the question as to whether and under what circumstances a function can be represented by a continuous curve, the student is referred to more advanced treatises. The functions which occur in such applications of the calculus as it is

proposed to deal with in this book, usually have graphs which can be easily drawn and which give a general view of the variation of the function.

It will be taken for granted that the student is already familiar with as much of the theory and construction of graphs as is now generally included in text-books on elementary algebra, including the plotting of graphs of the functions $y = ax + b$, $y = ax^2 + bx + c$, $x^2 + y^2 = a^2$, together with their simpler properties; and also with the graphs of the circular functions $\sin x$, $\cos x$, $\tan x$, &c.

In many examples in the Differential and Integral Calculus it is necessary, or at least advisable, to draw roughly the graph of a function, and some more examples of a rather less elementary type will now be considered.

9. Examples of graphs.

The graph of any function can be obtained by simply plotting a sufficient number of points, taking $x = 0, \pm 1, \pm 2, \dots$ in turn, with intermediate values when necessary, until enough points are obtained to show all the various branches of the graph, and then drawing a curve freely through them; but in most cases a great deal of information as to the shape and limitations of the curve can be obtained by examining the equation. This should always be done first, and the following examples are chosen so as to illustrate this.

(i) $y^2 = 16x$.

Here the first fact we may notice is that, corresponding to any positive value of x, there are two values of y which are equal in magnitude and opposite in sign (e.g. if $x = 4$, y may be either $+8$ or -8), i.e. taking any point on the axis of x to the $+$ side of the origin, we get two points of the graph by measuring equal distances upwards and downwards perpendicular to the axis of x; this shows that *the curve is symmetrical about the axis of* x.

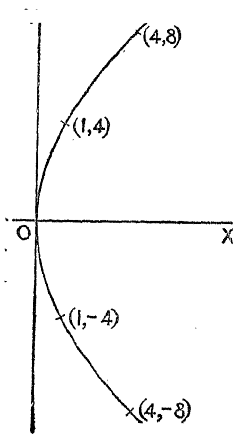


Fig. 1.

The next fact to notice is that if x be $-$, y^2 is $-$, and therefore y is imaginary, i.e. no points are obtained for negative values of x, and therefore *the curve lies entirely on the positive side of the axis of* y. It clearly goes through the origin, since $y = 0$ when $x = 0$, and

since values of x, however large, always give real values for y, it extends to an infinite distance. Now by taking a few numerical values, e.g. $x = 1, 2, 3, \ldots$, the graph can be drawn fairly accurately (Fig. 1).

(ii) $\frac{1}{16}x^2 + \frac{1}{9}y^2 = 1$.

Here $\frac{1}{9}y^2 = 1 - \frac{1}{16}x^2$ and, as in the preceding case, the curve is symmetrical about the axis of x, since to any value of x correspond two values of y equal in magnitude and opposite in sign. Similarly, to any value of y correspond two values of x equal in magnitude and opposite in sign, therefore the curve is symmetrical about the axis of y also.

The next fact to notice is that if x is numerically > 4, $\frac{1}{9}y^2$ is $-$ and therefore y is imaginary; similarly, if y is numerically > 3, $\frac{1}{16}x^2$ is $-$ and x is imaginary; therefore the curve lies entirely within the rectangle formed by the straight lines $x = \pm 4$, $y = \pm 3$.

[The symbol $|x|$ is used to denote the numerical value of x, so that $|x| < 3$ means that x is between -3 and $+3$, and therefore $x^2 < 9$; $|x| > 4$ means that x is either > 4 or < -4, and therefore $x^2 > 16$.]

Taking $x = 0, 1, 2, 3, 4$, and remembering that the curve is symmetrical about both axes, the graph can easily be drawn (Fig. 2).

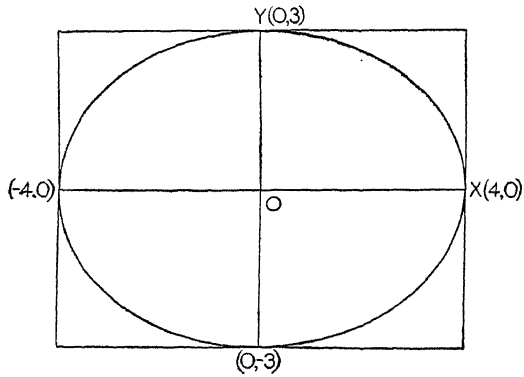


Fig. 2.

(iii) $y = 1/x$ or $xy = 1$.

The graph of this equation is not symmetrical about either axis of coordinates. In this case, for all values of x a change in the sign of x produces a change in the sign of y without altering the numerical value of y; if x is $+$, y must be $+$, and if x is $-$, y

must be $-$. The graph therefore lies entirely in the 1st and 3rd quadrants; it cannot extend into the 2nd and 4th quadrants because there x and y have opposite signs. [Similarly the graph of $y = -1/x$ lies entirely in the 2nd and 4th quadrants.]

The graph is said to be *symmetrical about the origin*; for if any point (x, y) on it be joined to the origin and the joining line be produced to an equal distance on the other side of the origin, i.e. to the point $(-x, -y)$, this point is also on the curve; in other words, any chord of the curve through the origin is bisected at the origin. [This property is evidently true of the graph of any odd function of x.]

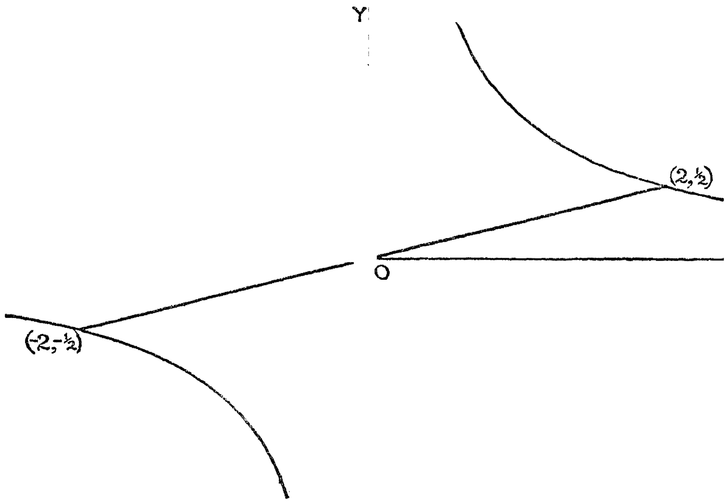


Fig. 3.

As x gets greater and greater, y gets less and less [when $x = 1$, 10, 100, 1,000,000, $y = 1$, ·1, ·01, ·000001 respectively, and so on], and can be made as small as we please by taking x sufficiently large; but, however large x be taken, y never becomes quite equal to zero. Therefore the curve is constantly approaching the axis of x, but never quite reaches it, i.e. the axis of x is an asymptote to the curve.

An *asymptote* to a curve is a tangent whose point of contact is at an infinite distance, i. e. a line which is continually approaching a curve, but yet which never quite meets it.

The equation may also be written $x = 1/y$, and therefore, by a similar argument, the axis of y is also an asymptote (Fig. 3).

(iv) $y = x^3$; $y = x^4$; and generally $y = x^n$.

These curves evidently go through the origin, and also through the point (1, 1), since (0, 0) and (1, 1) satisfy the equation $y = x^n$ whatever be the value of n.

In the first equation, if x changes sign, y changes sign also; therefore as in the preceding example (iii), the graph is symmetrical about the origin and lies in the 1st and 3rd quadrants only.

If x is between 0 and 1, y (i. e. x^3) is less than x, and therefore the graph is nearer to the axis of x than to the axis of y. But if $x > 1$,

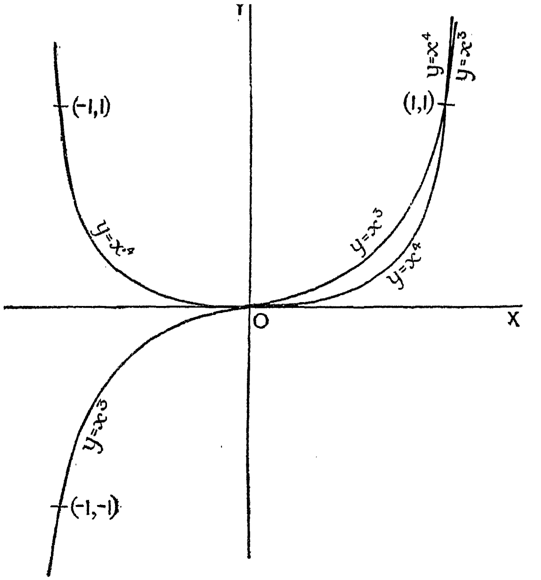


Fig. 4.

y is greater than x, and the graph is nearer the axis of y; moreover, as soon as x passes the value 1, y increases rapidly, and the curve rises steeply.

In the second equation, the values of x corresponding to given values of y always occur in pairs, equal in magnitude and opposite in sign; therefore the curve is symmetrical about the axis of y, and y, being equal to an even power of x, cannot be $-$. Therefore the curve is confined to the first two quadrants. In the first quadrant, between $x = 0$ and $x = 1$, the second graph is below the first, since for such values of x, $x^4 < x^3$; if $x > 1$, the second graph is above the first, since x^4 is then greater than x^3. The two graphs cross each other at (1, 1) (Fig. 4).

In the general case, $y = x^n$, if n be an odd integer, the graph is similar to that of $y = x^3$; if n be an even integer, the graph is similar to that of $y = x^4$. All the curves go through the origin and through the point (1, 1). The greater the value of n, the flatter the curve is near the origin, and the steeper after passing (1, 1);

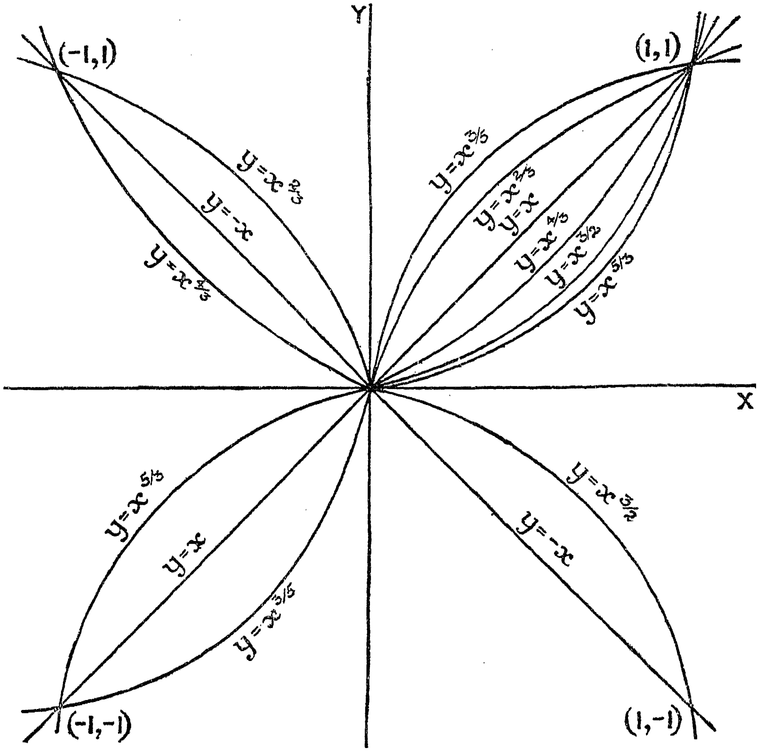


Fig. 5.*

i.e. the graph for any value of n is below that for any smaller value of n between the origin and (1, 1), and above it after passing through (1, 1).

If $y = x^{1/n}$, where n is a positive integer, then $x^n = y$, and the graphs bear the same relation to the axis of x as those described above bear to the axis of y; i. e. the graphs of $y = x^n$ and $y^n = x$ are symmetrical about the bisector of the angles XOY, $X'OY'$ between the axes, or, as it is often expressed, one

* The figure shows the relative positions of the graphs, but their distances from the lines $y = \pm x$ are rather exaggerated, in order that they may be distinguished one from another more readily.

graph is the reflexion of the other in the bisector of the angle XOY. [This property is true of the graphs of all inverse functions, e.g. $y = x^n$ and $y = x^{1/n}$, the two functions just mentioned; $y = \sin x$ and $y = \sin^{-1} x$; $y = a^x$ and $y = \log_a x$; &c.]

If $y = x^{p/q}$, where p and q are positive integers, the graphs are obtained in a similar manner. They all go through the origin and the point (1, 1). If p and q be both odd, the graph lies in the 1st and 3rd quadrants; if p be odd and q even (as in $y = x^{3/2}$, i.e. $y^2 = x^3$), the graph, being symmetrical about the axis of x, is in the 1st and 4th quadrants; and if p be even and q odd (as in $y = x^{2/3}$, i.e. $y^3 = x^2$), the graph, being symmetrical about the axis of y, is in the 1st and 2nd quadrants (Fig. 5). If $p/q > 1$, the graph between the origin and (1, 1) is below the straight line $y = x$; if $p/q < 1$, it is above the straight line $y = x$.

(v) $y = \dfrac{x^2}{1+x^2}$.

In this case it is evident, since x^2 is always + for real values of x, that y is always +, and the curve is confined to the first two quadrants; next, that since x^2 is always +, $1 + x^2$ must be $> x^2$, and therefore y is always < 1. Hence the graph lies entirely in the strip between the axis of x and the parallel straight line $y = 1$.

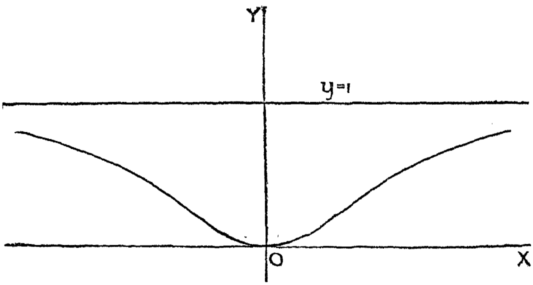


Fig. 6.

Again, the values of x corresponding to assigned values of y always occur in pairs equal in magnitude and opposite in sign, since the equation only contains even powers of x; therefore the graph is symmetrical about the axis of y. The curve goes through the origin, and in the neighbourhood of the origin y is much less than x; e.g. if $x = \cdot 1$, $y = \cdot 01/1\cdot 01 = 1/101$; therefore near the origin the graph keeps close to the axis of x. As x gets larger and larger, y gets nearer and nearer to 1, as is evident when the equation is written in the form $y = 1/(1 + 1/x^2)$; the term $1/x^2$ in the denominator becomes less and less as x increases, and can be made as small as we please. Therefore y can be made as near to 1 as we please

by increasing x sufficiently; hence the line $y = 1$ is an asymptote to the curve (Fig. 6).

(vi) $y = \dfrac{x^2}{1-x^2}$.

As in the preceding example, the curve goes through the origin, and is symmetrical about the axis of y. In this case we have another point to consider: are there any finite values of x which make y infinite? It is plain that y becomes infinitely large if $x^2 = 1$, i.e. if $x = +1$ or -1; these lines are obviously asymptotes.

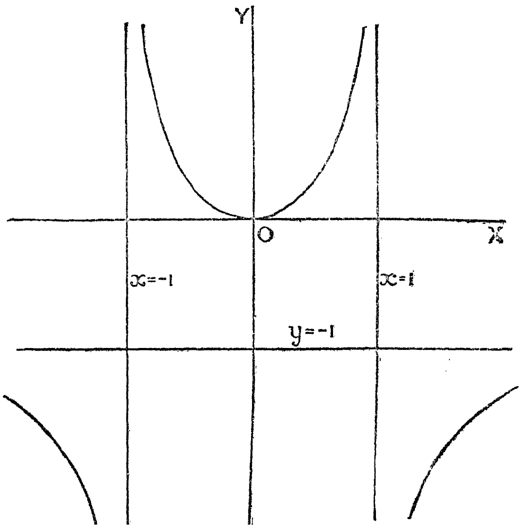


Fig. 7.

If x is slightly less than 1, y is very large and $+$; if x is slightly greater than 1, y is numerically very large and $-$. Hence the curve rises from the origin to the asymptote $x = 1$, and then on the other side of the asymptote reappears from the other end of it.

When $x > 1$, y is $-$, and since it can be put into the form $1/(1/x^2-1)$ it approaches the value $1/(-1)$ as x increases, and can be made as nearly equal to -1 as we please, since $1/x^2$ can be made as small as we please by taking x large enough. Hence the line $y = -1$ is also an asymptote (Fig. 7).

(vii) $y^2 = x^2 . \dfrac{3+x}{2-x}$.

This curve is symmetrical about the axis of x, but not about the axis of y. If $x > 2$, y^2 is $-$ and y imaginary; therefore the curve

does not extend to the right-hand or positive side of $x = 2$. If $x < -3$ (i.e. between -3 and $-\infty$), y^2 is $-$ and y imaginary; therefore the curve does not extend to the left-hand or negative side of $x = -3$. Hence it lies entirely between $x = -3$ and $x = 2$. Again, y becomes infinitely large as x approaches the value 2; therefore $x = 2$ is an asymptote, and evidently no other value of x except 2 can make y infinite. Also $y = 0$ when $x = 0$ and when $x = -3$.

Hence the curve, being symmetrical about the axis of x, consists of a loop between $x = -3$ and the origin, and approaches the asymptote $x = 2$ both upwards and downwards.

The width of the loop can be obtained roughly by plotting the points for which $x = -1, -2$. When $x = -1$, $y = \pm \cdot 8$ nearly; when $x = -2$, $y = \pm 1$; also when $x = 1$, $y = \pm 2$ (Fig. 8).

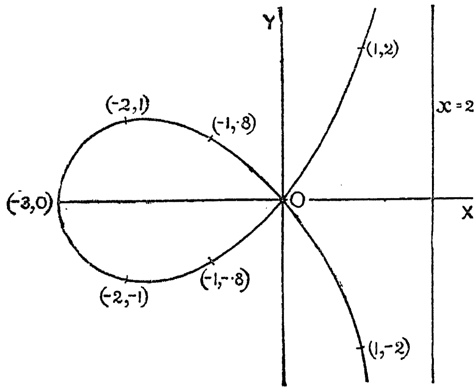


Fig. 8.

(viii) $y = \frac{1}{2}x + 1/x$.

This curve is not symmetrical about either axis, but y changes sign without changing its numerical value when x changes sign, and both are $+$ or both $-$. Therefore the curve is symmetrical about the origin, and lies in the 1st and 3rd quadrants only.

Next, y becomes infinitely large as x approaches 0; therefore the axis of y is an asymptote as in Example (iii).

There is also another asymptote obtained as follows: as x increases, $1/x$ decreases and can be made as small as we please by taking x large enough; hence the equation of the curve becomes, very nearly, $y = \frac{1}{2}x$ when x is very large, i.e. the curve approaches more and more nearly to coincidence with the straight line $y = \frac{1}{2}x$ as x increases indefinitely; hence $y = \frac{1}{2}x$ is an asymptote. Since

y is always a little more than $\frac{1}{2}x$ (when x is $+$), the curve lies (in the first quadrant) above the asymptote. Therefore, being symmetrical about the origin, it consists of two branches in the two acute angles between the axis of y and the straight line $y = \frac{1}{2}x$ (Fig. 9). The curve is a hyperbola whose asymptotes are not at right angles.

It will be seen later (Chapter VI) how the exact width of the loop in the preceding example, and the exact position of the points nearest to OX in the present example, can be determined.

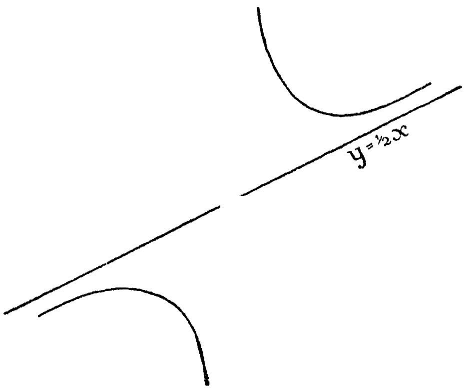


Fig. 9.

10. Questions connected with curve-drawing.

From these examples it will be seen that the following are the chief questions the student should ask himself when starting to draw the graph of a function:

(i) Is the graph symmetrical about either or both axes?

(It is symmetrical about the axis of x if its equation contains only even powers of y; and about the axis of y if its equation contains only even powers of x. Note that the graph of any *even* function of x (Art. 5) is symmetrical about the axis of y.)

(ii) Is the graph symmetrical about the origin?

(It is symmetrical about the origin if a change in the sign of x causes a change in the sign of y without altering its numerical value. If x and y are both $+$ or both $-$, it lies in the 1st and 3rd quadrants; if one is $+$ and the other $-$, it lies in the 2nd and 4th quadrants. Note that the graph of any *odd* function of x is symmetrical about the origin.)

(iii) Are there any values of x which make y^2 negative and therefore y imaginary (or any values of y which make x imaginary)?

(This often limits to a great extent the range of values of x which have to be considered in the actual plotting.)

(iv) Where does the curve cut the axes?

(It cuts the axis of x where $y=0$, and the axis of y where $x=0$. It goes through the origin if $y=0$ when $x=0$.)

(v) What values of x make y infinite, and what values of y make x infinite?

(This gives the asymptotes parallel to the axes. If y is given as an explicit function of x, it is often useful to solve the equation for x (or x^2) in terms of y. E.g. in Example (v) solving for x^2, we get $x^2 = y/(1-y)$; whence x is imaginary if $y > 1$ and infinite when $y=1$, and therefore $y=1$ is an asymptote.)

(vi) What is the value of y when x becomes infinitely large (or of x when y becomes infinitely large)?

(If y tends to a constant finite value as in Examples (v) and (vi), this gives an asymptote parallel to the axis of x; if y tends to an expression of the form $ax+b$, as in Example (viii), this gives an oblique asymptote.)

(vii) If the curve goes through the origin, then, in the neighbourhood of the origin, is y very small or very large compared with x, or is the ratio y/x finite?

(In the first case, the curve keeps close to the axis of x on leaving the origin, as in Example (v); in the second case, it keeps close to the axis of y, as in $y^3 = x^2$, where $y^3/x^3 = 1/x$, therefore y/x is very large when x is very small.

Again, in $y^2 = x^2/(1+x^2)$ we have $y^2/x^2 = 1/(1+x^2)$; therefore near the origin y/x is nearly 1, and the direction of the curve at the origin bisects the angle between the axes.)

As will be seen later, a determination of the maximum and minimum values of the ordinate, and of the points of inflexion of a curve, by an elementary application of the principles of the calculus, is often of very great assistance in drawing the graph of a function.

[For examples see p. 23.]

APPENDIX TO CHAPTER I

CONIC SECTIONS*

We have discussed in Art. 9, Ex. (i)–(iii) particular cases of the equations of the parabola, ellipse, and hyperbola. As we shall frequently have occasion to refer to these curves and their equations, a short discussion of the equations is here appended, for the benefit of the student who has done but little Analytical Geometry.

A *conic section* or *conic* may be defined as the locus of a point which moves in a plane in such a way that its distance from a fixed point in the plane (called the *focus*) bears a constant ratio e (called the *eccentricity*) to its perpendicular distance from a fixed straight line in the plane (called the *directrix*). If $e=1$, the conic is a parabola; if $e<1$, an ellipse; if $e>1$, a hyperbola.

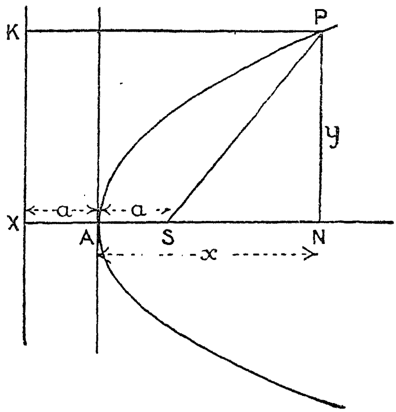


Fig. 10.

(a) The parabola.

The equation of Ex. (i) is a particular case of the simplest form of the equation of a parabola.

Let $2a$ be the distance SX (Fig. 10) of the focus S of a parabola from the directrix XK; the middle point A of SX is equidistant from S and the directrix, and is therefore a point on the locus.

Let (x, y) be the coordinates of any point P on the curve, referred to AS and the perpendicular to AS through A as axes, and let PK, PN be perpendicular to the directrix and axis of x respectively.

Then

$$y^2 = PN^2 = SP^2 - SN^2 = PK^2 - SN^2 = XN^2 - SN^2 = (a+x)^2 - (a-x)^2 = 4ax;$$

i. e. $y^2 = 4ax$ is the equation of the curve.

Geometrically, this takes the form $PN^2 = 4AS \, . \, AN$.

If the axis of the parabola be the axis of y, the relation $PN^2 = 4AS \, . \, AN$

* A useful collection of formulae, geometrical and analytical, is given in Workman, *Memoranda Mathematica* (Clarendon Press, 5s. net).

becomes $x^2 = 4ay$, A being taken as the origin. If in Fig. 11 A be the point whose coordinates are (h, k) and the axis of the parabola be parallel to the axis of y, $PN = x-h$, $AN = y-k$, and the equation becomes $(x-h)^2 = 4a(y-k)$,

i.e. $$y = \frac{1}{4a}x^2 - \frac{h}{2a}x + \frac{h^2}{4a} + k,$$

which is of the form $y = Ax^2 + Bx + C$.

Conversely, any equation of the form $y = ax^2 + bx + c$ may be written

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a},$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y - \frac{4ac - b^2}{4a}\right),$$

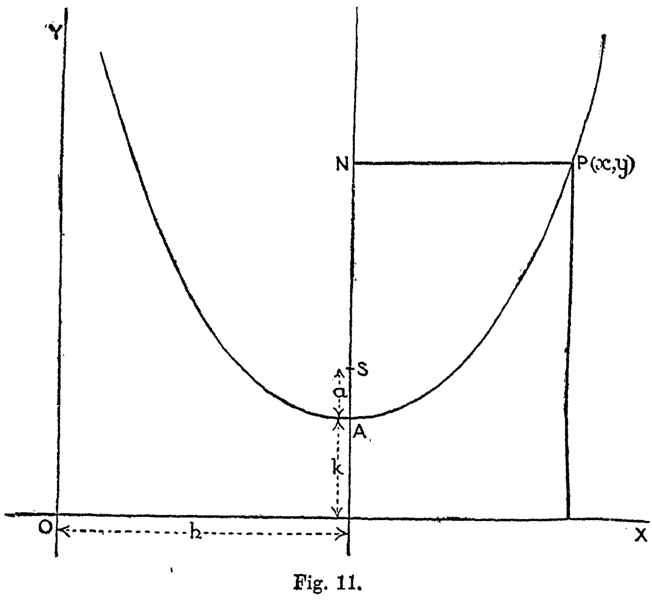


Fig. 11.

and therefore represents a parabola of latus rectum $1/a$, whose axis is parallel to the axis of y, and whose vertex is the point $-b/2a$, $(4ac-b^2)/4a$.

Also, y is numerically very large when x is numerically very large, and is $+$ or $-$ according as a is $+$ or $-$; therefore the vertex of the parabola is the lowest or highest point of the curve according as a is $+$ or $-$.

(b) The ellipse.

The equation of Ex. (ii) is a particular case of the equation of an ellipse in its simplest form.

Let SX (Fig. 12) be the perpendicular from the focus S of an ellipse to the directrix; if SX be divided internally at A and externally at A' in the ratio $e:1$, so that $SA = e.AX$ and $SA' = e.A'X$, then A and A' will be points

on the ellipse. Take C, the middle point of AA', as origin and CX as axis of x.

Since $SA' = e \,.\, A'X$ and $SA = e \,.\, AX$, we obtain, by adding,

$$SA' + SA = e(A'X + AX);$$

and, by subtracting, $\quad SA' - SA = e(A'X - AX);$

i. e. $\quad 2\,CA = e \,.\, 2\,CX$ and $2\,CS = e \,.\, 2\,CA$,

or denoting CA by a,

$$CS = ae \text{ and } CX = a/e.$$

Let (x, y) be the coordinates of any point P on the curve; and let PN, PK be drawn perpendicular to CX and the directrix respectively.

Then $SN^2 + NP^2 = SP^2 = e^2 \,.\, PK^2$ (from the definition of an ellipse)

$$= e^2 \,.\, NX^2.$$

$$\therefore \quad (ae - x)^2 + y^2 = e^2(a/e - x)^2,$$

$$a^2e^2 - 2\,aex + x^2 + y^2 = a^2 - 2\,aex + e^2x^2,$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2);$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

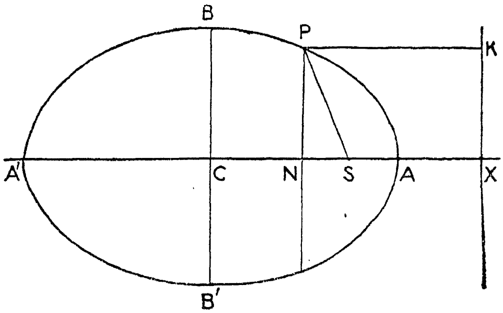


Fig. 12.

Denoting $a^2(1 - e^2)$ by b^2, we have the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

as the equation of the ellipse.

Putting $x = 0$ in this equation, we have $y^2/b^2 = 1$ and $y = \pm b$.

$\therefore$ b is the length of the intercept which the curve makes on the axis of y.

Since the curve is symmetrical about the axis of y, there will clearly be another focus S' and another directrix $K'X'$, symmetrical about CB with S and KX.

(c) **The hyperbola.**

The equation of Ex. (iii) is a particular case of the equation of a rectangular hyperbola in its simplest form.

If we proceed in the case of the hyperbola exactly as in the case of the ellipse, we get two points A, A' on the curve on *opposite* sides of X, since e is now >1; the relations $CS = ae$ and $CX = a/e$ will still be true, and just as before we shall arrive (remembering that $e > 1$) at the equation

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1.$$

Denoting $a^2(e^2-1)$ by b^2, we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

as the equation of the hyperbola.

Putting $x = 0$ in this equation, $y^2/b^2 = -1$; therefore the curve does not cut the axis of y in real points.

It is symmetrical about both axes as in the case of the ellipse; the equation may be written

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1,$$

whence, if $|x| < a$, y is imaginary. All values of $|x| > a$ give real values

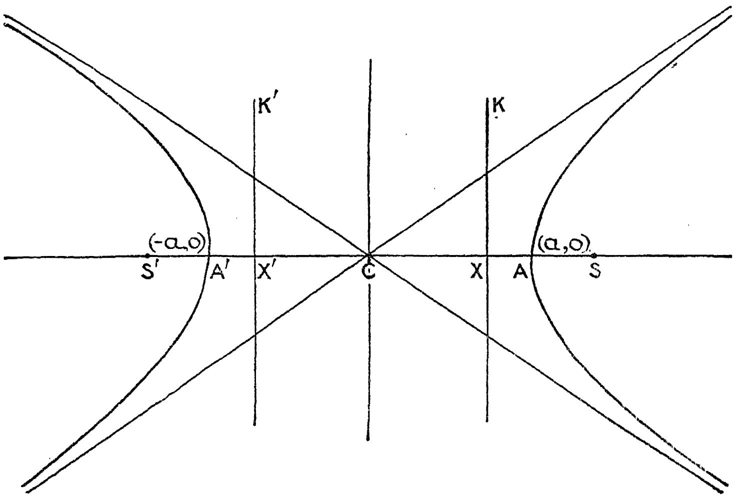


Fig. 13.

of y; therefore the curve consists of two branches extending from $(\pm a, 0)$ to infinity as in Fig. 13. Since the curve is symmetrical about the axis of y, the hyperbola also has another focus S' and another directrix $X'K'$ symmetrical about this axis with S and XK.

If $b^2 = a^2$, i.e. if $e^2 - 1 = 1$ and $e = \sqrt{2}$, the hyperbola is said to be *equilateral* or *rectangular*. The preceding equation then becomes $x^2 - y^2 = a^2$.

If the axes are turned through an angle of 45°, this equation takes another very simple and convenient form.

Change of Axes. The effect of rotating the axes about the origin through any angle θ is obtained as follows:

Let OX, OY (Fig. 14) be the original axes, OX', OY' the new axes, and let the angle $XOX' = YOY' = \theta$. If (x, y) be the coordinates of P referred to the original axes, and (x', y') the coordinates of P referred to the new axes, then

$$x = OM = OK - HM' = OM'\cos\theta - M'P\sin\theta = x'\cos\theta - y'\sin\theta;$$
$$y = MP = KM' + HP = OM'\sin\theta + M'P\cos\theta = x'\sin\theta + y'\cos\theta.$$

Taking the case of the *rectangular* hyperbola, in order to bring the curve from the position of Fig. 13 into the position of Fig. 3 relative to the axes, it is necessary to turn the axes through an angle of 45° in the clockwise direction. Therefore, putting $\theta = -45°$ in the preceding results, we have

$$\cos\theta = 1/\sqrt{2},\ \sin\theta = -1/\sqrt{2},\ x = (x'+y')/\sqrt{2},\ y = (-x'+y')/\sqrt{2},$$

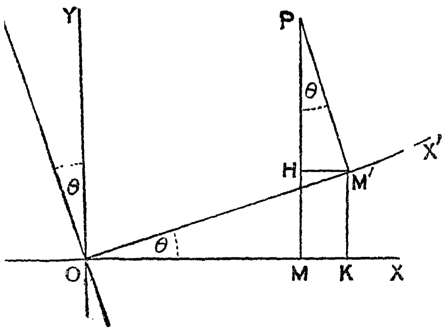


Fig. 14

and the equation becomes $\frac{1}{2}(x'+y')^2 - \frac{1}{2}(-x'+y')^2 = a^2$, which reduces to $2x'y' = a^2$.

Thus the equation of a rectangular hyperbola takes the form

$$xy = \tfrac{1}{2}a^2$$

when referred to its asymptotes as axes.

When the equation of the hyperbola is obtained in this form, the existence of its asymptotes follows at once as in the particular case on p. 9.

[Every hyperbola has a pair of asymptotes, and its equation can be obtained in a form similar to the preceding by taking the asymptotes as axes, but it is only in the case of the 'rectangular' hyperbola that the asymptotes are at right angles, and it is to this property that the name is due.]

(d) General equation of the second degree.

It is proved in text-books on Analytical Geometry (e.g. A. C. Jones's *Algebraical Geometry*, Ch. VI) that the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$

always represents a conic,* and that the equation $ax^2 + 2hxy + by^2 = 0$ repre-

* This fact often furnishes guidance in drawing the graph of a function.

sents two straight lines through the origin parallel to the asymptotes of the conic. A conic is a parabola (including the case of two coincident straight lines), an ellipse (including the case of a circle), or a hyperbola (including the case of two intersecting straight lines) according as the asymptotes are coincident, imaginary, or real. Hence an equation of the second degree represents a parabola, ellipse, or hyperbola according as the factors of the terms of the second degree are coincident, imaginary, or real. Therefore equation (i) represents a parabola, ellipse, or hyperbola according as h^2-ab is zero, negative, or positive.

For example, the equation

$$y=\frac{ax+b}{cx+d},$$

when cleared of fractions, becomes

$$cxy-ax+dy-b=0,$$

and it follows immediately from the preceding condition that the graph is a hyperbola.

Similarly,

$$y=\frac{ax^2+bx+c}{mx+n},$$

being of the second degree, represents a conic. Therefore since it obviously has a real asymptote $x=-n/m$, it must represent a hyperbola.

Again, if the term xy be absent from an equation of the second degree, the equation represents an ellipse or a hyperbola according as the coefficients of x^2 and y^2 have the same sign or different signs.

(e) Polar coordinates.

Any quantities which determine the position of a point in a plane are called coordinates of the point. The position of a point in a plane is fixed relative to two fixed straight lines at right angles in the plane, if the distances of the point from these two lines are given. These are the coordinates which we have used in the preceding chapter, and which are known as rectangular, or sometimes as Cartesian coordinates (from the fact that they were first introduced by Descartes). We will now consider briefly the system of coordinates which comes next in order of simplicity and importance.

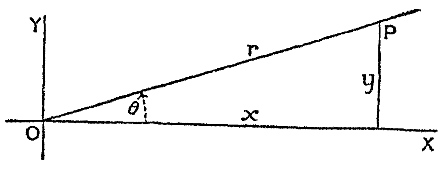


Fig. 15.

If (Fig. 15) O be a fixed point in a fixed straight line OX, the position of a point P is determined relative to O and OX if the length of OP and the magnitude of the angle XOP be given. These quantities are denoted by r

and θ respectively, and are called the *polar coordinates* of the point P. OP is called the *radius vector* of P, and XOP the *vectorial angle.*

If the coordinates (r, θ) of a point P satisfy a given equation, different positions of the point P can be plotted, by assigning values to θ and calculating the corresponding values of r; and their locus will be a curve. The given equation is called the *polar equation* of the curve. If a straight line OY be drawn perpendicular to OX, there are very simple relations between the polar coordinates of the point P and its rectangular coordinates referred to the axes OX, OY. It is evident that $x = r\cos\theta$, $y = r\sin\theta$; these equations give the rectangular coordinates in terms of the polar coordinates. Conversely the equations $r = \sqrt{(x^2+y^2)}$, $\theta = \tan^{-1}(y/x)$ give the polar coordinates in terms of the rectangular coordinates.

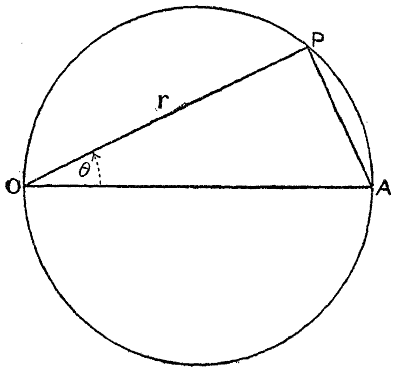


Fig. 16.

Polar equation of a circle.

The equation of a circle in polar coordinates admits of a very simple form if a point on the circumference be taken as origin, and the diameter through the point as initial line.

For, if a be the radius, it follows immediately from Fig. 16 that

$$r = OP = OA\cos\theta = 2a\cos\theta,$$

which is the polar equation of the circle.

Examples of the plotting of curves from their polar equations will be found in Chapter XVII.

Examples II.

Draw the graphs of the following functions:

1. $y = x^2$; $y = -x^2$; $y = 2+x^2$; $y = 2-x^2$.
2. $y^2 = x^3$; $y^2 = \frac{1}{4}x^3$; $y^2 = -x^3$; $y^3 = x^4$; $y^4 = x^3$.
3. $x^2+y^2 = 16$; $x^2+4y^2 = 16$; $4x^2+y^2 = 16$; $x^2-y^2 = 16$.
4. $y = 1/x^2$; $y = 1/x^3$; $y^2 = 1/x$; $y^3 = 1/x$.
5. $y = \dfrac{x}{1+x^2}$; $y = \dfrac{x}{1-x^2}$.
6. $y^2 = x^2(16-x^2)$; $y^2 = x^2/(16-x^2)$.
7. $y^2 = \dfrac{}{3-x}$, $y^2 = \dfrac{x^3}{3-x}$.
8. $y = \frac{1}{3}x$; $y = \frac{1}{3}x - 2/x$; $\frac{1}{3}x - 2/x + 1$.
9. $y^2 = \dfrac{x^2(4-x)}{3+x}$; $y^2 = \dfrac{x^2(4-x^2)}{3+x}$.
10. $y = \dfrac{(2-x)^2}{x}$.

11. $y = \dfrac{x^2+1}{x^2+4}$; $\quad y^2 = \dfrac{x^2+1}{x^2+4}$.

12. $y = \dfrac{1}{x^2+4}$, $\quad = \dfrac{1}{x^2-4}$.

13. $y^2 = 4 - x^2$; $\quad y^2 = \dfrac{1}{4-x^2}$.

14. $y^2 = \dfrac{4-x}{x}$; $\quad y^2 = \dfrac{x}{4-x}$.

15. $y = \quad x-4 \qquad x-4$

16. $y^2 = x^2\dfrac{x+3}{x-3}$; $\quad y^2 = \dfrac{x^2(x+3)}{3-x}$.

17. $y^2 = x^2(5-x)$; $\quad y^2 =$

18. $x^2 - 4y^2 = 9$; $\quad 4y^2 - x^2 = 9$.

19. $y^2 = x^2(8-x)(x-3)$; $\quad y^2 = x(x-3)(x-8)$; $\quad y^2 = x(x-3)^2$.

20. Prove that in the parabola $y^2 = 4ax$, the length of the chord through the focus perpendicular to the axis (in any conic, this chord is called the *latus rectum*) is $4a$.

21. Prove that the length of the latus rectum of an ellipse or hyperbola is $2b^2/a$.

22. Prove that in an ellipse, $SP + S'P = 2a$, and that in a hyperbola, $SP \sim S'P = 2a$.

23. Draw the graphs of $xy = 12$; $\quad x^2y^2 = 12$; $\quad x^2y^3 = a^5$; $\quad x^3y^2 = a^5$.

24. Draw the graphs of
$$y = \sin x; \quad y = 2\sin x; \quad y = \sin(\tfrac{1}{3}\pi + x); \quad y = \sin(x - \tfrac{1}{4}\pi).$$

25. Draw the graphs of $y = 2\sin^2 x$; $\quad y = 1 + \cos 2x$.

26. Draw the graph of $y = \sin x + \cos x$. [Draw the graphs of $\sin x$ and $\cos x$ on the same diagram, and then a third graph whose ordinate at any point is the sum of the ordinates of the first two graphs at the same point.] Also of $y = \sin x - \cos x$.

27. Draw the graphs of $y = \sin x + \sin 2x$; $\quad y = \sin x + \cos 2x$;
$y = \cos x + \cos 2x$; $\quad y = \cos x + \sin 2x$.

CHAPTER II

LIMITS AND CONTINUOUS FUNCTIONS

11. Mean rate of increase of a function.

A change in the value of the argument (x) of a function will generally produce a change in the value of the function (y). If the change in y bears a constant ratio to the change in x, i.e. if a given change in x always produces the same change in y, the function y is said to change at a constant rate; if not, the function changes at a variable rate. The ratio of the increment in the function to the increment in the argument is called the *average* or *mean rate of increase* of the function with respect to its argument for that particular increment. Geometrically, if P and Q be two points on the graph of the function, and if PM be drawn perpendicular to the ordinate of Q (Fig. 17), MQ/PM represents the mean rate of increase of the function for the increment PM of the argument.

From an inspection of the graph of a function, we can obtain a rough idea as to how the function is changing in the neighbourhood of any given value; where it increases rapidly, where slowly, where the mean rate of increase is changing rapidly, and so on.

For instance, from the graph of $y = x^4$ (Fig. 4), it is evident that, in the neighbourhood of the origin, a small increase in x produces a much smaller increase in y; that the same increase in x in the neighbourhood of the point (1, 1) produces a larger increase than before in y; and that the same increase in x sometime after passing (1, 1) produces a very much larger increase in y; moreover, when x is negative, the same (algebraical) increase in x will produce a decrease in y. Hence the function x^4 increases slowly compared with x in the neighbourhood of the origin, rapidly compared with x after passing the value 1; decreases as x increases when x is negative; and the mean rate of increase in the neighbourhood of a point is continually changing as the point moves along the curve.

Again, from the graph of $y = \log_{10} x$, shown in Fig. 17, it is obvious at once that logarithms increase very rapidly as x increases from 0 to 1, slowly after x passes the value 1, and more and more slowly

as x goes on increasing. For instance, an increase of ·09 in the value of x from ·01 to ·1 produces an increase of 1 in the value of y (from -2 to -1), whereas an increase of 9000 in the value of x from 1000 to 10000 also produces only the same increase in the value

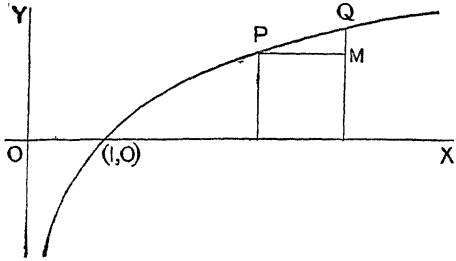


Fig. 17.

of y (from 3 to 4). The smaller the value of x, the more rapidly is y increasing with respect to x; the greater the value of x, the more slowly is y increasing with respect to x.

The important fact to notice is that the mean rate of increase of a function for a given interval varies from value to value. It is never constant save in one case, viz. when the graph is a straight line. This may be seen as follows:

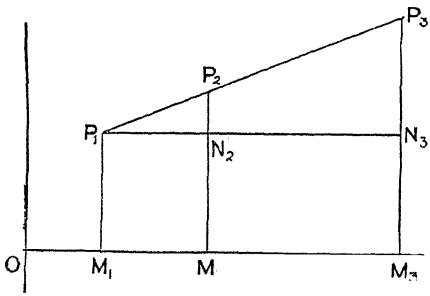


Fig. 18.

If the mean rate of increase of y with respect to x is constant, then in Fig. 18,

$$N_2P_2/P_1N_2 = N_3P_3/P_1N_3,$$

since these represent the average rates of increase of y for the increments M_1M_2, M_1M_3 of x; whence, from the properties of similar triangles, it follows that P_1, P_2, P_3 are collinear. Since any three points on the graph are collinear, the graph must be a straight line.

Hence $y = ax + b$ is the only function of x whose mean rate of increase is constant. [The rate of increase of y with respect to x in this case is equal to a, since any increase in the value of x produces an increase of a times as much in the value of y; if x is increased by h, y becomes $a(x+h)+b$, i.e. y increases by ah.]

We shall in Art. 19 explain what is meant by the 'rate of increase of a function for a particular value of its argument'.

In all other functions except the linear function $ax+b$, this rate of increase is constantly changing. For each value of x, there is usually a definite rate of increase of y per unit increase of x; but as soon as the value of x is altered, this rate of increase is also thereby altered. It is the object of the Differential Calculus to find an exact measure of this rate of change of a function with respect to its argument for any value of the argument, and this measure is given by what is called the differential coefficient of the function.

Before proceeding to the formal definition and the methods of evaluation of the differential coefficient of a function, it is necessary first to get clear ideas of a *limit* and a *continuous function*. These will now be considered in turn.

12. Limits.

Let y be a function of x; then to every value of x corresponds a value (real or imaginary) of y. If x takes in succession a series of values which gradually approach a fixed number a, then it may happen that the corresponding values of y gradually approach a fixed number b, and we may be able to make y as near b as we please by taking x near enough to a. This number b is then said to be the limiting value, or more briefly, the *limit* of y as x approaches a. The values of y may behave in the same manner if x takes a succession of values which increase indefinitely; in this case, b is said to be the limit of y when x becomes infinite.

More precisely, if, as x approaches a value a, y approaches a value b in such a way that $|y-b|$ can be made less than any assignable quantity by taking x sufficiently near a (and remains less for all values of x which are still nearer to a), then b is said to be the limiting value of y as x approaches the value a; this may be written *

$$\operatorname*{Lt}_{x \to a} y = b.$$

* The symbol $\operatorname*{Lt}_{x=a} y = b$ is used in many books, but that given above is preferable because in many cases x cannot be taken *equal* to a; it can only be taken as near to a as we please without actual coincidence with it.

Similarly, if as x increases indefinitely, y approaches a value b in such a way that $|y-b|$ can be made less than any assignable quantity by taking x sufficiently large (and remains less for all values of x which are still larger), then b is said to be the limiting value of y as x becomes infinite; this is written

$$\operatorname*{Lt}_{x \to \infty} y = b.$$

As a rule, in the case of simple functions, the limiting value, when it exists, is the same whether x approaches a from above or below, but it is possible that the limit may be different in these two cases. E.g. the limit of the principal value (Art. 102) of $\tan^{-1}(1/x)$ as $x \to 0$ is $+\frac{1}{2}\pi$ if x approaches 0 from the positive side, and $-\frac{1}{2}\pi$ if x approaches 0 from the negative side (see Fig. 29), since the angle in the first quadrant whose tangent is $1/x$ can be made as nearly equal to $+\frac{1}{2}\pi$ as we please by taking x sufficiently small and positive; and the angle in the fourth quadrant whose tangent is $1/x$ can be made as nearly equal to $-\frac{1}{2}\pi$ as we please by taking x sufficiently small and negative.

These results might be written

$$\operatorname*{Lt}_{x \to 0} \tan^{-1}(1/x) = -\tfrac{1}{2}\pi; \qquad \operatorname*{Lt} \tan^{-1}(1/x) = +\tfrac{1}{2}\pi.$$

For another (geometrical) example, see Art. 14 (1), Fig. 22.

Therefore, strictly speaking, the side from which x approaches a should be specified. If not, it may be taken that the limit is the same in both cases.

13. Examples of limits.

(1) Find $\operatorname*{Lt}_{x \to 3} \dfrac{x^2-9}{x-3}$.

The value of this fraction is obtained at once by direct substitution for any value of x except $x = +3$. Denoting the fraction by y, we have when $x = 0$, $y = 3$; when $x = 1$, $y = 4$; when $x = 2$, $y = 5$, but when $x = 3$, numerator and denominator both become zero, and we get $y = 0/0$, which is quite indeterminate [since any finite number multiplied by 0 gives 0]. Instead of taking x equal to 3, take a series of values for x which get nearer and nearer to 3 and ultimately differ from 3 by as small a quantity as we please (i.e. in the words of the definition, let x approach the value 3).

E.g. if $x = 2{\cdot}9$, $y = \dfrac{(2{\cdot}9)^2-9}{2{\cdot}9-3} = 2{\cdot}9+3 = 5{\cdot}9$;

if $x = 2{\cdot}99$, $y = \dfrac{(2{\cdot}99)^2-9}{2{\cdot}99-3} = 2{\cdot}99+3 = 5{\cdot}99$;

if $x = 2{\cdot}9$ $\quad \dfrac{(2{\cdot}999)^2-9}{2{\cdot}999-3} = 2{\cdot}999+3 = 5{\cdot}999$;

and so on.

Similarly, taking values of x which approach 3 from the other side, we get

$$\text{if } x = 3{\cdot}1, \quad y = \frac{(3{\cdot}1)^2-9}{3{\cdot}1-3} = 3{\cdot}1+3 = 6{\cdot}1$$

$$\text{if } x = 3{\cdot}01, \quad y = \frac{(3{\cdot}01)^2-9}{3{\cdot}01-3} = 3{\cdot}01+3 = 6{\cdot}01;$$

$$\text{if } x = 3{\cdot}001, \quad y = \frac{(3{\cdot}001)^2-9}{3{\cdot}001-3} = 3{\cdot}001+3 = 6{\cdot}001;$$

and so on.

Both sets of values of y are approaching the number 6, and can evidently be made to differ from 6 by as small a quantity as we please by taking x sufficiently near to 3.* Hence the limit of $(x^2-9)/(x-3)$, as $x \longrightarrow 3$ from either above or below, is 6.

The result is obtained at once by dividing the numerator of the given fraction by the denominator; this gives $x+3$, which evidently approaches the value 6 as x approaches 3. But the student will know from algebra that the division by $x-3$ is not permissible unless $x-3$ is different from zero; it is not permissible when $x = 3$, and therefore we still have no value for the fraction when $x = 3$. [See further Art. 17 (5).]

The above discussion furnishes a good illustration of the way in which a fraction may tend to a finite value when its numerator and denominator both tend to zero and, although in this case the value (of the limit when $x \longrightarrow 3$, not the value when $x = 3$) might have been obtained more simply by cancelling out the non-vanishing factor $x-3$, yet there are many cases in which there is no such common factor. It will be seen that differential coefficients are limits of fractions whose numerator and denominator both $\longrightarrow 0$.

(2) *Recurring decimals* furnish good illustrations of the meaning and nature of limits. We find, by arithmetic, the 'value' of $\cdot\dot{1}$ to be $\frac{1}{9}$.

Now $\cdot\dot{1} = \cdot 1111 \ldots = \dfrac{1}{10} + \dfrac{1}{10^2} + \dfrac{1}{10^3} + \ldots$ to infinity, and what is really meant is that the sum of n terms of this series, as $n \longrightarrow \infty$, approaches the limit $\frac{1}{9}$.

* Generally, taking $x = 3 \pm \epsilon$, we get $y = \dfrac{(3\pm\epsilon)^2-9}{(3\pm\epsilon)-3} = (3\pm\epsilon)+3 = 6\pm\epsilon$; the difference between y and 6 is equal to the difference between x and 3, and therefore, in order to make y differ from 6 by less than any assigned quantity, it is only necessary to make x differ from 3 by less than the same assigned quantity.

The difference
between $\frac{1}{9}$ and the first term ·1 is $1/90$;
" " sum of the first 2 terms ·11 is $1/900$;
" " " " 3 terms ·111 is $1/9000$;
" " " " 10 terms is $1/(9\times 10^{10})$;
" " " " 100 terms is $1/(9\times 10^{100})$;
and so on.

The difference between $\frac{1}{9}$ and the sum of n terms of the series can be made less than any quantity that may be specified, however small it may be, by taking a sufficient number of terms.

(3) Find $\mathop{\mathrm{Lt}}\limits_{n\to\infty}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}\right]$.

The series in the brackets is a geometrical progression; its sum to n terms is, by the ordinary formula $a(1-r^n)/(1-r)$, equal to $1-1/2^n$. As n becomes very large, $1/2^n$ approaches the value zero, and can be made as small as we please by taking n sufficiently large; hence the sum of n terms of the series can be made as near 1 as we please.

Therefore $$\mathop{\mathrm{Lt}}\limits_{n\to\infty}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}\right]=1.$$

Geometrically, if AB (Fig. 19) be a straight line of unit length, and if AB be bisected in P_1, P_1B in P_2, P_2B in P_3, P_3B in P_4 and so on,

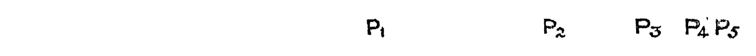


Fig. 19.

the sum of n terms of the given series is represented by
$$AP_1+P_1P_2+P_2P_3+P_3P_4+\dots+P_{n-1}P_n, \quad \text{i. e. } AP_n,$$
and it is obvious that, as n increases, P_n tends to coincide with B. P_n may be made as near to B as we please by performing a sufficient number of bisections, but since there is always a distance between P_n and B equal to half the last segment bisected, no finite number of bisections, however great, can make P_n coincide with B. B is the 'limiting position' of P_n, and AB is the limit of AP_n as $n\to\infty$, i.e. 1 is the limit of
$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\dots+\frac{1}{2^n} \text{ as } n\to\infty.$$

It follows, from the definition of a limit, that the sum of n terms of the series can be made to differ from 1 by less than any assignable quantity. In this case, it is quite easy to determine how many terms must be taken

in order to make the sum differ from 1 by a given small amount. If the difference is to be less than σ,

$$\text{then} \quad 1/2^n \text{ is to be less than } \sigma.$$

$$\therefore \quad 2^n \text{ must be } > 1/\sigma, \text{ i.e. } > \sigma^{-1}.$$

$$\therefore \quad \text{taking logarithms, } n \log 2 > -\log \sigma, \text{ and } n > -\log \sigma / \log 2.$$

If $\sigma = 10^{-1000}$, this gives $n > 3321{\cdot}9 \ldots$.

Therefore the sum of 3322 terms of the series will differ from 1 by a quantity less than 10^{-1000}, and evidently any larger number of terms will have a sum still nearer to 1.

(4) Find $\underset{x \to 2}{\mathrm{Lt}} \dfrac{\sqrt{(3-x)} - \sqrt{(x-1)}}{6-3x}$.

If 2 be substituted for x in this expression, the numerator and denominator both become zero, and we again get the meaningless expression 0/0. If the numerator of the given fraction be rationalized by multiplying numerator and denominator by $\sqrt{(3-x)} + \sqrt{(x-1)}$, the result is

$$\frac{(3-x)-(x-1)}{(6-3x)\{\sqrt{(3-x)}+\sqrt{(x-1)}\}} = \frac{2(2-x)}{3(2-x)\{\sqrt{(3-x)}+\sqrt{(x-1)}\}}$$

$$= \frac{2}{3\{\sqrt{(3-x)}+\sqrt{(x-1)}\}},$$

provided x is not exactly equal to 2 [see Example (1)].

As x *approaches* the value 2, this approaches the value $\dfrac{2}{3(1+1)}$, i.e. $\frac{1}{3}$. Therefore the limit of the given expression, as $x \to 2$, is $\frac{1}{3}$, but the expression has no value when x is equal to 2 exactly, or it is undefined for the value $x = 2$.

(5) *Limits of x/a and a/x when $x \to 0$ and when $x \to \infty$.*

It is evident that the value of the fraction x/a diminishes with x, and can be made as small as we please by taking x sufficiently small; this is expressed by the statement $\underset{x \to 0}{\mathrm{Lt}}\, x/a = 0$.*

Similarly, the value of the fraction x/a can be made as large as we please by taking x sufficiently large; this may be expressed as $\underset{x \to \infty}{\mathrm{Lt}}\, x/a = \infty$.

Infinity, not being a definite value, is not a limit in the sense of the definition at the beginning of this article: strictly, x/a has no 'limit' as $x \to \infty$, but $\mathrm{Lt}\, x/a = \infty$ is a convenient symbolic

* In this case the limit coincides with the value when x is actually *equal* to 0.

statement of the fact that x/a can be made as large as we please by taking x sufficiently great.

Similarly, we may say that $\operatorname*{Lt}_{x \to a} x/(x-a) = \infty$.

Again, the value of the fraction a/x can be made less than any assignable quantity by taking x sufficiently great, and greater than any assignable quantity by taking x sufficiently small. These facts are expressed symbolically as follows:

$$\operatorname*{Lt}_{x \to \infty} a/x = 0; \qquad \operatorname*{Lt}_{x \to 0} a/x = \infty.$$

(6) Find $\operatorname*{Lt}_{n \to \infty} x^n/n!$ [x a fixed number].

This may be written

$$\frac{x}{1} \times \frac{x}{2} \times \frac{x}{3} \times \qquad \times \frac{x}{n}.$$

Now, however large x may be, since it is fixed, and n undergoes *unlimited* increase, these factors continually diminish, and after a time will be very small. As soon as $n > 2x$, x/n will be $< \frac{1}{2}$, and all the succeeding factors, since they continually diminish, will be $< \frac{1}{2}$. If A be the product of all the factors up to this stage, then after m more factors (each $< \frac{1}{2}$), the total product will be $< A/2^m$. Now since A, although it may be a large number, is yet *finite*, and $1/2^m$ can be made as small as we please by increasing m sufficiently, it follows that the value of this product may be made as small as we please by taking m, and therefore n, sufficiently large,

$$\text{i.e. } \operatorname{Lt} x^n/n! = 0.$$

(7) *Limiting values of rational algebraical fractions when* $x \longrightarrow 0$ *and when* $x \longrightarrow \infty$.

First consider a fraction whose numerator and denominator are each of the second degree.

Let $y = \dfrac{ax^2 + bx + c}{a'x^2 + b'x + c'}$.

It is evident that the terms which contain x can be made as small as we please by taking x sufficiently small; therefore the numerator and denominator approach the values c and c' respectively as $x \longrightarrow 0$. In fact in this case we can put $x = 0$ exactly, and obtain the limit when $x \longrightarrow 0$, agreeing with the actual value of the fraction when $x = 0$, as c/c'.

To find the limit when $x \to \infty$, we have, on dividing numerator and denominator by x^2,

$$y = \frac{a+b/x+c/x^2}{a'+b'/x+c'/x^2}.$$

The terms with x or x^2 in the denominator can be made as small as we please by taking x sufficiently large; therefore the numerator and denominator approach the values a and a' respectively as $x \to \infty$.

Hence $$\underset{x \to \infty}{\mathrm{Lt}}\ y = a/a'.$$

Any such fraction can be treated in this manner. The limiting value when $x \to 0$ is obtained by substituting $x = 0$ in numerator and denominator.* The limit when $x \to \infty$ is found by dividing numerator and denominator by the highest power of x which occurs in the fraction. If the numerator is of lower degree than the denominator, the value of the fraction will tend to zero as $x \to \infty$; if the numerator is of higher degree than the denominator, the value of the fraction will increase indefinitely as $x \to \infty$; if numerator and denominator are of the same degree n, the limiting value will be a/a', where a and a' are the coefficients of x^n in numerator and denominator respectively.

(8) $\mathrm{Lt}\ \dfrac{x^n - a^n}{x-a}$

The investigation of this limit is divided into three cases according as n is a positive integer, a positive fraction, or negative.

(i) Let n be a positive integer.

Then, by ordinary division,

$$\frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1}.$$

As $x \to a$, each of these terms, and there are n of them, approaches the value a^{n-1}; †

$$\therefore \quad \text{the limit} = na^{n-1}.$$

(ii) Let n be a positive fraction p/q, where p and q are positive integers.

Put $x = y^q$ and $a = b^q$; $\therefore\ x^{p/q} = (y^q)^{p/q} = y^p$, and similarly $a^{p/q} = b^p$.

* Provided c and c' (the constant terms) are not both zero, in which case it would be necessary first to divide out numerator and denominator by some power of x.

† Here, and in the succeeding cases, the results of Art. 15 are assumed.

Also when $x \to a$, $y \to b$;

$$\therefore \operatorname*{Lt} \frac{x^{p/q}-a^{p/q}}{x-a} = \operatorname*{Lt}_{y \to b} \frac{y^p-b^p}{y^q-b^q} = \operatorname*{Lt}_{y \to b} \frac{\dfrac{y^p-b^p}{y-b}}{\dfrac{y^q-b^q}{y-b}} = (\text{by case i}) \frac{pb^{p-1}}{qb^{q-1}}$$

$$= \frac{p}{q} b^{p-q} = \frac{p}{q}(b^q)^{p/q-1} = \frac{p}{q} a^{p/q-1}.$$

(iii) Let n be $-$ and equal to $-m$, where m is $+$.

$$\text{Then } \operatorname*{Lt}_{x \to a} \frac{x^{-m}-a^{-m}}{x-a} = \operatorname*{Lt}_{x \to a} \frac{\dfrac{1}{x^m}-\dfrac{1}{a^m}}{x-a} = \operatorname*{Lt}_{x \to a} \frac{\dfrac{a^m-x^m}{x^m a^m}}{x-a}$$

$$= \operatorname*{Lt} -\frac{1}{x^m a^m} \cdot \frac{x^m-a^m}{x-a} = (\text{by the preceding cases}) - \frac{1}{a^{2m}} \cdot ma^{m-1}$$

$$= -ma^{-m-1}.$$

Therefore, for all rational values of n,

$$\operatorname*{Lt} \frac{x^n-a^n}{x-a} = na^{n-1}.*$$

The importance of this limit lies in the fact that the differential coefficient of any power of x can be at once deduced from it (Art. 27).

(9)† $\operatorname*{Lt} \left(1+\frac{1}{m}\right)^m$

This is a limit of extreme importance, and a full discussion of it is reserved until later (Chapter X). In the meantime, we may take the particular case when m is supposed to become indefinitely great through a succession of positive integral values.

Since m is a positive integer, we get, on expanding by the Binomial Theorem, a series of $m+1$ terms for $(1+1/m)^m$, viz.

$$\left(1+\frac{1}{m}\right)^m = 1 + m \cdot \frac{1}{m} + \frac{m(m-1)}{1 \cdot 2} \cdot \frac{1}{m^2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{m^3} + \dots$$

$$= 1+1+\frac{1}{1 \cdot 2}\left(1-\frac{1}{m}\right) + \frac{1}{1 \cdot 2 \cdot 3}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right) + \dots$$

$$\text{to } m+1 \text{ terms.} \qquad \text{(i)}$$

As m increases, every term of this series after the first two increases, and moreover, additional terms, all of which are positive, are added on; hence $(1+1/m)^m$ increases as m increases.

* Notice that Example (1) is a particular case of this limit, viz. $a = 3$, $n = 2$.
† This may be deferred until Chapter X is reached.

Again, the sum of this series is evidently less than the sum of the series

$$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots+\frac{1}{m!}\tag{ii}$$

since every term of the series (i) after the second is less than the corresponding term of the series (ii); and the sum of the series (ii) again is less than the sum of the series

$$1+1+\frac{1}{2}+\frac{1}{2.2}+\ldots+\frac{1}{2^{m-1}},\tag{iii}$$

since every term of (ii) after the third is less than the corresponding term of (iii). The last series, after the first term, is a geometrical progression whose common ratio is $\frac{1}{2}$; hence its sum is equal to

$$1+\frac{1-(\frac{1}{2})^m}{1-\frac{1}{2}},\ \text{i.e. } 1+2\left(1-\frac{1}{2^m}\right),\ \text{i.e. } 3-\frac{1}{2^{m-1}}.$$

Hence $$\left(1+\frac{1}{m}\right)^m<3-\frac{1}{2^{m-1}},$$

and therefore, *a fortiori*, <3, however great m may be.

We have now shown that $(1+1/m)^m$ continually increases with m and yet is always less than 3; hence apparently we may conclude (and it can be formally proved) that $(1+1/m)^m$ approaches a definite limit which is not greater than 3.

If we evaluate $(1+1/m)^m$ for increasing numerical values of m, we obtain a better idea of the magnitude of this limit. For example,

if $m=10$, $(1+1/m)^m = 1{\cdot}1^{10} = 2{\cdot}5937$;
if $m=50$, $(1+1/m)^m = 1{\cdot}02^{50} = 2{\cdot}6916$;
if $m=100$, $(1+1/m)^m = 1{\cdot}01^{100} = 2{\cdot}7048$;
if $m=1000$, $(1+1/m)^m = (1{\cdot}001)^{1000} = 2{\cdot}7169$;
if $m=10000$, $(1+1/m)^m = (1{\cdot}0001)^{10000} = 2{\cdot}7181$;
if $m=100000$, $(1+1/m)^m = (1{\cdot}00001)^{100000} = 2{\cdot}7183$, and so on;

from which it appears that, as m increases indefinitely, $(1+1/m)^m$ approaches a limit which is a little greater than 2·718.

This limit is a perfectly definite but incommensurable number (i.e. its value cannot be expressed in the form a/b, where a and b are integral) which is denoted by the letter e. It is one of the most important numbers in mathematics, and is continually occurring in all its branches, both pure and applied. Its value to ten places of decimals is 2·7182818285..., and it has actually been computed to more than 500 places of decimals.

(10) *Examples from Trigonometry.*

$$\mathop{\mathrm{L}}\frac{\sin x}{x}.$$

This is a very important limit, since the differential coefficients of all the circular functions can be deduced from it.

Let AOP (Fig. 20) be an angle of x ($< \frac{1}{2}\pi$) radians at the centre of a circle of radius r; and let the tangent at A cut OP produced in T. Draw PN perpendicular to OA.

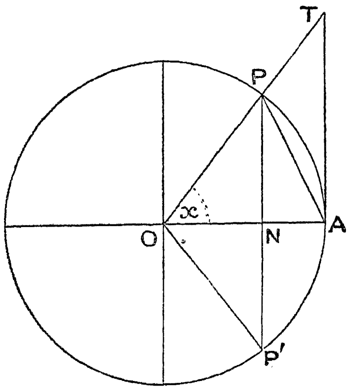


Fig. 20.

It is obvious that

$$\text{area of } \Delta\, AOP < \text{area of sector } AOP < \text{area of } \Delta\, AOT,$$

i.e. $$\tfrac{1}{2} r \,.\, r \sin x < \tfrac{1}{2} r^2 x < \tfrac{1}{2} r \,.\, r \tan x;$$

whence, dividing by $\frac{1}{2} r^2$, $\sin x < x < \tan x$,

and, dividing by $\sin x$, $\quad 1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}.$

Hence, inverting and therefore reversing the inequality signs,

$$1 > \frac{\sin x}{x} > \cos x.$$

Now, as x approaches the value 0, $\cos x$ approaches the value 1 and can be made to differ from 1 by as small a quantity as we please by taking x sufficiently small.

Therefore $(\sin x)/x$, which is between 1 and $\cos x$, also approaches the value 1, and can be made to differ from it by as small a quantity as we please by taking x sufficiently small; hence

$$\mathop{\mathrm{Lt}}_{x \to 0} \frac{\sin x}{x} = 1.$$

It is interesting to notice how the ratio $(\sin x)/x$ approaches 1 as $x \to 0$.

For an angle of 5°, $\sin x = \cdot 0871557$, $x = \cdot 0872665$, $(\sin x)/x = \cdot 99873$;
" " 2°, $\sin x = \cdot 0348995$, $x = \cdot 0349066$, $(\sin x)/x = \cdot 99980$;
" " 1°, $\sin x = \cdot 0174524$, $x = \cdot 0174533$, $(\sin x)/x = \cdot 99995$;
" " 30′, $\sin x = \cdot 0087265$, $x = \cdot 0087266$, $(\sin x)/x = \cdot 99999 + \ldots$
" " 10′, $\sin x = \cdot 0029089$, $x = \cdot 0029089$, in this case at least the first seven figures coincide.

It must be carefully noticed that it is the ratio of the sine of an angle to the *circular measure* of the *same* angle which approaches unity as the angle is indefinitely diminished. Thus $\mathrm{Lt}\,(\sin 2x)/x$ is not 1, but it is evidently the same as $2 \times \mathrm{Lt}\,(\sin 2x)/2x$ as $x \to 0$, and the second factor of this tends to the limit 1.

Therefore $$\mathrm{Lt}\,\frac{\sin 2x}{x} = 2.$$

Similarly, $$\mathrm{Lt}\,\frac{\sin ax}{x} = \mathrm{Lt}_{x\to 0}\,\frac{\sin ax}{ax} \times a = 1 \times a = a,$$

$$\mathrm{Lt}_{x\to 0}\,\frac{\sin x^\circ}{x^\circ} = \mathrm{Lt}_{x\to 0}\,\frac{\sin x^\circ}{\pi x/180} \times \frac{\pi}{180} = 1 \times \frac{\pi}{180} = \frac{\pi}{180},$$

$$\mathrm{Lt}_{x\to 0}\,\frac{\sin ax}{\sin bx} = \mathrm{Lt}_{x\to 0}\,\frac{(\sin ax)/ax}{(\sin bx)/bx} \times \frac{a}{b} = \frac{1}{1}^{*} \times \frac{a}{b} = \frac{a}{b},$$

and so on.

Geometrically, it follows from this limit that, when an arc of a circle is indefinitely diminished, the ratio of the chord to the arc approaches the limit 1.

For the length of the arc PAP' (Fig. 20) which subtends an angle $2x$ radians at the centre O is $2rx$; and the length of the chord of the arc $= 2PN = 2r \sin x$,

$$\therefore \quad \frac{\text{chord}}{\text{arc}} = \frac{2r\sin x}{2rx} = \frac{\sin x}{x}.$$

As the arc is indefinitely diminished, $x \to 0$, and this ratio $\to 1$.

This ratio rapidly approaches its limiting value, so that for a small angle, the length of the chord is a good approximation to the length of the arc.

Two important limits involving the cosine can be deduced from the preceding limit.

We have $$(1 - \cos x)(1 + \cos x) = \sin^2 x,$$

$$\therefore \quad 1 - \cos x = \frac{\sin^2 x}{1 + \cos x},$$

* This assumes the results of Art. 15, q.v.

$$\frac{1-\cos x}{x}=\frac{\sin^2 x}{x(1+\cos x)}=\frac{\sin x}{x}\times\frac{\sin x}{1+\cos x}$$

$$\text{which} \to 1\times\frac{0}{2},^{*} \quad \text{i.e. } 0 \text{ as } x\to 0;$$

and $\dfrac{1-\cos x}{x^2}=\left(\dfrac{\sin x}{x}\right)^2\times\dfrac{1}{1+\cos x}$ which $\to 1^2\times\dfrac{1}{2}$, i.e. $\dfrac{1}{2}$ as $x\to 0$.

Hence $\quad \underset{x\to 0}{\mathrm{Lt}}\dfrac{1-\cos x}{x}=0; \qquad \underset{x\to 0}{\mathrm{Lt}}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}.$

Also $\quad \underset{x\to 0}{\mathrm{Lt}}\dfrac{\tan x}{x}=\underset{x\to 0}{\mathrm{Lt}}\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}=1\times 1=1.$

14. Geometrical examples of limits.

(1) *Tangent to a curve.* Let TPT' (Fig. 21) be a tangent to a circle at a point P, and let Q, Q' be points on the circle, one on either side of P; join PQ, PQ' and produce them. If the points Q and Q' be supposed to move along the circle towards P, the angles TPQ, $T'PQ'$ will diminish, and will be very small when Q and Q' are very near to P; they can be made as small as we please by taking Q and Q' sufficiently near to P. Hence the tangent $T'PT$ is the limiting position of the chords PQ, $Q'P$ as Q and Q' approach the limiting position P. This is the definition of a tangent at a point of a curve in general, viz. *The tangent to a curve at a point P is the limiting position of a chord PQ of a curve, when Q approaches indefinitely near to P.*

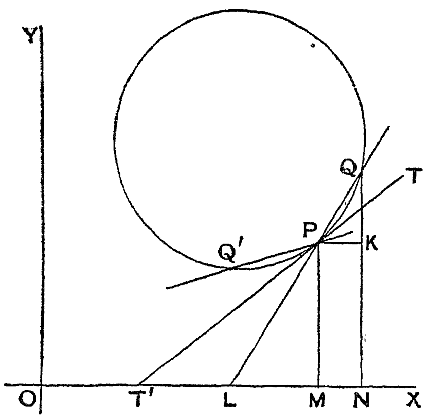


Fig. 21.

It will be noticed that with this definition it is possible for the tangent to a curve at a point to cross the curve at the point (see Art. 59).

If PM, QN be perpendiculars from P and Q to the axis of x, and PK perpendicular to QN, then $KQ/PK=\tan QPK=$ the tangent of the angle PLX which PQ makes with the axis of x; this is called the *slope* of the secant PQ.

As Q approaches P, QP approaches TP and the angle PLX approaches the limiting value $PT'X$. KQ and PK both become indefinitely small, but

* This assumes the results of Art. 15, q. v.

their ratio KQ/PK, being $\tan PLX$, tends to the limiting value $\tan PT'X$, i.e. the slope of the tangent.

In the case of the circle, and usually in the case of any curve, the limit is the same from whichever side the point Q approaches the point P. It is possible, however, for the limit to be different in the two cases. This is the case at a point such as P shown in Fig. 22, where a curve is drawn consisting of two branches intersecting at an angle.

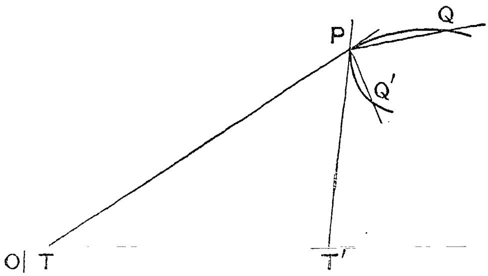


Fig. 22.

If Q approaches P from above, the chord QP approaches the limiting position PT, and its slope the limiting value $\tan PTX$; if Q' approaches P from below, the chord PQ' approaches the limiting position PT', and its slope the limiting value $\tan PT'X$. In such a case the slope is said to be *discontinuous* at the point P (Art. 17 (1)).

(2) *Perimeter and area of a circle.* Let a regular polygon with n sides be inscribed in a circle of radius r, and let tangents be drawn at its angular points, forming a regular circumscribed polygon with n sides. Then it is evident that the perimeter of the inscribed polygon increases, and that of the circumscribed polygon decreases as n increases.

A side of either polygon subtends an angle $2\pi/n$ radians at the centre of the circle, so that the length of a side of the inscribed polygon (Fig. 23)

$$= 2\,PM = 2\,OP \sin MOP = 2r\sin(\pi/n),$$

and the length of a side of the circumscribed polygon

$$= 2\,RQ = 2\,OQ \tan QOR = 2r\tan(\pi/n).$$

Hence

$$\frac{\text{perimeter of inscribed polygon}}{\text{perimeter of circumscribed polygon}} = \frac{PQ}{RS} = \frac{2r\sin(\pi/n)}{2r\tan(\pi/n)} = \cos(\pi/n).$$

Now, as $n \to \infty$, $\cos(\pi/n) \to 1$, therefore the limit of the ratio of the perimeters is 1. Hence the limit of the inscribed perimeter is the same as the limit of the circumscribed perimeter. This common limit of the two perimeters is defined as the perimeter or circumference of the circle.

This gives an excellent illustration of the meaning of a 'limit'. We have the two series of perimeters each gradually approaching the same definite value as n increases, so that either of them may be made to differ from it by as small a quantity as we please by taking n large enough; but no

matter how great n may be, the inscribed and circumscribed perimeters never coincide. The limit, the perimeter of the circle, separates the inscribed and circumscribed perimeters; it is greater than the perimeter of any inscribed polygon and less than the perimeter of any circumscribed

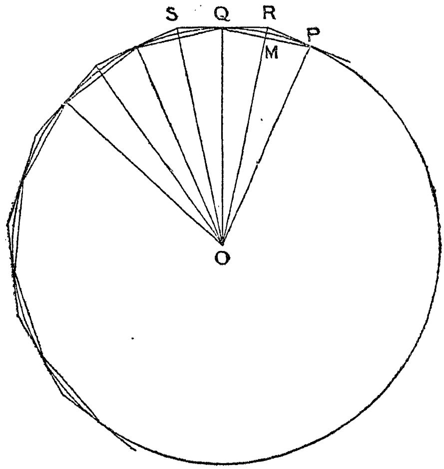


Fig. 23.

polygon, however great the number of sides may be.* The value of the limit is $2\pi r$.

In the same way, the area of the inscribed polygon increases and the area of the circumscribed polygon decreases as n increases.

The inner area

$$= n \cdot \Delta POQ = nPM \,.\, MO = nr^2 \sin(\pi/n) \cos(\pi/n),$$

and the outer area

$$= n \,.\, \Delta ROS = nRQ \,.\, QO = nr^2 \tan(\pi/n).$$

$$\therefore \quad \frac{\text{area of inner polygon}}{\text{area of outer polygon}} = \frac{nr^2 \sin(\pi/n) \cos(\pi/n)}{nr^2 \tan(\pi/n)} = \cos^2(\pi/n).$$

This approaches the limit 1 as $n \to \infty$; and therefore the limit of the area of the inscribed polygon is the same as the limit of the area of the cir-

* This is the principle of the method which was used by mathematicians for many hundreds of years up to the early part of the seventeenth century in their attempts to solve the problem of 'squaring the circle', which is equivalent to finding the value of π. They calculated the perimeters of inscribed and circumscribed polygons with large numbers of sides, and assumed the length of the circumference to be intermediate between them. In this way, Van Ceulen obtained the value of π to 32 places of decimals by calculating the perimeter of a polygon with the enormous number of 2^{62}, i.e. 4,611686,018427,387904 sides! The perimeter of the circle is greater than the perimeter of this inscribed polygon and less than the perimeter of the corresponding circumscribed polygon.

cumscribed polygon. This common limit is defined to be the area of the circle. Both areas get nearer and nearer, and can be made as near as we please, to the 'area of the circle' by taking n sufficiently large; but the area of the circle is greater th- the area of any inscribed polygon and less than the area of any circumscribed polygon, however great be the number of sides. It is equal to

$$\mathrm{Lt}\; nr^2 \tan(\pi/n)$$

$$nr^2 \cdot \frac{\tan(\pi/n)}{\pi/n}$$

$$= r^2 \times 1 \times \pi$$

$$= \pi r^2.$$

It is interesting and instructive to see how the perimeters and areas approach their limits, and a few of their values are appended. The polygons are inscribed in a circle of radius r.

		inscribed perimeter	*circumscribed perimeter*	*inscribed area*	*circumscribed area*
Polygon with	4 sides	$5{\cdot}6569r$	$8r$	$2r^2$	$4r^2$
„ „	8 „	$6{\cdot}1229r$	$6{\cdot}6274r$	$2{\cdot}8284r^2$	$3{\cdot}3137r^2$
„ „	16 „	$6{\cdot}2430r$	$6{\cdot}3652r$	$3{\cdot}0615r^2$	$3{\cdot}1826r^2$
„ „	32 „	$6{\cdot}2731r$	$6{\cdot}3035r$	$3{\cdot}1215r^2$	$3{\cdot}1517r^2$
„ „	64 „	$6{\cdot}2806r$	$6{\cdot}2883r$	$3{\cdot}1365r^2$	$3{\cdot}1441r^2$
„ „	128 „	$6{\cdot}2825r$	$6{\cdot}2844r$	$3{\cdot}1403r^2$	$3{\cdot}1422r^2$
„ „	256 „	$6{\cdot}2830r$	$6{\cdot}2835r$	$3{\cdot}1412r^2$	$3{\cdot}1418r^2$

We see that the first and second columns are closing in on $2\pi r$, i.e. $6{\cdot}2832r$, and the last two columns on πr^2, i.e. $3{\cdot}1416r^2$.

(3) *Area and length of any curve.* Let (Fig. 24) PQ be an arc of a curve, and PM, QN perpendiculars from P and Q to the axis of x; let MN be divided into n equal parts, each of length h, and let the ordinates at the points of division $M_1, M_2, \ldots$, meet the curve in $P_1, P_2, \ldots$. Through each of these points draw parallels to the axis of x to meet the adjacent ordinates on either side.

Then the sum of the inner rectangles $PM_1, P_1M_2, P_2M_3, \ldots$ increases, and the sum of the outer rectangles $P_1M, P_2M_1, P_3M_2, \ldots$ decreases, as n increases. Moreover, the difference between the two sets of rectangles is equal to the sum of the small rectangles $PP_1, P_1P_2, \ldots$, and this sum is equal to the area of a rectangle pq whose base is h, and height $NQ-MP$, as is obvious by moving them all parallel to the axis of x until they are between QN and the next ordinate; i.e. the difference between the two sets of rectangles $= h(NQ-MP)$. Now this can be made as small as we please by taking h sufficiently small, i.e. by making n sufficiently large. Hence both sets of rectangles tend to a common limit, as n increases indefinitely. This limit is defined as the area between the curve PQ, the axis of x and the ordinates

MP and NQ. It is greater than the sum of the inner rectangles and than the sum of the outer rectangles, however great n be taken.

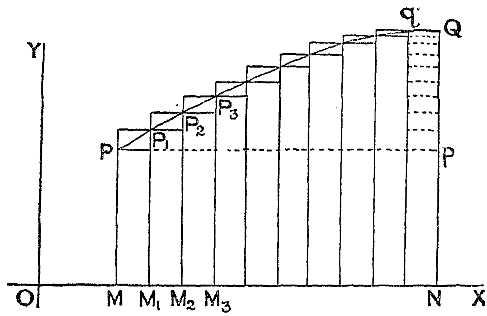


Fig. 24.

In proving the limits of the two sums identical, we have supposed the ordinates to increase continually or to decrease continually throughout the arc PQ. If this is not the case, the arc can be divided up into a finite number of parts, throughout each of which the ordinate either continually increases or continually decreases.

It is not essential that the parts into which MN is divided should be equal. It can be shown that the limit is the same however MN be divided up, *provided each of the parts tends to zero*, when the number of them is indefinitely increased.

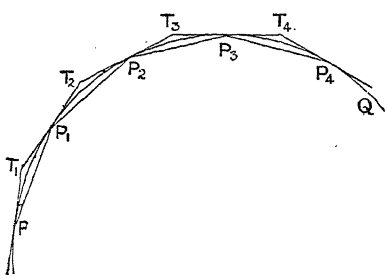


Fig. 25.

Similarly, if the chords PP_1, P_1P_2, P_2P_3, ... be drawn (Fig. 25), and if tangents be drawn to the curve at $P, P_1, P_2, \dots Q$, intersecting at T_1, T_2, T_3, ..., the sum of the chords PP_1, P_1P_2, P_2P_3 ... and the sum of PT_1, T_1T_2, T_2T_3, ... both tend, as $n \to \infty$, to a common limit, which is defined as the 'length of the curve' from P to Q.

(4) *Volume of a solid of revolution.* First consider the case of a right circular cylinder. By inscribing regular polygons in the circular ends and circumscribing regular polygons about them, and joining their angular points by lines parallel to the axis, two right prisms can be obtained of which the volume of the inner increases and the volume of the outer decreases as the number n of sides of the polygons is increased. Also the difference between their volumes can be made as small as we please by taking n large enough. Hence they tend to the same limit as $n \to \infty$, and this limit is defined to be the 'volume' of the cylinder.

Next let the area $MPQN$ of Fig. 24, together with the sets of rectangles,

rotate about the axis of x. The figure produced by the rotation of $MPQN$, which is such that the section of it by any plane perpendicular to the axis of x is a circle, is called a *solid of revolution*. Each rectangle traces out a thin flat cylinder of thickness h. The difference between the sum of the cylinders generated by the inner rectangles and the sum of the cylinders generated by the outer rectangles is equal to the volume generated by the rotation of pq about the axis of x. This volume is

$$\pi NQ^2 . h - \pi MP^2 . h, \text{ i.e. } \pi h(NQ^2 - MP^2),$$

and this can be made as small as we please by taking h sufficiently small, i.e. by making n sufficiently great. Hence both sets of cylinders tend to a common limit as $n \to \infty$. This limit is defined as the 'volume' of the solid of revolution.

As an example, let us find in this way the volume of a right circular cone, the solid of revolution formed by the rotation of a right-angled triangle about one of the sides containing the right angle.

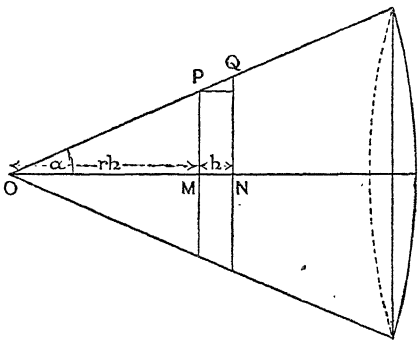


Fig. 26.

Taking this side as axis of x, and dividing it into n parts each of length h, consider the volume of the cylinder formed by the rotation of the inner rectangle which stands upon the $(r+1)^{\text{th}}$ segment of the

Its height (Fig. 26)

$MP = OM \tan\alpha = rh \tan\alpha$;

therefore its volume $\quad = \pi MP^2 . h = \pi r^2 h^3 \tan^2\alpha$,

and the sum of all such volumes is obtained by adding together these terms for all values of r from 1 to $n-1$;

i. e. sum of volumes formed by inner rectangles

$$\begin{aligned}
&= \pi h^3 \tan^2\alpha \, \{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\} \\
&= \pi h^3 \tan^2\alpha \, . \tfrac{1}{6}(n-1)\, n\, (2n-1) \\
&= \pi \tan^2\alpha \, (nh)^3 \, \tfrac{1}{6}(1-1/n)(2-1/n) \\
&= \pi \tan^2\alpha \, . \, b^3 \, . \tfrac{1}{6}(1-1/n)(2-1/n),
\end{aligned}$$

if b be the length nh of the axis of the cone.

As $n \to \infty$, $1/n \to 0$, and this expression tends (in increasing value) to the limit

$$\pi \tan^2\alpha \, . \, b^3 \, . \tfrac{1}{6} . 2, \text{ i.e. } \tfrac{1}{3}\pi b^3 \tan^2\alpha,$$

which may be written $\frac{1}{3}\pi a^2 b$, if a be the radius of the base.

Hence the volume of the cone is equal to $\frac{1}{3}$ of the area of the base $\times$ the height.

If the sum of the outer cylinders be taken, the volume, in exactly similar manner,

$$= \pi h^3 \tan^2\alpha \, (1^2 + 2^2 + 3^2 + \dots + n^2)$$
$$= \pi h^3 \tan^2\alpha \, . \, \tfrac{1}{6} n(n+1)(2n+1)$$
$$= \pi \tan^2\alpha \, . \, b^3 \, . \, \tfrac{1}{6} \, (1 + 1/n)(2 + 1/n),$$

which also tends (this time in decreasing value), as $n \to \infty$, to the limit

(5) *Area of surface of solid of revolution.*

First consider a frustum of a right circular cone. Let similar and similarly situated polygons with n sides be inscribed in the circular ends of the frustum; then, by joining corresponding vertices (Fig. 27) PQ, $P'Q'$, ..., we get a number of trapeziums such as $PP'Q'Q$. As $n \to \infty$, the sum of the areas of these trapeziums tends to a limit which is defined as the area of the curved surface of the frustum.

The area of $PP'Q'Q$

$= \frac{1}{2}$(sum of parallel sides) $\times$ (perpendicular distance between them)
$= \frac{1}{2}(PP' + QQ')\,MN.$

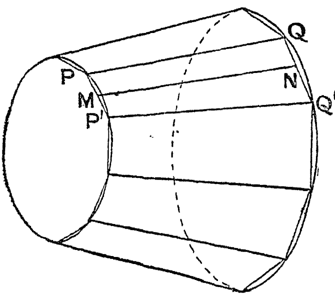


Fig. 27.

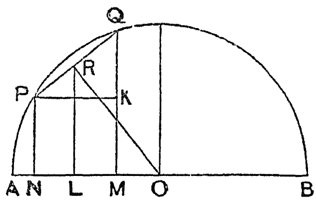


Fig. 28.

Therefore the sum of the areas of the trapeziums is

$$\tfrac{1}{2} n (PP' + QQ')\,MN = \tfrac{1}{2}\,(\text{sum of perimeters of polygons}) \times MN.$$

When $n \to \infty$, the perimeters tend to the circumferences of the circles, and MN tends to the limit PQ; therefore the area of the curved surface of the frustum is

$$\tfrac{1}{2} PQ \, (\text{sum of circumferences of ends}) = PQ \times \text{mean circumference}.$$

By drawing circumscribed polygons to touch the ends of the frustum at $P, P', \dots, Q, Q', \dots$, another set of trapeziums is obtained, the sum of whose areas tends to the same limit as $n \to \infty$.

We can now define the area of the curved surface of any solid of revolution. In Fig. 25, when PQ rotates about the axis of x, the chords PP_1, P_1P_2, P_2P_3, ... and also the lines PT_1, T_1T_2, T_2T_3, ... describe frusta of cones. The sums of the areas of the curved surfaces of these two series of frusta tend, as $n \to \infty$, to a common limit; this limit is defined as the 'area' of the curved surface of the solid of revolution.

As an example, let us find the area of the surface of a sphere, the solid of revolution formed by the rotation of a semicircle about its bounding diameter.

Let PQ (Fig. 28) be a side of a regular polygon inscribed in the semicircle (of radius r), and let R be the middle point of PQ; draw the ordinates PN, RL, QM, and join OR. Draw PK perpendicular to MQ.

The area of the frustum generated by the rotation of PQ is, as just proved, $PQ \times 2\pi RL$. Now the right-angled triangles QPK and ROL are similar, since the sides of one are perpendicular to the sides of the other. Therefore $PQ/PK = OR/RL$, i.e. $PQ \,.\, RL = PK \,.\, OR$. Hence the area traced out by PQ is

$$2\pi PQ \,.\, RL = 2\pi PK \,.\, OR = 2\pi NM \,.\, OR,$$

and the sum of the areas of the frusta is

$$\Sigma\,(2\pi NM \,.\, OR) = 2\pi OR \,.\, \Sigma\,(NM),$$

since OR is the same for every side of the polygon. Taking all the sides of the polygon from A to B, $\Sigma\,(NM) = AB = 2r$, and therefore the sum of the areas of the frusta $= 2\pi OR \times 2r$.

Now as $n \to \infty$, $OR \to$ the limit r, and therefore the area of the surface of the sphere is

$$\text{Lt}\, 2\pi OR \times 2r = 2\pi r \times 2r = 4\pi r^2.$$

15. General theorems on limits.

These have been tacitly assumed in the preceding examples.

(i) *The limit of the algebraical sum of a finite number of quantities is equal to the algebraical sum of their limits.*

For if $\text{Lt}\, y = b$ and $\text{Lt}\, z = c$, then $y = b + \alpha$ and $z = c + \beta$, where α and $\beta \to 0$;

$$\therefore \quad y \pm z = b \pm c + (\alpha \pm \beta),$$

and since $\alpha \pm \beta \to 0$, $y \pm z$ approaches the limit $b \pm c$.

Similarly for any *finite* number of quantities.

This theorem is not true for an infinite number of quantities, as is shown by the following example:

In the series

$$\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n}{n^2},$$

the limit of each term as $n \to \infty$ is 0. Therefore the sum of the limits is 0.

But the sum of the series

$$\frac{1}{n^2}(1 + 2 + 3 + \dots + n) = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \tfrac{1}{2}\left(1 + \frac{1}{n}\right),$$

and the limit of this, as $n \to \infty$, is $\tfrac{1}{2}$.

Hence the limit of the sum of an infinite series is not always equal to the sum of the limits of the separate terms.

(ii) *The limit of the product of a finite number of quantities is equal to the product of their limits.*

In this case
$$yz = (b+\alpha)(c+\beta),$$
$$\therefore\ yz-bc = \alpha c+\beta b+\alpha\beta.$$

When α and $\beta \to 0$, the right-hand side of the equation $\to 0$, therefore yz approaches the limit bc, and similarly for any finite number of factors.

(iii) *The limit of the quotient of two quantities is equal to the quotient of their limits, provided the limit of the denominator is not zero.*

For
$$\frac{y}{z}-\frac{b}{c}=\frac{b+\alpha}{c+\beta}-\frac{b}{c}=\frac{c\alpha-b\beta}{c(c+\beta)},$$

and here, again, the expression on the right-hand side $\to 0$ when α and $\beta \to 0$, *provided* c *is not zero.*

$$\therefore\ y/z \text{ tends to the limit } b/c.$$

Examples III.

Find the limiting values of the following:

1. $\dfrac{x^2-4}{x-2}$ when $x\to 2$ [as in Art. 13 (1)]; 2. $\dfrac{x^3-1}{x-1}$ when $x\to 1$;

3. $\dfrac{x^3-8}{x^2-4}$ when $x\to 2$; 4. $\dfrac{1}{3}+\dfrac{1}{9}+\ldots+\dfrac{1}{3^n}$ when $n\to\infty$;

5. $\cdot 3+\cdot 03+\cdot 003+\ldots$ to n terms when $n\to\infty$.

Find the limits of the following, when $x\to 0$ and when $x\to\infty$:

6. $\dfrac{ax+b}{cx+d}$, 7. $\dfrac{3x^2-5x+2}{5x^2+7x+6}$, 8. $\dfrac{x^2-ax+b}{px+q}$, 9. $\dfrac{x^2+a^2}{x^3+b^3}$,

10. $\dfrac{x^2(3x-2)}{(x-1)^2(4-x)}$; 11. $\dfrac{(2x-1)^3}{x(x+3)^2}$, 12. $\dfrac{ax^n}{(ax+b)^{n+1}}$ (n positive).

Find the limits of

13.* $\dfrac{\sqrt{(5x-4)}-\sqrt{x}}{x-1}$ when $x\to 1$; 14. $\dfrac{x}{\sqrt{(1+x)}-\sqrt{(1-x)}}$ when $x\to 0$;

15. $\dfrac{\sqrt{(3a-x)}-\sqrt{(x+a)}}{4x-4a}$ when $x\to a$; 16. $\dfrac{x^7-1}{x-1}$ and $\dfrac{\sqrt[n]{x}-1}{x-1}$ when $x\to 1$;

17. $\dfrac{x^{10}-a^{10}}{x-a}$, $\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$, and $\dfrac{x^{10}-a^{10}}{x^7-a^7}$ when $x\to a$;

18. $\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt[3]{x}-\sqrt[3]{a}}$ when $x\to a$; 19. $\sqrt{(1+x)}-\sqrt{x}$ when $x\to\infty$;

20. $\sqrt{(x^2+ax+b)}-x$ when $x\to\infty$; 21. $\dfrac{1+2+3+\ldots+n}{(n-1)^2}$ when $n\to\infty$;

* In this and the following examples, the positive value of the root is to be taken.

22. $\dfrac{1-\cos 2x}{x}$ when $x \to 0$; 23. $1-\cos 2x$ when $x \to 0$;

24. $\dfrac{\sin\frac{p}{q}\theta}{\theta}$ when $\theta \to 0$; 25. $\dfrac{\sin p\theta}{\sin q\theta}$ when $\theta \to 0$;

26. $\dfrac{\tan m\theta}{\theta}$ when $\theta \to 0$; 27. $\dfrac{\tan mx}{\tan nx}$ when $x \to 0$;

28. $\dfrac{1-\cos p\theta}{\theta^2}$ when $\theta \to 0$; $\dfrac{\cos a\theta - \cos b\theta}{\theta^2}$ when $\theta \to 0$.

29. Taking a circle as the limit of (i) an inscribed, (ii) a circumscribed regular polygon of n sides, when $n \to \infty$, prove that its area is πr^2.

30. Find, by Art. 14, the area of the curved surface of a right circular cone.

31. Find the area of the curved surface and the volume of a right circular cylinder.

32. Find the volume of a sphere by the method of Art. 14 (4).

33. Find $\operatorname{Lt} \dfrac{1-\cos m\theta}{1-\cos n\theta}$, $\theta \to 0$.

34. Find Lt $(\sec\theta - \tan\theta)$ as $\theta \to \frac{1}{2}\pi$. 35. Find $\operatorname{Lt} \dfrac{\sin m\theta}{\tan n\theta}$ as $\theta \to 0$.

36. Find, by the method of Art. 14 (4), the volume formed by rotation about the axis of x of the area between the parabola $y^2 = 4ax$, the axis of x and the latus-rectum.

37. Find $\underset{n \to \infty}{\operatorname{Lt}} \dfrac{1^2+2^2+3^2+\ldots+n^2}{n^3}$.

[See Art. 14 (4) for the sum of the series in the numerator.]

38. Find Lt $[1+2x+3x^2+\ldots n \text{ terms}]$ when $|x| < 1$.

16. Continuous functions.

Let y be a function of x; then a change in the value of x will produce a change in the value of y. The change in y due to a given increase in x may be positive, i.e. it may be an increase, as in the functions $y = x^3$, $y = 2^x$; it may be negative, i.e. a decrease, as in the functions $y = (1-x)^3$, $y = 10^{-x}$; it may be large, as in the function $y = x^{10}$ when x is large; it may be small, as in the function $y = \log x$ when x is large. But in all such cases it *usually* happens that, when the change in $x \to 0$ as a limit, the change in y also $\to 0$ as a limit, and when this is the case, the function y is said to be *continuous*.

A more precise definition is as follows:—Let $y = f(x)$ be a function of x; when x is changed to $x+h$, y becomes $f(x+h)$, i.e. an increase of h in the value of x produces an increase of $f(x+h)-f(x)$ in the value of y. Now let σ be any arbitrarily selected positive small quantity; if, for a particular value of x, it is possible, however small σ be taken, to find a positive quantity ϵ such that the increase

in y is numerically $< \sigma$ for all values of $|h|$ which are $< \epsilon$, then the function $f(x)$ is said to be continuous for that value of x. If this property holds for all values of x between a and b, the function $f(x)$ is said to be continuous from $x = a$ to $x = b$.

It is not easy to grasp at once what is involved in this definition; we will illustrate it by some examples.

(i) $y = x^2$.

If x becomes $x+h$, y becomes $(x+h)^2$, i.e. $x^2+2hx+h^2$; therefore the increase in $y = 2hx+h^2$.

Now $2hx+h^2 < \sigma$ if (adding x to both) $(h+x)^2 < \sigma+x^2$,

i.e. if $h+x < +\sqrt{(\sigma+x^2)}$,

i.e. if $h < \sqrt{(\sigma+x^2)} - x$.

This is $+$, since σ is $+$ and therefore $\sqrt{(\sigma+x^2)} > \sigma$, and is the '$\epsilon$' of the definition. Any value of h smaller than this number makes the increase in $y < \sigma$, however small σ be taken and whatever be the value of x.

Hence the function $y = x^2$ is continuous for all values of x.

(ii) $y = (x-3)/x$.

In this case, if x becomes $x+h$, y becomes $\dfrac{x+h-3}{x+h}$

$$\text{the increase in } y = \frac{x+h-3}{x+h} - \frac{x-3}{x} = \frac{3h}{x(x+h)}.$$

This will be $<\sigma$ if $\dfrac{3h}{x^2+xh} < \sigma$, in numerical value,

i.e. if $3h < \sigma x^2 + \sigma xh$,

i.e. if $h < \dfrac{\sigma x^2}{3-\sigma x}$

Now, however small σ is, h can always be taken smaller than this expression, *except in the one case* x $= 0$; the number on the right-hand side is then equal to zero, and no positive value for h can be found. Therefore the given function is continuous for all values of x, except $x = 0$; it is discontinuous when $x = 0$.

Similarly, any function of x is discontinuous for a value of x which makes it infinite.

(iii) $y = \tan x$.

If x becomes $x+h$, the increase in $\tan x$

$$= \tan(x+h) - \tan x$$

$$= \frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x$$

$$= \frac{\tan h\,(1+\tan^2 x)}{1 - \tan x \tan h}$$

$$= \frac{\tan h \sec^2 x}{1 - \tan x \tan h}$$

$$= \frac{\tan h}{\cos^2 x - \sin x \cos x \tan h}.$$

This will be $< \sigma$ if $\tan h < \sigma(\cos^2 x - \sin x \cos x \tan h)$,

i.e. $$\text{if } \tan h < \frac{\sigma \cos^2 x}{1+\sigma \sin x \cos x} \text{ in numerical value,}$$

and h can always be chosen so as to satisfy this condition *except when* $\cos x = 0$, i.e. except when x is an odd multiple of $\frac{1}{2}\pi$.

Therefore $\tan x$ is a continuous function of x, except when

$$x = \pm\tfrac{1}{2}\pi,\ \pm\tfrac{3}{2}\pi,\ \pm\tfrac{5}{2}\pi,\ \ldots.$$

This gives us an example of a function which is discontinuous at an infinite number of isolated points; it is continuous throughout the ranges $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, and so on, but not throughout the range 0 to π, or any range which includes one or more of the points mentioned above.

17. Properties of a continuous function.

(1) If y be a continuous function of x, an indefinitely small change in the value of x produces only an indefinitely small change in the value of y.

This is involved in the definition above, since σ is to be arbitrarily small, and this statement is sometimes given as a definition of a continuous function.

Examples of discontinuities. (i) $y = 1/x$ [see Fig. 3].

In this case, if $x = -\alpha$, $y = -1/\alpha$, and if $x = +\alpha$, $y = +1/\alpha$;

Therefore an increase of 2α in the value of x (from $-\alpha$ to $+\alpha$) produces an increase of $2/\alpha$ in the value of y. If α be indefinitely small, $1/\alpha$ is indefinitely large; therefore an indefinitely small change in the value of x as it passes through the origin produces an indefinitely large change in the value of y. Hence the function $1/x$ is discontinuous when $x = 0$. It is continuous throughout any range which does not include the origin, for if x increases to $x+h$, y changes from $1/x$ to $1/(x+h)$, i.e. y increases by

$$\frac{1}{x+h} - \frac{1}{x}, \text{ i.e. } \frac{-h}{x(x+h)},$$

and this $\to 0$ as $h \to 0$, provided x is not zero [cf. Art. 16 (ii)].

(ii) $y = \tan x$.

If x is very slightly $< \frac{1}{2}\pi$, $\tan x$ is very large and positive; if x is very slightly $> \frac{1}{2}\pi$, $\tan x$ is very large and negative. Therefore a very small increase in x from one side of $\frac{1}{2}\pi$ to the other produces a very large increase in $\tan x$; the function $\tan x$ is discontinuous when $x = \frac{1}{2}\pi$; and similarly when x is equal to any odd multiple of $\frac{1}{2}\pi$. Tan x has an infinite number of discontinuities, isolated values occurring at intervals of π [cf. Art. 16 (iii)].

(iii) $y =$ the principal value of $\tan^{-1}(1/x)$, i.e. the angle between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$ whose tangent is $1/x$.

As x increases from $-\infty$ to 0, $1/x$ decreases from 0 to $-\infty$, and $\tan^{-1}(1/x)$ decreases from 0 to $-\frac{1}{2}\pi$; and as x increases from 0 to $+\infty$, $1/x$ decreases from $+\infty$ to 0 and $\tan^{-1}(1/x)$ decreases from $+\frac{1}{2}\pi$ to 0.

When x passes through the value 0, y takes a sudden jump from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ without passing through the intermediate values: an indefinitely small increase in the value of x on passing through the origin produces a finite increase π in the value of y. The function is discontinuous when $x = 0$ (Fig. 29).

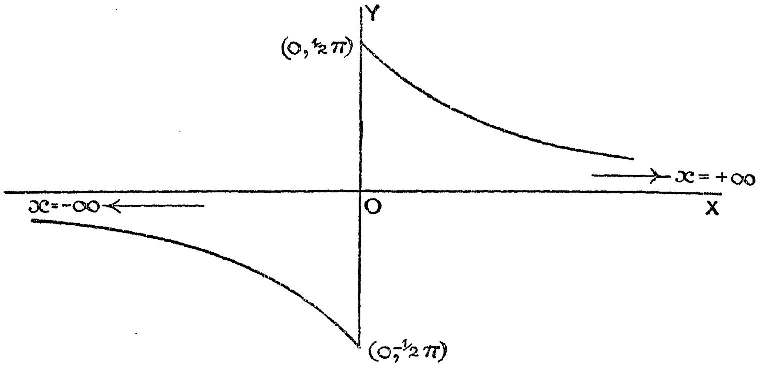


Fig. 29.

A similar kind of discontinuity has already been mentioned in Art. 14, Ex. 1, where, at the point P, the gradient undergoes an abrupt change from $\tan PT'X$ to $\tan PTX$ in passing through the point P (Fig. 22).

(iv) An example from Mechanics:—Consider the motion of two unequal masses connected by an inextensible string passing over a smooth pulley and hanging vertically. The larger mass M will descend with constant acceleration. Now suppose that at a certain instant the ascending mass suddenly picks up another mass, equal to the descending mass, say. At this instant its velocity will suddenly be diminished, and afterwards M will continue to descend for some time with constant retardation, come to rest, and then begin to ascend again.

If we draw the velocity-time graph of the motion of M (Fig. 30), the

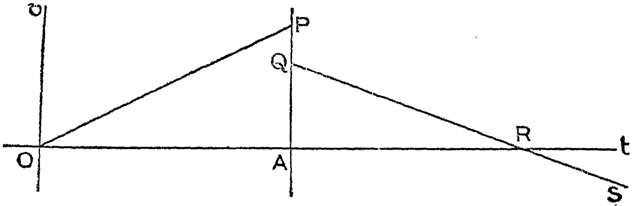


Fig. 30.

straight line OP corresponds to the first stage of the motion when the velocity is increasing; there will be a discontinuity at A corresponding to the instant when the additional mass is picked up (and the velocity suddenly reduced

from AP to AQ), and the straight line QRS corresponds to the second stage of the motion when the velocity is decreasing; QR belonging to the interval during which M continues to descend, R to the instant when it is momentarily at rest, and RS to the time when it is ascending again (and therefore the sign of its velocity is reversed).

(v) Again, if we represent graphically the relation between the weight x in lbs. of a parcel and the cost y in pence of sending it by Parcel Post [3*d.* for any weight up to 1 lb., 4*d.* for any weight between 1 and 2 lb., 5*d.* for 2–3 lb., 6*d.* for 3–5 lb., 7*d.* for 5–7 lb., and 1*d.* for every additional lb. or fraction of a lb. up to 11 lb.], as x increases from 0 to 1, y remains constant and equal to 3; as x increases through the value 1, y takes a sudden jump from 3 to 4 and remains equal to 4 until x reaches 2; y then takes another sudden jump from 4 to 5 and remains equal to 5 until x reaches 3, and so on. The graph consists of 9 straight portions parallel to the axis of x, of which the 4[th] and 5[th] are of length 2 units and all the others of length 1 unit, and it is discontinuous when $x = 1, 2, 3, 5, 7, 8, 9, 10$ (Fig. 31).

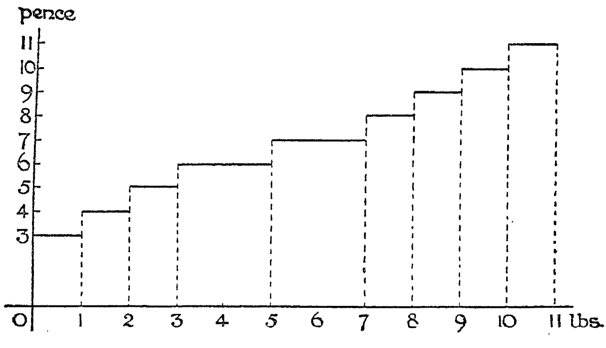


Fig. 31.

(2) The graph of a continuous function is a continuous curve without any breaks in it.

Compare the graphs in the preceding examples with those of functions which are everywhere continuous, e. g. $\sin x$, x^3, $x^2/(1+x^2)$ (Figs. 4, 6).

(3) In passing from any one value to any other value within a range throughout which it is continuous, a function must pass at least once through every intermediate value.

Let c (Fig. 32) be a value intermediate between two values a and b represented by the ordinates AM, BN. Draw the graph of the function, and the line $y = c$; then graphically it is obvious that a continuous curve cannot be drawn from A to B without crossing

the line $y = c$ at least once. It may of course cross it any odd number of times.

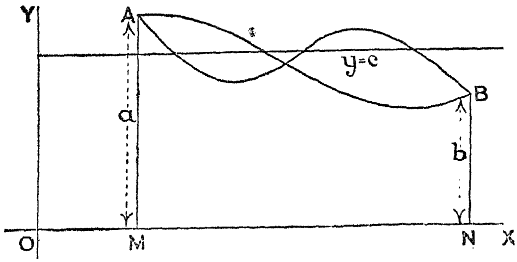


Fig. 32.

(4) A very important particular case of the preceding is that *a continuous function cannot change sign without passing through the value zero;* i.e. graphically, a continuous curve, in passing from one side of the axis of x to the other side, must cut that axis at some intermediate point.

This is obviously not necessarily true for a discontinuous function; e.g. $\sec x$, in changing from $+1$ (when $x = 0$) to -1 (when $x = \pi$) does not pass through the value 0; it has a discontinuity (when $x = \frac{1}{2}\pi$) between these points. Similarly, $1/x$, in changing sign, does not vanish; it is discontinuous at the origin [(1) (i)].

This theorem is very useful in dealing with algebraical equations. It will be seen [(6) below] that the expression

$$ax^n + bx^{n-1} + \dots + k \text{ [where } n \text{ is a positive integer]}$$

is continuous; therefore, if such an expression be positive when $x = \alpha$ and negative when $x = \beta$, it must be equal to zero for some intermediate value of x. Hence the equation

$$ax^n + bx^{n-1} + \dots + k = 0$$

will have at least one root (it must have an *odd* number) between two values of x which make the left-hand side take opposite signs. For instance, in the equation

$$x^3 - 3x^2 + 4x - 10 = 0,$$

$x = 2$ makes the left-hand side equal to -6, and $x = 3$ makes it equal to $+2$; therefore the equation has at least one root between 2 and 3.

(5) *If* $f(x)$ *be continuous when* $x = a$, *then its value when* $x = a$ *is equal to the value of* $\mathrm{Lt}\, f(x)$ as $x \to a$ *from either side,*

i.e. $$\operatorname{Lt} f(x) = f(a).$$

This may be, and often is, taken as the definition of continuity, i.e. a function $f(x)$ is said to be continuous when $x = a$, if $\underset{\substack{x \to a \\ a \leftarrow x}}{\operatorname{Lt}} f(x) =$ the definite number $f(a)$. It is not true if $f(a) = \infty$ and $\underset{x \to a}{\operatorname{Lt}} f(x) = \infty$, for infinity is not a definite value. This latter statement simply means that the value of $f(x)$ can be made larger than any assignable value by taking x sufficiently near a, not that it gets nearer and nearer to a certain definite value.

It is not true in such cases as (1) (iii) and (iv); here the function tends to a different limit as x approaches a from the one side or the other, and whatever value be assigned to the function when $x = a$, it cannot be equal to both these limits.

Returning to the example of Art. 13 (1), it was found that the limit of $(x^2-9)/(x-3)$, as $x \to 3$, is 6 whether x approach the value 3 from above or below; when $x = 3$, the value of the function is at present undefined (since the zero factor $x-3$ cannot then be cancelled out), and can be assigned at will. If the value 6 be assigned to the function when $x = 3$, then the function will be continuous when $x = 3$, since the value will then coincide with the limits on either side. If any other value than 6 were assigned, the function would be discontinuous.

If we draw the graph of $y = (x^2-9)/(x-3)$, we see that it consists of two straight lines, since either x is equal to 3 and then y is indeterminate, or x is not equal to 3 and then $y = x + 3$. In the first case, $x = 3$ is the equation of a straight line parallel to the axis of y; in the second case, $y = x+3$ is the equation of a straight line equally inclined to the axes. The graph therefore consists of these two straight lines (Fig. 33). For any value of x other than 3, we get a single point on the graph, giving one definite value of the function for that particular value of x; but when $x = 3$, we have an unlimited number of points since the whole of the line $x = 3$ constitutes part of the graph, and therefore y is quite indeterminate.

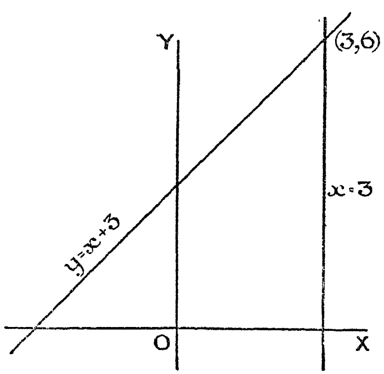


Fig. 33.

If the value 6 be assigned to the function when $x = 3$, we are selecting the point on the line $x = 3$ where it is cut by the other line $y = x+3$, but this is quite an arbitrary selection.

Similarly, $\underset{x \to 0}{\mathrm{Lt}} \dfrac{\sin x}{x} = 1$, whether x approach zero from the positive or the negative side. The function is undefined for the value $x = 0$. If the value 1 be assigned to the function when $x = 0$, $(\sin x)/x$ will be continuous when $x = 0$, since the value then coincides with the limits on either side.

(6) It can be proved, by a method similar to that of Art. 15, that

(i) The algebraical sum and the product of any finite number of continuous functions are themselves continuous.

(ii) The quotient of two continuous functions is continuous except for values of the variable which make the denominator zero.

The functions which are met with in elementary applications of the calculus are usually either continuous for all values of x, or have discontinuities only at a number of isolated points, e. g. $\tan x$, $x/(x^2-4)$; and such functions are of course continuous throughout any range which does not include a point of discontinuity. For instance, the function $x/(x^2-4)$ is continuous between $x = -\infty$ and $x = -2$, between $x = -2$ and $x = +2$, and between $x = +2$ and $x = +\infty$; it is discontinuous for two values of x only, when $x = -2$ and when $x = +2$, both of which values make the function infinite.

Examples IV.

1. Prove, from the definition, that the following functions are everywhere continuous:

$$a+bx;\quad 2+x-x^2;\quad \cos x;\quad 1/(1+x^2);\quad \sin^2 x.$$

2. Deduce, from Art. 17 (6) that the following functions are continuous: x^3; x^4; $ax^n + bx^{n-1} + \ldots + k$ [n a positive integer]; $x/(x^2+1)$; $\sin^3 x$; $\sin x/(4+\cos x)$; $\sin^m x \cos^n x$ [m and n positive integers].

3. Where are the following functions discontinuous?

$$\frac{x^2}{2x-3},\quad \frac{x}{(x+1)^3},\quad \cot x;\quad \sec x;\quad \operatorname{cosec} 2x;\quad \frac{1}{x^4-13x^2+36};\quad \frac{1+\sin^2 x}{\cos x}$$

$$\tan 3x;\quad \frac{2+\sin x}{1+\cos x};\quad \text{the principal value of } \cot^{-1} x.$$

4. What value must be assigned to the function $(x^3+27)/(x+3)$ when $x = -3$ in order that it may be everywhere continuous?

5. Prove that the equation $x^3+8x^2-5x-3=0$ has a root between 0 and 1.

6. Show that the equation $x^6-7x^4+9x^2-1=0$ has one root between 0 and 1, and another between -1 and 0.

7. Prove that the equation $24x^4-68x^3-26x^2+153x-63=0$ has roots between -2 and -1, 0 and 1, 1 and 2, 2 and 3.

8. Prove that $(x-a)/(\sqrt{x}-\sqrt{a})^2$ is discontinuous when $x = a$.

9. Prove that $2^{1/x}$ is discontinuous when $x = 0$; draw its graph.

10. Show that the principal value of $\tan^{-1}\{1/(x+1)\}$ is discontinuous when $x = -1$; draw its graph.

CHAPTER III

DIFFERENTIATION OF SIMPLE ALGEBRAICAL FUNCTIONS

18. Rate of Increase of a Function.

We now proceed to consider how to find the rate of increase, with respect to x, of a given function of x. As already pointed out (Art. 11), this rate of increase is constant only for the linear function $ax+b$; for all other functions, it varies from value to value of the function.

In the first place, instead of considering a number of disconnected values of y corresponding to disconnected values of x, as we do in actually plotting a graph from its equation, we imagine x to be growing or increasing continuously just as, measuring from a particular instant, time goes on, or as, starting from a particular position, a train travels onward, so that, in changing from one value to another, x passes through all the intermediate values, just as the train in passing from one point to another passes all intermediate points. As x changes thus, the function y will generally change in a similar manner, sometimes increasing, sometimes decreasing, sometimes changing rapidly, sometimes slowly, sometimes for an instant stationary (Art. 54), but occasionally, at a point of discontinuity, taking a sudden jump from one value to another (Art. 17).

A given increase in the value of x from a to b produces an increase in the value of the function y; and it is obvious that the resulting increase in the value of y depends not solely upon the increase in x, but also upon the actual value of x before the increase (Art. 11). The ratio of the increase in the function to the increase in x gives the *average* rate of increase of the function per unit increase of x throughout this particular interval a to b; but the average rate of increase throughout a finite interval will probably be quite different from the rate of increase at, say, the commencement of the interval, just as the average velocity of a train during any interval of time may be quite different from its actual velocity at the commencement of the interval.

Consider further the analogy with the motion of a train. The value of the function for any particular value of x corresponds to the distance of the train from some fixed point of the line (which distance is a function of the time) at any particular instant; the rate of increase of the function with respect to x corresponds to the rate of increase of this distance with respect to the time, i. e. to the velocity of the train. This velocity may be constant for a time, but probably it will not be constant for a very long time; it may be large or small, increasing or decreasing rapidly or slowly; and so with the function (except that its rate of change is never constant for a finite range of values of x, unless it is linear, and then it is always constant).

In the case of the train, if we take any interval of time and divide the distance therein travelled by the length of the interval, we get the 'average velocity' during that interval. This average velocity may be quite different from the velocity at the commencement of the interval, but the distance of the train from the fixed point will be a continuous function of the time, so that a very small increase in the time will produce only a very small alteration in the distance, and if the length of the interval $\to 0$, this average velocity will approach some fixed limiting value. This limit is defined as the velocity at the commencement of the interval.

This is what is meant when it is stated that the velocity of a train is, at a particular instant, 30 miles per hour (which is 44 feet per second); it does not mean that in the next minute it will go half a mile, because even in a minute there is time for the velocity to change appreciably; but in a second the distance that the train goes will be nearly 44 feet, and in $\frac{1}{10}$ of a second the distance will be still more nearly 4·4 feet, and so on, because in a second the velocity will change very little, and in $\frac{1}{10}$ of a second still less.

Again, if the train travels 30 miles between 5 o'clock and 6 o'clock, the average velocity during that hour is 30 miles per hour, but this gives us no information as to the velocity at 5 o'clock. If the distance travelled between 5 o'clock and 5.10 be divided by $\frac{1}{6}$ hour, we get the average velocity between 5 and 5.10; if the distance travelled between 5 o'clock and 1 minute past be divided by $\frac{1}{60}$ hour, we get a result nearer to the velocity at 5 o'clock, and if we could measure the distance travelled between 5 o'clock and 1 second past 5, this, divided by $\frac{1}{3600}$ hour, would be nearer still. This series of average velocities through diminishing intervals of time, all commencing at 5 o'clock, tends to a limiting value, and this limiting value is 'the velocity at 5 o'clock'.

We proceed in exactly the same manner with any function of x. We find the average rate of increase of the function with respect to x for a given increase in x; and then we find the limit to which this average rate of increase tends when the increase in $x \to 0$, i.e. *the actual rate of increase for any value of* x *is the limit of the average rate of increase throughout a range commencing at that value, when the range is indefinitely diminished.* As the range decreases, we get values for the average rate of increase which approach nearer and nearer to the actual rate of increase at the beginning of the range, and (from the definition of a limit) we can get as near as we please to this actual rate of increase by taking the range sufficiently small. This limit is called the *differential coefficient* of the function with respect to x. In the illustration above, the velocity of the train is the differential coefficient, with respect to the time, of its distance from some fixed point of the railway line.

19. The function $y = x^2$.

Let us consider in detail the very simple function $y = x^2$.

If $x = 10$, $y = 100$; if x becomes 11, y becomes 121. If x becomes 10·1, y becomes 102·01; if x becomes 10·01, y becomes 100·2001.

In the first case, the average rate of increase of y per unit increase of $x = 21/1 = 21$; in the second case, it is 2·01/·1, i.e. 20·1; in the third case, it is ·2001/·01 = 20·01. These numbers 21, 20·1, 20·01, ... tend to the limit 20.

Generally, if x becomes $10 + h$, y becomes $100 + 20h + h^2$. Therefore

$$\frac{\text{increase in } y}{\text{increase in } x} = \frac{20h + h^2}{h} = 20 + h.$$

Clearly, as the increase h in x gets less and less, this ratio gets nearer and nearer to 20, and we can make it as near to 20 as we please by taking h sufficiently small; therefore when $x = 10$, the limit of the average rate of increase of the function x^2 is 20.

This means that, when x has the value 10, x^2 is increasing at the rate of 20 units per unit increase of x, just as the statement, that at a given instant a train is travelling at 20 miles per hour, means that its distance from some fixed point on the line is at that instant increasing at the rate of 20 miles per unit increase of the time (measured in hours).

Similarly, in the general case, if x is increased to $x + h$, y becomes $(x + h)^2$, i.e. $x^2 + 2hx + h^2$. Therefore the ratio

$$\frac{\text{increase in } y}{\text{increase in } x} = \frac{2hx + h^2}{h} = 2x + h,$$

which, when $h \to 0$, tends to the limit $2x$. That is, the rate of increase of the function x^2 with respect to x is $2x$, or *the differential coefficient of* $\mathrm{x}^2 = 2\mathrm{x}$.

It should be noticed that if x is increased by a *very small* amount, y is increased by *approximately* $2x$ times as much; approximately, not exactly, because x is here stated to increase by a *very* small amount but not an *indefinitely* small amount. The smaller the increase in x, the more nearly is the statement true (because the actual amount of the increase in y is $(2x+h)$ times the increase in x, and the smaller the increase in x, the nearer is this to $2x$).

20. Geometrical Illustrations.

The preceding results can be illustrated geometrically:

(i) If the length of the straight line OX represents x, y will be represented by the area of the square OM which has OX as side; if OX is increased to OX', the resulting increase in y is represented by the shaded area in Fig. 34. This is equal to twice the rectangle MX' + the square MM'.

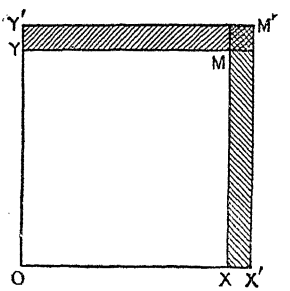


Fig. 34.

If XX' is very small, the square MM' is very small compared with the rectangle MX' (the ratio of their areas $= XX'/MX$, which can be made as small as we please by taking XX' small enough).

Therefore the increase in y is represented approximately by twice the area of the rectangle MX', which

$$= 2MX \, . \, XX' = 2x \times \text{the increase in } x.$$

(ii) Again, referring to the graph of $y = x^2$ (Fig. 35), let P be any point (x, y) on the graph, and let Q be the point on the graph whose abscissa is $x+h$. Draw PN, QN' perpendicular to the axis of x, and PM perpendicular to QN. Then MQ represents the increase in y due to the increase NN' in x, and the average rate of increase of the function in the interval NN'

$$= \frac{\text{increase in } y}{\text{increase in } x} = \frac{MQ}{PM} = \tan QPM = \tan PKX;$$

so that the average rate of increase of the function between any two values is represented geometrically by the tangent of the angle which the chord joining corresponding points on the graph makes with the positive direction of the axis of x.

As NN', the increase in x, is taken smaller and smaller, the point Q moves nearer and nearer to P, and when the increase in x is indefinitely small, Q is indefinitely near to P. The limiting position of the chord PQ when Q approaches indefinitely near to P is (Art. 14 (1)) the tangent to the curve at P, and therefore the limiting value of $\tan PKX$ is $\tan PTX$, if the tangent at P meets the axis of x in T. This is called the *slope* [or sometimes the *gradient*] of the curve at the point P. Hence, *for any value of* x, *the rate of increase of the function per unit increase of* x *is represented geometrically by the slope of the graph of the function at the corresponding point.*

This result is true in general. In the case of the function at present under consideration, $y = x^2$, the slope of the graph at any point (x, y) is $2x$.

Taking numerical cases, when $x = 3$, $y = 9$ and the slope $= 6$;

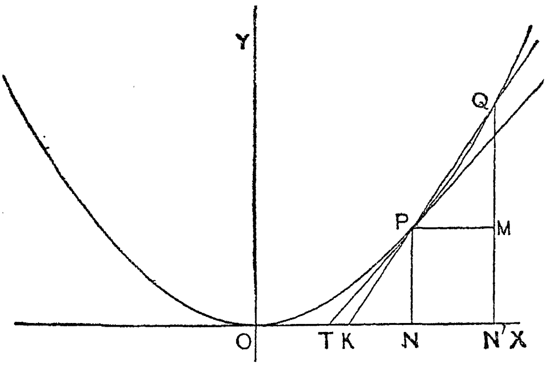


Fig. 35.

therefore the tangent at the point (3, 9) is inclined to the axis of x at an angle whose tangent is 6, i.e. a little more than $80\frac{1}{2}°$.

When $x = -2$, $y = 4$, and the slope $= -4$; hence the tangent at the point $(-2, 4)$ is inclined to the axis of x at an angle whose tangent is -4, i.e. $104° 2'$.

21. Another illustration.

As a further illustration of the meaning and use of the differential coefficient of the function $y = x^2$, let us consider the following example:

The radius of a circle is increasing at the rate of 1 inch per second; find the rate of increase of the area of the circle at the instant when the radius is 20 inches. (The circle is supposed to be continuously increasing in the same way as the circular ripples caused by dropping a stone into water.)

At any particular instant, the radius and the area of the circle will have definite values; both are functions of the time. The differential coefficient of the area A with respect to the radius r gives the rate of increase of the area per unit increase of the radius. The differential coefficient of r^2 with respect to r is $2r$, and the increase in πr^2 is obviously π times the increase in r^2; therefore the differential coefficient of the area πr^2 with respect to r is $2\pi r$. Hence the rate of increase of the area $= 2\pi r$ per unit increase of r. From this it follows* that

$$\begin{aligned}\text{the rate of increase of the area per sec.} &= 2\pi r \times \text{rate of increase of } r \text{ per sec.}\\ &= 2\pi r \times 1\\ &= 40\pi \text{ sq. in. per sec.}\end{aligned}$$

at the instant when $r = 20$ in.

It should be noticed that this does not mean that in the next second the area will increase by 40π sq. in., because as soon as r is a little greater, the rate of increase, $2\pi r$, will also be a little greater. The fact that the rate of increase of the area of a circle is equal to $2\pi r$ times the rate of increase of the radius is verifiable geometrically, because if the radius is increased from r to $r+h$ where h is small, a very narrow strip is added on to the circle all round it. If h is very small, this is practically the same as a rectangle whose width is h, and whose length is the circumference of the circle, $2\pi r$ (it is really rather more); therefore, approximately, the increase in the area $= 2\pi rh = 2\pi r \times$ increase in the radius, and hence the ratio $\dfrac{\text{increase in area}}{\text{increase in radius}}$ tends to the limit $2\pi r$ as the increase in $r \to 0$.

The differential coefficient of x^2, and illustrations of it, have been discussed at some length, because it is of the utmost importance that the student should grasp at the outset the meaning of a differential coefficient, and should clearly understand what is involved in such a statement as 'the differential coefficient of x^2 is $2x$'.

We next proceed to the definition of the differential coefficient of a function in general. The following examples should first be worked through.

√Examples V.

1. The side of a square is increasing at the rate of 1 foot per minute; find the rate of increase of (i) the area, (ii) the perimeter, (iii) the diagonal of the square, at the instant when the side is (*a*) 1 yard, (*b*) 2 yards, (*c*) 10 yards.
2. Find the inclinations to the axis of x of the tangents to the curve $y = x^2$ at the points $(1\frac{1}{2}, 2\frac{1}{4})$, $(4, 16)$, $(-3, 9)$.
3. At what point of the curve $y = x^2$ is the tangent inclined to the axis of x at (i) 20°, (ii) 60°, (iii) 135°?
4. Find the slope of the curve $4y = 3x^2$ (i) at $(2, 3)$, (ii) at $(-4, 12)$. Where is the tangent inclined to the axis of x at 45°, 70°, 120°?

* See also Art. 34.

5. Find the average rate of increase of the function $y = x^3$ as x increases (i) from 10 to 11, (ii) from 10 to 10·1, (iii) from 10 to 10·01, (iv) from 10 to $10+h$. To what limit do these increases tend? Show that the limit is the same when x increases from $10-h$ to 10.

6. Find the differential coefficient of x^3, and verify geometrically as in Art. 20 (i).

7. Find the inclinations to the axis of x of the tangents to the curve $y = x^3$ at the points $(\frac{1}{2}, \frac{1}{8})$, $(1, 1)$, $(-2, -8)$.

8. Where is the tangent to $y = x^3$ inclined to the axis of x at 45°? Find the slope of $8y = x^3$ at $(2, 1)$.

9. At what angle do the curves $y = x^2$ and $y = x^3$ intersect?

10. The side of a cube is increasing at the rate of 1 inch per second; find the rate of increase of (i) the volume, (ii) the superficial area, (iii) a diagonal of the cube, at the instant when the side is 1 foot.

11. The radius of a sphere is increasing at the rate of 1 foot per minute; find the rate of increase of (i) the volume, (ii) the superficial area of the sphere, at the instant when the radius is 1 yard.

12. The height of a cone is 15 inches and remains constant, while the radius of the base is increasing at the rate of 6 inches per minute; at what rate is the volume of the cone increasing, at the instant when the diameter of the base is 1 yard?

13. At what point of the parabola $y = x^2$ is the curve twice as steep as at the end of the latus-rectum?

14. The area of the surface of a sphere is increasing at the rate of 1 square inch per second; at what rate is the volume increasing, at the instant when the radius is 3 inches? (Find the rate of increase of the radius first.)

15. A point moves along the curve $y = x^2$ in such a way that its velocity parallel to the axis of x is constant and equal to 2 foot-seconds; find its velocity parallel to the axis of y (i) when $x = 3$, (ii) when $y = 16$, (iii) when $x = -2$.

16. A point moves along the curve $y = x^3$ so that its velocity parallel to the area of y is constant and equal to 12 foot-seconds; find its velocity parallel to the axis of x (i) when $x = 1$, (ii) when $x = -2$.

17. Each face of a cube is increasing in area at the rate of 2 square inches per second. At what rate per second is (i) the side, (ii) the volume increasing, at the instant when the side is 10 inches in length?

18. The volume of a sphere is increasing at the rate of 5 cubic inches per second; at what rate is (i) the radius, (ii) the superficial area increasing, at the instant when the radius is 6 inches?

19. The area of a circle is increasing at the rate of 4 square inches per second; at what rate is the circumference increasing, at the instant when the radius is 8 inches?

20. The side of a cube is equal to the radius of a sphere, and both are increasing at the same rate. Show that the volume of the sphere is increasing more than four times as fast as the volume of the c[illegible] and the area of the surface of the sphere more than twice as fa[illegible] area of the surface of the cube.

22. Definition of a differ

Let y be a continuous

of x will produce an in

decrease], in the value of y. These increments are generally denoted by the symbols δx and δy respectively; δy is + or − according as y increases or decreases, and similarly for δx. [Notice carefully that the δ in δx and δy is not a quantity, but a symbol; δx has nothing to do with $\delta \times x$, but merely stands for 'the increment of x'.]

If δx, the increase in x, is indefinitely small, δy, the resulting increase in y, will also be indefinitely small, since y is a continuous function of x [Art. 17 (1)]; but usually the ratio $\delta y/\delta x$, i.e. the average rate of increase of y with respect to x, tends to a definite finite limit * as $\delta x \longrightarrow 0$. This limit is called the *differential coefficient* (sometimes the *derivative*) of y with respect to x, and is denoted by the symbol $\dfrac{dy}{dx}$ or dy/dx.

It must be carefully borne in mind that dy/dx is not a fraction whose numerator and denominator are dy and dx respectively, but it is the 'limiting value' of the fraction $\delta y/\delta x$; the d/dx is a symbol which, placed in front of y denotes the result of performing a certain operation (described above) upon the function y, in the same way that the symbol $\sqrt{}$, placed in front of a number y, denotes the result of performing a certain operation upon the number y, viz. the extraction of its square root.

This particular symbol is used in order that it may be possible to indicate both the function y whose differential coefficient or rate of change is to be evaluated, and also the variable x with respect to which it is differentiated, i.e. the variable whose variation causes the change in y. For instance, the velocity v of a moving point may be regarded both as a function of the time t it has been in motion, and also as a function of the distance s it has travelled. Hence dv/dt represents the rate of increase of the velocity [illegible] per unit increase of time, i.e. the acceleration, and dv/ds stands for the rate of increase of the velocity per unit increase of distance [illegible], which is quite different.

Similarly, if V cubic inches be the [illegible] volume of a sphere of radius r inches and surface S square inches, [illegible] dV/dr is the rate of increase of the volume per unit (inch) increase of radius, dS/dr is the rate of increase of the superficial area per unit increase of radius, dV/dS is the rate of increase of the volume per unit (square inch) increase of surface. Again, dr/dV and dS/dV represent the rates of increase [illegible]

* [illegible] functions considered [illegible] this book, this limit exists, and is the [illegible]

of the radius and surface respectively per unit (cubic inch) increase of volume.

Sometimes the differential coefficient is denoted by the symbol $D_x y$, or simply Dy, if there can be no doubt as to what is the independent variable. Sometimes the actual function, in terms of x, is written after the symbol d/dx; e.g. the differential coefficients* of x^n and $\sin x$ may be written

$$Dx^n \text{ or } \frac{d(x^n)}{dx} \text{ or } \frac{d}{dx}x^n, \text{ and } D\sin x \text{ or } \frac{d(\sin x)}{dx} \text{ or } \frac{d}{dx}\sin x.$$

If a function of x be denoted by the symbol $f(x)$, its differential coefficient is usually denoted by the symbol $f'(x)$, and is often called the *derived function.*

The differential coefficient of a function gives an exact measure of the rate of change of the function with respect to the variable for any particular value of the variable. In exactly the same sense that the velocity of a moving point is said to be so many miles per hour or so many feet per second at a particular instant, so the rate of increase of a function of x, for any particular value x_1 of x, is equal to the value of its differential coefficient when $x = x_1$, per unit increase of x.

23. Geometrical meaning of the differential coefficient.

This has been found in the case of the function x^2 in Art. 20 (ii). The reasoning given there is quite general, and applies to all functions whose graphs are continuous curves; the form of the curve is immaterial. Hence, if the graph of a function be drawn, the differential coefficient of the function is represented geometrically by the tangent of the angle which the tangent to the curve makes with the axis of x; i.e. *if the tangent to the graph at a point* (x, y) *makes an angle* ψ *with the positive direction of the axis of* x, *the corresponding value of* dy/dx *is equal to* $\tan\psi$.

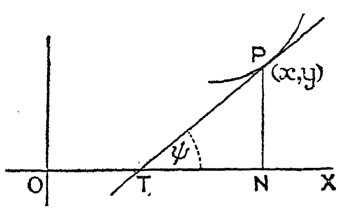


Fig. 36.

This result can also be stated in the alternative form: *the value of* dy/dx, *for any value of* x, *is equal to the slope of the graph at the corresponding point.*

* The letters d. c. will often be used as an abbreviation for the term 'differential coefficient'.

The statement that the limiting value of $\delta y/\delta x$ is the same whether δx be $+$ or $-$ is equivalent to the statement that the tangent is the same from whichever side the point Q approaches the point P.

This would not be the case at a point such as P in Fig. 37, in which PB, PC represent the positions of the tangent at P according as Q approaches P from the left or the right. In such a case as this, the ordinate is continuous, but the slope discontinuous, at the point P, i.e. the function is continuous, but its rate of increase is discontinuous. Such points, however, do not occur in the graphs of elementary functions.

It must be carefully noticed that $dy/dx = \tan\psi$, where ψ is the angle which the tangent at (x, y) makes with the *positive* direction

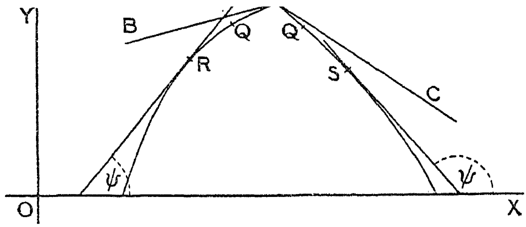


Fig. 37.

of the axis of x. For instance, at the point R in Fig. 37, ψ is an acute angle, therefore $\tan\psi$ and dy/dx are positive. At S, ψ is an obtuse angle, therefore $\tan\psi$ and dy/dx are negative (see Art. 25).

24. Differentials.

It has been stated (Art. 22) that the limit, when $\delta x \rightarrow 0$, of $\delta y/\delta x$ is usually a definite quantity denoted by dy/dx. Hence it follows that, in such cases, $\delta y/\delta x = dy/dx + \epsilon$, where $\epsilon \rightarrow 0$ as $\delta x \rightarrow 0$, i.e.

$$\delta y = \frac{dy}{dx}\cdot\delta x + \epsilon\,.\,\delta x.$$

Therefore, if δx be very small, it follows that

$$\delta y = \frac{dy}{dx}\,.\,\delta x \text{ approximately,} \tag{i}$$

and the smaller δx is, the more nearly does this become true, since ϵ gets less and less with δx, i.e. the term $\epsilon\,.\,\delta x$ gets smaller and smaller in comparison with the other two terms, or, as it is usually expressed, $\epsilon\,.\,\delta x$ is *a small quantity of higher order* than δx or δy.

Hence, *if* x *be increased by a very small amount,* y *will increase by* (*approximately*) dy/dx *times as much.*

The expression on the right-hand side of (i) is called the *differential* of y. [The name 'differential coefficient' for the function dy/dx is due to the fact that it occurs as the coefficient of δx in the differential of y.]

It must be carefully noticed that equation (i) does not mean that the two expressions become approximately equal because they are both very small, but that their ratio tends to the limit 1; their difference becomes very small compared with either of them. For the amount of error involved in the use of equation (i) see Chap. XIII.

Orders of small quantities.

The ratio of two quantities which are both indefinitely small may be finite, or it may be indefinitely small, or it may be indefinitely great. This leads to the notion of 'orders of small quantities'.

Two variables α and β, each of which tends to the limit zero, are said to be indefinitely small quantities or 'infinitesimals' of the same order if the ratio β/α be finite. If this ratio $\to 0$, β is said to be an infinitesimal of higher order than α; if it $\to \infty$, β is said to be an infinitesimal of lower order than α. If the limit of the ratio β/α^2 be finite, then β is called an infinitesimal of the second order if α be taken as an infinitesimal of the first order. Similarly, if the limit of β/α^3 be finite, β is an infinitesimal of the third order; and generally, if Lt β/α^n be finite, β is an infinitesimal of the n^{th} order. For instance, if the radius r of a very small sphere be an infinitesimal of the first order, the superficial area A of the sphere will be an infinitesimal of the second order since $A/r^2 = 4\pi$, a finite number; and the volume V of the sphere will be an infinitesimal of the third order, since $V/r^3 = \frac{4}{3}\pi$, a finite number.

Examples from geometry.

If TB be the tangent to a circle at B, and TQQ' the chord perpendicular to TB meeting the circle in Q and Q', we know from elementary geometry that $TB^2 = TQ \cdot TQ'$. Therefore, if T move along the tangent towards B so that $TB \to 0$, TQ will be an infinitesimal of the second order, if TB be taken as an infinitesimal of the first order, since $TQ/TB^2 = 1/TQ'$, of which the limit is $1/d$, if d be the diameter of the circle.

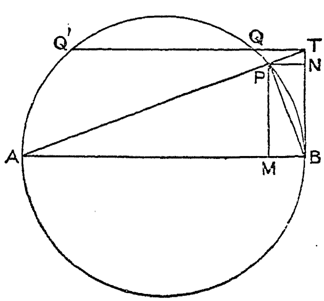


Fig. 38.

Again, if AB be a diameter of the circle, and PM perpendicular to AB from the point P where TA cuts the circle again, it is obvious that PM, PB, MB all tend to 0 as P approaches indefinitely near to B, but they do not on that account all become nearly equal. In fact, by similar triangles, $MB/BP = BP/AB$, which $\to 0$ as P approaches B; therefore MB becomes

indefinitely small compared with BP, i.e. it is an infinitesimal of higher order. Since $MB/BP^2 = 1/AB$, which is finite, it follows that, if BP be regarded as of the 1st order of small quantities, MB will be of the 2nd order. Also $MP/BP = AP/AB$, which tends to 1 as P approaches B; therefore MP and BP are small quantities of the same order, and moreover become ultimately equal. Since BP and MP are of the same order and MB is of the 2nd order compared with BP, it follows that MB is also of the 2nd order compared with MP, as is easily seen independently by taking the relation $MP^2 = AM \,.\, MB$.

Again, if PN be perpendicular to TB, $TN/PN = PM/AM$, which $\to 0$ as P approaches B; therefore TN is an infinitesimal of higher order than PN.

Also $\qquad TN \,.\, NB = PN^2 = BM^2 = MP^4/AM^2 = BN^4/AM^2.$

Therefore $TN/BN^3 = 1/AM^2$, which tends to the finite limit $1/d^2$.

Hence, since BN is of the 1st order, TN is an infinitesimal of the 3rd order.

Examples from Trigonometry are furnished by the results of Art. 13 (10), where it was shown that the limits, as $x \to 0$, of $(\sin x)/x$ and $(1 - \cos x)/x^2$ are respectively 1 and $\frac{1}{2}$. It follows that, if x be taken as an infinitesimal of the 1st order, $\sin x$ will also be an infinitesimal of the 1st order, and $1 - \cos x$ an infinitesimal of the 2nd order, or, as it is sometimes expressed, 'of the second order of small quantities.'

The geometrical meaning of the differential of y should be noticed.

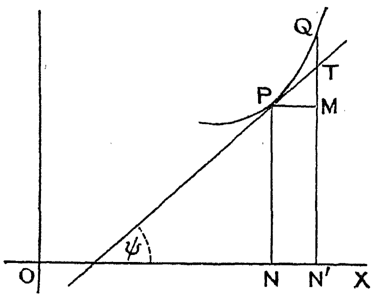


Fig. 39.

If NN' represents δx, the resulting δy is represented by MQ. Now, if the tangent at P makes an angle ψ with OX and meets MQ in T, the differential of y

$$= \frac{dy}{dx} \,.\, \delta x = \tan\psi \,.\, PM = MT.$$

Equation (i) therefore is equivalent to the statement that, if Q be taken very near to P, MQ and MT become approximately equal, their difference TQ (this will be the $\epsilon \,.\, \delta x$ above) becoming very small compared with either of them, i.e. TQ will be an infinitesimal of order higher than the first.

25. Sign of the differential coefficient.

We have, from the preceding article,

$$\frac{\delta y}{\delta x} = \frac{dy}{dx} + \epsilon, \text{ where } \epsilon \longrightarrow 0 \text{ as } \delta x \longrightarrow 0.$$

Suppose that dy/dx is not zero. Then it follows that, if δx be taken sufficiently small, the sign of $dy/dx + \epsilon$ will be the same as the sign of dy/dx, since ϵ can be made as small as we please (and therefore certainly numerically smaller than dy/dx) by taking δx small enough. Hence, for such values of δx, the sign of $\delta y/\delta x$ is the same as the sign of dy/dx. Therefore, if δx be $+$, δy will be $+$ or $-$ according as dy/dx is $+$ or $-$. Now y increases or decreases according as δy is $+$ or $-$; therefore y increases as x increases if dy/dx is $+$, and y decreases as x increases if dy/dx is $-$.

Geometrically, if dy/dx is $+$, $\tan\psi$ is $+$, and the angle ψ is an acute angle as at the point R in Fig. 37; in this case the ordinate y increases as the abscissa x increases. If dy/dx is $-$, $\tan\psi$ is $-$, and ψ is an obtuse angle as at the point S in Fig. 37; in this case, the ordinate y decreases as the abscissa x increases.

The case when $dy/dx = 0$ will be discussed later (Art. 53).

Hence, *a function of* x *increases or decreases when* x *increases according as its d. c. is* $+$ *or* $-$; and, conversely, *the d. c. of a function is* $+$ *or* $-$ *according as the function increases or decreases when* x *increases.*

26. General method of finding differential coefficients from first principles.

Theoretically, the method followed in finding the differential coefficient of x^2 may be employed for any function of x, as follows:

Let y be any function of x, denoted by $f(x)$. If x is increased to $x+h$, then y becomes $f(x+h)$;

$$\therefore \text{ the increase in } y = f(x+h) - f(x),$$

and the ratio $$\frac{\delta y}{\delta x} = \frac{f(x+h)-f(x)}{h}.$$

The d. c. of $f(x)$ is the limit to which this tends as $h \longrightarrow 0$, and the method to be adopted in evaluating this limit depends upon the nature of the function $f(x)$. This process of calculating the d. c. directly from the definition is generally referred to as *differentiation from first principles.* It consists of four stages: we first take an arbitrary increment for x, next calculate the corresponding increment of y, then find their ratio, and lastly the limit to which this ratio tends as the increments $\longrightarrow 0$.

As a matter of fact this method is employed only in a few simple cases; it would generally be long and tedious in other than the simplest functions.

These few simple cases are known as *standard forms*, and general rules are obtained which enable the d. c.'s of more complicated functions to be deduced from these standard forms. At the same time it is advisable, and the student is strongly recommended, to work out a number of differential coefficients from first principles; the process serves to fix in mind the meaning of the differential coefficient which is otherwise rather apt to be forgotten. A few examples are appended.

(i) $y = 1/x^3$.

If x is increased to $x+h$, y becomes $1/(x+h)^3$;

$$\therefore \text{ the increase in } y = \frac{1}{(x+h)^3} - \frac{1}{x^3} = \frac{x^3-(x+h)^3}{x^3\,(x+h)^3}$$
$$= \frac{-3x^2h - 3xh^2 - h^3}{x^3\,(x+h)^3}$$

Dividing this by h, the increase in x,

we have $$\frac{\delta y}{\delta x} = \frac{-3x^2 - 3xh - h^2}{x^3\,(x+h)^3}$$

which, as $h \longrightarrow 0$, approaches the limit $\dfrac{-3x^2}{x^3 \cdot x^3}$, $\quad -\dfrac{3}{x^4}$.

Therefore the differential coefficient of $1/x^3$ is $-3/x^4$.

It could have been foreseen that a negative result would be obtained in this case, since it is obvious that $1/x^3$ decreases as x increases; hence its d. c. is $-$ (Art. 25).

Consider a numerical illustration. It follows from this result and Art. 24, that if x is increased by a very small amount, $1/x^3$ will *decrease* by approximately $3/x^4$ times as much, and the smaller the increase in x, the more accurately will this statement be true.

Now, if $x = 2$, $y = \frac{1}{8} = {\cdot}125$, and $dy/dx = -3/2^4$; therefore, if x be increased by $\cdot001$, y will decrease by (approximately) $3/2^4 \times {\cdot}001$, i.e. $\cdot0001875$.

Therefore, when $\quad x = 2{\cdot}001, \quad y = {\cdot}125 - {\cdot}0001875$
$$= {\cdot}1248125,$$
which can be verified by working out the value of $1/(2{\cdot}001)^3$; the exact result to 7 figures thereby obtained is $\cdot1248127-$, giving a difference from the previous result of less than $\cdot0000002$. This slight discrepancy is due to the fact that the increase in the value of x, although small, is not indefinitely small; if a smaller increase in x were taken, the results would agree even more closely.

(ii) $y = \dfrac{4x+5}{3x+4}$.

In this case, when x is increased to $x+h$, y becomes $\dfrac{4(x+h)+5}{3(x+h)+4}$;

$$\text{the increase in } y = \frac{4x+4h+5}{3x+3h+4} - \frac{4x+5}{3x+4}$$

$$= \frac{h}{(3x+3h+4)(3x+4)} \text{ (after simplification).}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{1}{(3x+3h+4)(3x+4)},$$

which, when $h \longrightarrow 0$, tends to the limit $\dfrac{1}{(3x+4)^2}$.

Since this, being a perfect square, is always $+$, it follows that the given function always increases when x increases. If its graph be drawn, it will be seen that as x increases from $-\infty$ to $-\frac{4}{3}$, y increases from $\frac{4}{3}$ to ∞; when $x = -\frac{4}{3}$, y is infinite and the function is discontinuous; as x increases from $-\frac{4}{3}$ to ∞, y increases from $-\infty$ to $\frac{4}{3}$. [The graph is a rectangular hyperbola whose asymptotes are the straight lines $x = -\frac{4}{3}$, and $y = \frac{4}{3}$.]

As a numerical illustration, find, approximately, the value of y when $x = 2{\cdot}0135$.

When $\qquad x = 2$, $y = 1{\cdot}3$ and $dy/dx = \frac{1}{100}$.

The increase in $y = dy/dx \times$ increase in x, approximately

$= {\cdot}01 \times {\cdot}0135 = {\cdot}000135$.

$\therefore \quad y = 1{\cdot}300135$ nearly, when $x = 2{\cdot}0135$.

(iii) $y = \sqrt{x}$.

When x is increased to $x+h$, y becomes $\sqrt{(x+h)}$;

$$\therefore \quad \text{the increase in } y = \sqrt{(x+h)} - \sqrt{x},$$

and
$$\frac{\delta y}{\delta x} = \frac{\sqrt{(x+h)} - \sqrt{x}}{h}.$$

Before making h tend to zero, it is necessary to transform this expression into such a form that the numerator and denominator do not both $\longrightarrow 0$ with h; in the case of algebraical expressions, this generally means that the expression must be transformed until the h (which is not exactly 0) divides out.

In this case, the desired result is obtained by rationalizing the numerator; multiply numerator and denominator by $\sqrt{(x+h)} + \sqrt{x}$.

Then
$$\frac{\delta y}{\delta x} = \frac{x+h-x}{h\{\sqrt{(x+h)} + \sqrt{x}\}} = \frac{1}{\sqrt{(x+h)} + \sqrt{x}},$$

which, as $h \longrightarrow 0$, approaches the limit $1/(\sqrt{x} + \sqrt{x})$.

$$\therefore \quad \text{the d.c. of } \sqrt{x} = \frac{1}{2\sqrt{x}}.$$

As a numerical illustration, find $\sqrt{257}$.

Since $256 = 16^2$, this can be written $16\sqrt{\frac{257}{256}}$, i.e. $16\sqrt{(1+\frac{1}{256})}$.

To find $\sqrt{(1+\frac{1}{256})}$, we take $y = \sqrt{x}$. If $x = 1$, $y = 1$, and $dy/dx = \frac{1}{2}$.

Hence, if x be increased by $\frac{1}{256}$, y will increase by, approximately, $\frac{1}{2} \times \frac{1}{256}$.

$\therefore \quad \sqrt{(1+\frac{1}{256})} = 1 + \frac{1}{512}$ approximately,

and $\sqrt{257} = 16(1+\frac{1}{512}) = 16 + \frac{1}{32} = 16{\cdot}03125$ approximately.

The true value is $16{\cdot}031219\ldots$.

Examples VI.

Find from first principles, the differential coefficients of:

1. x^4. 2. $(1-x)^2$. 3. $1/x$. 4. $1/x^2$. 5. $1/(p-qx)$.
6. $(4x-5)/(3x-2)$. 7. $(a+bx)/(c+dx)$. 8. $2x^2-7x$.
9. ax^2+bx+c. 10. $x/(x^2+1)$. 11. $1/\sqrt{x}$. 12. $1/\sqrt{(a-bx)}$.
13. $\sqrt{(a^2-x^2)}$. 14. $(x^2+a^2)^{3/2}$. 15. $1/\sqrt{(1-x^2)}$.
16. Use the d. c. of $1/\sqrt{x}$ to find the approximate value of $1/\sqrt{401}$.
17. Use the d. c. of $1/x^2$ to find the approximate value of $1/(10{\cdot}07)^2$.
18. Find the slope of the graph of $y = 1/\sqrt{x}$ at the point $(4, \frac{1}{2})$.
19. " " " $y = 1/x^2$ " " $(10, {\cdot}01)$.
20. " " " $y = x/(x^2+1)$ " $(2, {\cdot}4)$.
21. Find the d. c. of $3x^2-7x+8$, and deduce the approximate numerical value of this expression when $x = 2{\cdot}015$.
22. Find the d. c. of $(7x-4)/(10+5x)$, and deduce the approximate numerical value of this expression when $x = 18{\cdot}03$.
23. Find the d. c. of $1/(x^2-1)$, and deduce the approximate numerical value of this expression when $x = 8{\cdot}96$.
24. Prove that the function $(3-5x)/(7x-2)$ decreases as x increases [save in passing through the value $x = \frac{2}{7}$].
25. Prove that the function $(a+bx)/(c+dx)$ increases or decreases as x increases according as $bc-ad$ is $+$ or $-$ [save in passing through the value $x = -c/d$].
26. Find the slope of the graph of $y = ax^2+bx+c$ at any point. At what point is the tangent to the graph parallel to the axis of x?
27. Where is the slope of $y = x/(x^2+1)$ zero? Draw the graph. What are the greatest and least values of the function?
28. Where does the function $y = x^3-3x$ increase, and where does it decrease, as x increases from $-\infty$ to $+\infty$?
29. Express in symbols the following statements:
    (i) The rate of change of x per second is equal to n times the rate of change of z per second.
    (ii) The rate of change of y per unit increase of x is n times the rate of increase of y per unit increase of z.
    (iii) The rate of increase of y is equal to the sum of the rates of increase of u and v with respect to x.
30. Express in symbols:
    (i) The velocity of the point (x, y) parallel to the axis of x is equal to n times the velocity parallel to the axis of y.
    (ii) The acceleration of a point moving in a straight line with velocity v is proportional to its velocity.
    (iii) The retardation of such a point is proportional to the time.
31. Express in symbols:
    (i) The rate of increase of the area of a circle per unit increase of the radius is proportional to the radius.
    (ii) The rate of increase of the volume of a cone of constant height, per unit increase of the radius of the base, is proportional to the radius.
    (iii) The rate of increase of the volume of a cube is proportional to the square of the rate of increase of the area of the surface, with respect to the length x of the edge.

(iv) The rate of increase of the volume per second is proportional to the rate of increase of the area per second.

32. Express in symbols:

(i) The slope of a curve at any point is proportional to the abscissa.

(ii) The ordinate of a curve at any point varies as the square of the slope.

(iii) At any point P of a curve, the slope of the curve is equal to half the slope of the line joining P to the origin.

27. Differential coefficient of x^n.

We have found the d. c. of x^n for several particular values of n [for $n = 2$ in Art. 19; for $n = -3$ and $n = -\frac{1}{2}$ in Art. 26]; we now proceed to the general case.*

If x is increased to $x+h$, y becomes $(x+h)^n$;

$$\therefore \quad \text{the increase in } y = (x+h)^n - x^n;$$

$$\therefore \quad \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{h \to 0}{\text{Lt}} \frac{(x+h)^n - x^n}{h} = \underset{h \to 0}{\text{Lt}} \frac{(x+h)^n - x^n}{(x+h) - x}.$$

It has been shown in Art. 13 (8) that the limit of $(x^n - a^n)/(x-a)$, when $x \longrightarrow a$, is na^{n-1} for all rational values of n; applying this to the expression just obtained, since $x+h \longrightarrow x$ when $h \longrightarrow 0$, we have

$$\underset{h \to 0}{\text{Lt}} \frac{(x+h)^n - x^n}{(x+h) - x} = nx^{n-1}$$

Therefore, whether n be $+$ or $-$, integral or fractional,

the d. c. of $\mathrm{x^n = nx^{n-1}}$.

E. g. the d. c. of $x^6 = 6x^5$; d. c. of $x^{20} = 20x^{19}$;

d. c. of $\sqrt[3]{x}$, i.e. of $x^{1/3} = \frac{1}{3}x^{-2/3} = 1/(3\sqrt[3]{x^2})$;

d. c. of $1/x^4$, i.e. of $x^{-4} = -4x^{-5} = -4/x^5$;

d. c. of $1/\sqrt{x}$, i.e. of $x^{-1/2} = -\frac{1}{2}x^{-3/2} = -1/(2\sqrt{x^3})$.

Two particular cases † are of special importance:

(i) if $y = \sqrt{x} = x^{1/2}$, then $\dfrac{dy}{dx} = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$. [Cf. Art. 26 (3).]

(ii) if $y = 1/x = x^{-1}$, then $\dfrac{dy}{dx} = -x^{-2} = -\dfrac{1}{x^2}$.

28. An important approximation follows from the preceding result.

We have $$\underset{h \to 0}{\text{Lt}} \frac{(x+h)^n - x^n}{h} = nx^{n-1};$$

* For another method of differentiating x^n, which does not require the use of the limit of Art. 13 (8), see Art. 32.

† These two cases occur so frequently that it is advisable to commit them to memory as separate standard forms.

$\therefore$ as in Art. 24, $\dfrac{(x+h)^n-x^n}{h}=nx^{n-1}+\epsilon$, where $\epsilon\to 0$ as $h\to 0$.

i.e. $$(x+h)^n = x^n+nx^{n-1}h+\epsilon h.$$

$\therefore$ when h is very small we have, neglecting terms of the 2nd order (Art. 24),

$$(x+h)^n = x^n+nx^{n-1}h, \text{ approximately.}$$

If $x=1$, then $(1+h)^n = 1+nh$ approximately, when h is very small.

e.g. $$\begin{aligned}\sqrt[3]{1002} = (1000+2)^{1/3} &= 10\,(1+\tfrac{1}{500})^{1/3}\\ &= 10\,(1+\tfrac{1}{1500}) \text{ using the approximation just obtained}\\ &= 10\,(1+\cdot00067)\\ &= 10\cdot0067 \text{ nearly.}\end{aligned}$$

The illustrations given in Art. 26, Ex. (i) and (iii) are also particular cases of this approximation.

Examples VII.

Write down the differential coefficients of:

1. $x^5,\ x^9,\ x^{30},\ x^{75}$.
2. $\sqrt[5]{x},\ \sqrt[10]{x},\ \sqrt[3]{x^2},\ \sqrt[4]{x^7},\ \sqrt[7]{x^4},\ \sqrt[n]{x}$.
3. $1/x^3,\ 1/x^7,\ 1/x^{10},\ 1/x^{50},\ 1/x^n$.
4. $1/\sqrt[3]{x^2},\ 1/\sqrt[5]{x},\ 1/\sqrt[4]{x^3},\ 1/\sqrt[n]{x},\ 1/\sqrt[q]{x^p}$.
5. Find approximately the values of $\sqrt[3]{127},\ \sqrt[4]{623},\ \sqrt[10]{1030}$.
6. Also of $1/\sqrt{99},\ 1/\sqrt[3]{995},\ 1/\sqrt[5]{245}$.
7. Also of $(\frac{51}{50})^{10},\ 1\cdot001^{20},\ (\frac{99}{100})^{12}$.

29. General Theorems on Differential Coefficients.

Theorem (i). *The d. c. of a constant is zero.*

By the term 'constant' here we mean a quantity which has the same value for all values of x. An increase in the value of x produces no change in the value of a constant, therefore $\delta y/\delta x$ is in this case a fraction whose numerator is zero and whose denominator (although ultimately very small) is not zero. Its limit is therefore zero.

Graphically, if y is a constant, the graph is a straight line parallel to the axis of x; its slope is always zero, i.e. dy/dx is zero (Art. 23).

Theorem (ii). *The d. c. of the algebraical sum of a finite number of functions is equal to the algebraical sum of their d. c.'s.*

If $y=u+v-w$ where $u,\ v,\ w$ are functions of x, the total increase in y, due to an increase δx in x, is equal to the algebraical sum of the increases in $u,\ v$, and w,

i.e. $$\delta y = \delta u+\delta v-\delta w,$$

$$\therefore\quad \frac{\delta y}{\delta x}=\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x}-\frac{\delta w}{\delta x};$$

$\therefore$ when $\delta x \to 0$, we have, by Art. 15 (i),

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx};$$

and similarly for the algebraical sum of any *finite* number of functions of x.

For the conditions under which the d. c. of the sum of an *infinite* number of functions can be obtained by differentiating each term and taking the sum of the infinite series formed by the d. c.'s, the student is referred to more advanced works.

Theorem (iii). *The d. c. of* ay, *where* a *is a constant, is equal to* a $\times$ *the d. c. of* y.

The increase in ay is evidently a times the increase in y, i. e. $\delta(ay) = a \,.\, \delta y$;

$$\therefore \quad \frac{\delta(ay)}{\delta x} = a\frac{\delta y}{dx},$$

and therefore, in the limit, $\dfrac{d}{dx}(ay) = a\,\dfrac{dy}{dx}$.

Geometrically, this theorem shows that, if all the ordinates of a curve are increased in the constant ratio $a:1$, the slope of the curve at any point is increased in the same ratio.

These three theorems, together with the result of Art. 27, enable us to write down at once the d. c. of any rational integral function of x.

E.g. the d. c. of $ax^2+bx+c = 2ax+b$;

the d. c. of $x^4-3x^3+5x^2-7x+6 = 4x^3-9x^2+10x-7$;

and generally, the d. c. of

$ax^n+bx^{n-1}+cx^{n-2}+\ldots+kx+l = nax^{n-1}+(n-1)bx^{n-2}+(n-2)cx^{n-3}+\ldots+k.$

Again, the d. c. of $(2-x^2)^2$, i.e. of $4-4x^2+x^4 = -8x+4x^3$;

the d. c. of $(x^{1/2}+x^{-1/2})^3$, i.e. of $x^{3/2}+3x^{1/2}+3x^{-1/2}+x^{-3/2}$

$$= \tfrac{3}{2}x^{1/2}+\tfrac{3}{2}x^{-1/2}-\tfrac{3}{2}x^{-3/2}-\tfrac{3}{2}x^{-5/2}.$$

It must be carefully noticed that, although the d. c. of x^3 with respect to x is $3x^2$, the d. c. of such a function as $(x^2+1)^3$ with respect to x is not $3(x^2+1)^2$; this is the d. c. with respect to x^2+1, i. e. it is the limit of the ratio of the increase in $(x^2+1)^3$ to the increase in x^2+1, not the limit of the ratio of the increase in $(x^2+1)^3$ to the increase in x. In order to find the d. c. of $(x^2+1)^3$ with respect to x, it can either be expanded as in the last two examples given or, as is better, it can be dealt with by Theorem (vii) on p. 79, which gives a general method of dealing with such cases.

Examples VIII.

Differentiate the following functions with respect to x:

1. x^2-7x+5.	2. $3x^2-8x-4$.	3. px^2+qx+r.
4. $2x^3-9x+6$.	5. ax^3+bx^2+cx+d.	6. $3x^{10}-6x^5+x$.
7. $x^4-2a^2x^2+a^4$.	8. $x^{2n}+2a^nx^n+a^{2n}$.	9. $(x-5)^2$.
10. $(\sqrt{x}+\sqrt{a})^2$.	11. $(1-x)^3$.	12. $(ax-b)^3$.
13. $\frac{3}{x^2}-\frac{4}{x}+1$.	14. $\frac{x^2+1}{x}$	15. $\left(x-\frac{1}{x}\right)^2$.
16. $\frac{x+3a}{}$	17. $\left(\frac{x-1}{x}\right)^2$	18. $\left(\frac{a^2+x^2}{x^4}\right)^3$.
19. $(\sqrt{x}-2)^3$.	20. $(1-\sqrt{x})^2/x$.	21. $(a-b/x)^3$.

30. Theorem (iv). To find the differential coefficient of a product of two functions of x.

It is obvious, by taking two simple numerical factors such as 5×8, that the total increase in the product of two factors is not obtained by multiplying together the increases of the separate factors; and therefore the d. c. of a product is not equal to the product of the d. c.'s of its factors.

Let $y=uv$, where u and v are both functions of x. When x becomes $x+\delta x$, u and v, being functions of x, will change let them become $u+\delta u$ and $v+\delta v$ respectively;

∴ their product, y, will become

$$(u+\delta u)(v+\delta v)=uv+u\,.\,\delta v+v\,.\,\delta u+\delta u\,.\,\delta v.$$

∴ δy, the increase in y, $=u\,.\,\delta v+v\,.\,\delta u+\delta u\,.\,\delta v$,

and
$$\frac{\delta y}{\delta x}=u\,.\,\frac{\delta v}{\delta x}+v\,.\,\frac{\delta u}{\delta x}+\frac{\delta u}{\delta x}\,.\,\delta v. \qquad \text{(i)}$$

In the limit, when δx and therefore δu, δv, δy all $\rightarrow 0$ (u, v, and therefore y being supposed continuous functions of x), $\delta y/\delta x$, $\delta u/\delta x$, $\delta v/\delta x$ tend to limits denoted by dy/dx, du/dx, dv/dx respectively, so that the foregoing relation (i) becomes

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx},$$

since, in the last term of (i), the first factor $\delta u/\delta x$ tends to the finite limit du/dx, and the second factor δv tends to the limit 0.* Therefore this term tends to the limit zero.

This result must be remembered, and it is most convenient to remember it in the verbal form:

the d. c. of a product = 1st factor × d. c. of 2nd + 2nd factor × d. c. of 1st.

* δx, δu, δv each separately $\rightarrow 0$, but the ratio of any two of them tends to a finite limit.

Examples:

d. c. of $(x^3+1)(x^2+2) = (x^3+1)2x+(x^2+2)3x^2 = 5x^4+6x^2+2x$;

d. c. of

$$\sqrt{x}(ax^2+bx+c) = \sqrt{x}(2ax+b)+(ax^2+bx+c)\cdot\frac{1}{2\sqrt{x}} = \frac{5ax^2+3bx+c}{2\sqrt{x}};$$

d. c. of $(ax^2+bx+c)^2$, i.e. $(ax^2+bx+c)(ax^2+bx+c)$

$$= (ax^2+bx+c)(2ax+b)+(ax^2+bx+c)(2ax+b)$$
$$= 2(ax^2+bx+c)(2ax+b).$$

The preceding working can be illustrated graphically as follows: If the values of u and v are represented by lengths OX and OY measured along two straight lines at right angles, y is represented by the area of the completed rectangle $OXZY$; if XX', YY' denote δu and δv respectively, then δy is represented by the increase in the area of the rectangle, i.e. by the shaded area in Fig. 40.

i.e. $\delta y = ZY'+ZX'+ZZ' = u.\delta v+v.\delta u+\delta u.\delta v$ as before.

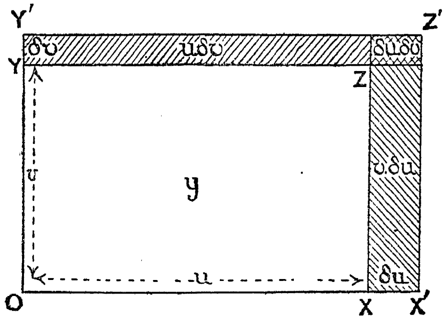


Fig. 40.

When δu and δv are very small, the last term is of the second order of small quantities and can be neglected in comparison with the first two terms; i.e. the area ZZ' is very small compared with the areas ZY' and ZX'.

Therefore approximately (i.e. to the first order of small quantities)

$$\delta y = u.\delta v+v.\delta u$$

whence, dividing by δx, and taking the limit,

$$\frac{dy}{dx} = u\frac{dv}{dx}+v\frac{du}{dx}.$$

31. Theorem (v). Differential coefficient of a product of any number of functions of x.

The rule for finding the d. c. of a product of two functions of x can be extended so as to apply to the product of any (finite) number of functions of x.

If $y = uvw$, where u, v, w are all functions of x, then, regarding this as the product of the two factors uv and w, we have

$$\begin{aligned}\frac{dy}{dx} &= uv\frac{dw}{dx} + w\frac{d(uv)}{dx}\\ &= uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)\\ &= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx},\end{aligned}$$

and similarly for any finite number of factors.

Hence the d. c. of a product is obtained by multiplying the d. c. of each factor in turn by all the other factors, and adding the results.

This result, in the case of three factors, can be illustrated geometrically. See Examples IX. 25.

A very important result follows from these rules. The d. c. of y^2 with respect to x may be found by regarding it as the product of two factors, each y, whence the d. c. of y^2

$$= y\frac{dy}{dx} + y\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Similarly, if n be any integer, by taking the product of n factors each y, we get *the d. c. of* y^{n} *with respect to* $\mathrm{x} = \mathrm{ny}^{\mathrm{n}-1}\,\mathrm{dy/dx}$. (See also Art. 34.)

The d. c. with respect to x of a function of the form $x^n y^m$, where y is a function of x, can now be written down, as follows:

d. c. of $x^3 y$ with respect to $x = x^3 \cdot \dfrac{dy}{dx} + y \cdot 3x^2$;

d. c. of $x^2 y^2$ with respect to $x = x^2 \cdot 2y\dfrac{dy}{dx} + y^2 \cdot 2x$;

d. c. of $x^4 y^3$ with respect to $x = x^4 \cdot 3y^2\dfrac{dy}{dx} + y^3 \cdot 4x^3$, and so on.

32. Alternative method of differentiating x^n.

It should be noticed that, from these rules for differentiating a product, the d. c. of x^n may be deduced for positive or negative, integral or fractional values of n, without having recourse to the limit of Art. 13 (8).

For (i) if n be a + integer,

$$x^n = x \times x \times \ldots n \text{ factors}.$$

By the preceding article, since the d. c. of each factor is 1, and this, when multiplied by all the other factors, gives x^{n-1}, we have x^{n-1} repeated n times, hence

$$\frac{d}{dx}x^n = nx^{n-1}.$$

(ii) If n be a positive fraction p/q, we have $y = x^{p/q}$; $\quad \therefore\ y^q = x^p$;
$\therefore$ differentiating with respect to x,

$$qy^{q-1}\frac{dy}{dx} = px^{p-1};$$

$$\therefore\ \frac{dy}{dx} = \frac{p}{q}x^{p-1} \div y^{q-1} = \frac{p}{q}x^{p-1} \div (x^{p/q})^{q-1} = \frac{p}{q}x^{p-1-p+p/q} = \frac{p}{q}x^{p/q-1}.$$

(iii) If n be $-$ and equal to $-m$,

$$y = x^{-m} = 1/x^m; \quad \therefore\ x^m y = 1;$$

differentiating with respect to x,

$$x^m\frac{dy}{dx} + y\,.\,mx^{m-1} = 0;$$

$$\therefore\ \frac{dy}{dx} = -\frac{myx^{m-1}}{x^m} \qquad myx^{-}$$

$$= -mx^{-m}x^{-1} = -mx^{-m-1}.$$

Hence we have proved that the d. c. of $x^n = nx^{n-1}$, for all rational values of n.

Examples IX.

Differentiate with respect to x the following products:

1. $(x^2-4)(x^4+3)$.	2. $x^n(1+\sqrt{x})$.	3. $(x^n+a^n)(x^m+a^m)$.
4. $\sqrt{x}(x^3+3x^2)$.	5. $(ax+b)(x^2+cx+c^2)$.	6. $\sqrt{x}(a/x+b/x^2)$.
7. $x(x^2-1)(x^2+4)$.	8. $(3x+2)^2$.	9. $(3x+2)^3$.
10. $(3x+2)^n$.	11. $\sqrt{x}(x-1)(x-2)$.	12. $(a-bx+cx^2)^2$.
13. $(a-bx+cx^2)^3$.	14. $(a-bx+cx^2)^n$.	15. xy^2; x^2y^4; x^3y^2.
16. x^ny; xy^n.	17. x^ny^3; x^3y^n.	18. x^ny^n.
19. x^2+xy+y^2.	20. $x^4+x^2y^2+y^4$.	21. $ax^3+bx^2y+cxy^2+dy^3$.
22. $(ay+b)^2$.	23. $(ay+b)^3$.	24. $(ay+b)^n$.

25. Illustrate the result of Art. 31 for three factors, by taking a rectangular block with edges u, v, w respectively, and proceeding as in Art. 30.

33. Theorem (vi). To find the differential coefficient of a quotient of two functions of x.

Let $y = u/v$, where u and v are both functions of x. Proceeding as in the case of a product, when x becomes $x+\delta x$, let u and v become $u+\delta u$ and $v+\delta v$ respectively.

Therefore y becomes $\dfrac{u+\delta u}{v+\delta v}$,

and
$$\delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{\quad -u\,\delta v}{v(v+\delta v)}$$

Therefore
$$\frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v+\delta v)};$$

and in the limit, when δx, and therefore also δu, δv, δy, $\rightarrow 0$, this becomes

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

This result must be remembered, and it can be put in the following convenient verbal form:

$$d.\ c.\ of\ a\ quotient = \frac{den^r \times d.\ c.\ of\ num^r - num^r \times d.\ c.\ of\ den^r}{(denominator)^2}$$

It should be noticed that this result can be deduced from the preceding result as follows:

If $$y = u/v, \text{ then } u = vy.$$

$$\frac{du}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + \frac{u}{v}\frac{dv}{dx};$$

$$v\frac{du}{dx} = v^2\frac{dy}{dx} + u\frac{dv}{dx};$$

whence $$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$ as before.

Examples:

The d. c. of $\dfrac{x^2}{3x+2} = \dfrac{(3x+2)2x - x^2 . 3}{(3x+2)^2} = \dfrac{3x^2+4x}{(3x+2)^2}$.

The d. c. of $\dfrac{x^3-3x+1}{x^3+3x+1} = \dfrac{(x^3+3x+1)(3x^2-3)-(x^3-3x+1)(3x^2+3)}{(x^3+3x+1)^2}$;

which reduces to $\dfrac{12x^3-6}{(x^3+3x+1)^2}$.

The d. c. of $\dfrac{\sqrt{x}}{ax+b} = \dfrac{(ax+b)/(2\sqrt{x}) - \sqrt{x} . a}{(ax+b)^2} = \dfrac{b-ax}{2\sqrt{x}(ax+b)^2}$.

The d. c. of $\dfrac{y^2}{x^2} = \dfrac{x^2 . 2y\dfrac{dy}{dx} - y^2 . 2x}{x^4} = \dfrac{2y\left(x\dfrac{dy}{dx} - y\right)}{x^3}$.

Examples X.

Differentiate the following functions with respect to x:

1. $\dfrac{3x+4}{5x-3}$. 2. $\dfrac{1-2x}{2-x}$. 3. $\dfrac{ax+b}{bx+a}$.

4. $\dfrac{x^2+4}{x^2-4}$. 5. $\dfrac{x^2-2x+4}{x^2+2x+4}$. 6. $\dfrac{x^n+1}{x^n-1}$.

7. $\dfrac{8x}{x^2+1}$. 8. $\dfrac{\sqrt{x}}{1-x}$. 9. $\dfrac{x}{1+\sqrt{x}}$.

10. $\dfrac{1+\sqrt{x}}{1-\sqrt{x}}$. 11. $\dfrac{ax^2+bx+c}{ax^2-bx+c}$. 12. $\dfrac{(x-1)^2}{x^2+1}$.

13. $\dfrac{x(x-2)}{3x-1}$. 14. $\dfrac{(x+2)(x-3)}{x(x-1)}$. 15. $\dfrac{x(x-1)}{(x-2)(x-3)}$.

16. x/y. 17. y/x. 18. x^2/y^2.

19. y^3/x^2. 20. x^n/y^n. 21. y^n/x^n.

34. Theorem (vii). To find the differential coefficient of a function of a function.

Consider the function $y=(x^2-3)^{10}$. This is an expression of the kind known as a function of a function; x^2-3 is a function of x, and y is a function of x^2-3. Other examples are $\log\sin x$, $\cos^n x$, $\sqrt{(a^3-x^3)}$, &c.

The d. c. of such an expression as $(x^2-3)^{10}$ cannot be conveniently worked out at once from first principles, but may be obtained in two stages by denoting x^2-3 temporarily by u, and writing $y=u^{10}$.

Generally, let y be a continuous function of u, where u is a continuous function of x. When x is increased to $x+\delta x$, u will become $u+\delta u$, and this change in the value of u will make y become $y+\delta y$; then it is an obvious identity that

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\times\frac{\delta u}{\delta x}.$$

When $\delta x \longrightarrow 0$, δu will $\longrightarrow 0$, since u is a continuous function of x, and $\delta u/\delta x$ will tend to the limit du/dx; and when $\delta u \longrightarrow 0$, δy will $\longrightarrow 0$, since y is a continuous function of u, and $\delta y/\delta u$ will tend to the limit dy/du; also $\delta y/\delta x$ will tend to the limit dy/dx. Therefore, by Art. 15 (ii), the preceding relation becomes

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}.$$

For instance, in the example mentioned above where

$$y=u^{10} \text{ and } u=x^2-3,$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=10u^9\times 2x=20x(x^2-3)^9.$$

The d. c. is the limit of (increase in y)/(increase in x), and by this method it is found in two stages:

It is equal to $\displaystyle \operatorname{Lt}\frac{\text{increase in } y}{\text{increase in } u}\times\operatorname{Lt}\frac{\text{increase in } u}{\text{increase in } x}$

i.e. to $\displaystyle \operatorname{Lt}\frac{\text{increase in } u^{10}}{\text{increase in } u}\quad\operatorname{Lt}\frac{\text{increase in } (x^2-3)}{\text{increase in } x}.$

i.e. to d. c. of u^{10} with respect to $u\times$ d. c. of (x^2-3) with respect to x,

i.e. to $\qquad 10u^9\times 2x.$

This theorem can be illustrated geometrically as follows:

Let y be a continuous function of u, $f(u)$, where u is a continuous function of x, $F(x)$.

Let AB (Fig. 41) be part of the graph of $u = F(x)$, drawn with reference to the axes OX, OU, and let AC be the corresponding part of the graph of $y = f(u)$, drawn with reference to the axes OU and OY (the continuation of XO). Let P be any point on AB, and Q the corresponding point on AC. If OM, the abscissa x of P, be increased by MN, the ordinate is thereby increased from OH to OK, and this increase in one of the coordinates of Q produces an increase in the other coordinate of Q from OE to OF. Now

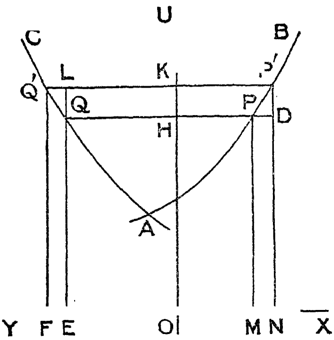


Fig. 41.

$$\frac{\delta y}{\delta x} = \frac{EF}{M} = \frac{LQ'}{PD} = \frac{LQ'}{LQ} \times \frac{P'D}{PD}$$

$$= \text{slope of } QQ' \times \text{slope of } PP'.$$

When P' approaches indefinitely near P, Q' approaches indefinitely near Q, and PP', QQ' become the tangents at P and Q respectively. Therefore, taking the limits,

$$\frac{dy}{dx} = \text{slope of } AC \text{ at } Q \times \text{slope of } AB \text{ at } P = \frac{dy}{du} \times \frac{du}{dx}.$$

This is a rule which has constantly to be applied, and the student must, by doing many examples, make himself so thoroughly familiar with it that he will always avoid such mistakes as giving the d. c. of $(2x-1)^3$ as $3(2x-1)^2$ instead of $3(2x-1)^2 \times 2$.

Examples:

if $y = (x^2-a^2)^n$, i.e. u^n where $u = x^2 - a^2$;

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} \,.\, 2x = 2nx(x^2-a^2)^{n-1};$$

if $y = \sqrt{(5-4x)}$, i.e. $\sqrt{u}$ where $u = 5-4x$;

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times -4 = -\frac{2}{\sqrt{(5-4x)}}$$

if $y = \frac{1}{(1-x^3)^4}$, i.e. u^{-4} where $u = 1-x^3$;

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -4u^{-5} \times -3x^2 = \frac{12x^2}{(1-x^3)^5}.$$

After a little practice, it will be unnecessary to insert the u explicitly, and the results can be written down at once as below.

The d. c. of $(4x+2)^3 = 3\,(4x+2)^2 \times$ d. c. of $4x+2 = 3(4x+2)^2 \times 4.$

,, $\quad (1-x)^3 = 3\,(1-x)^2 \times$ d. c. of $1-x = 3(1-x^2) \times -1.$

,, $\quad (x^2+a^2)^3 = 3\,(x^2+a^2)^2 \times$ d. c. of $x^2+a^2 = 3\,(x^2+a^2)^2 \times 2x.$

,, ,, $\quad \left(\dfrac{x^2}{x-1}\right)^3 = 3\left(\dfrac{x^2}{x-1}\right)^2 \times$ d. c. of $\dfrac{x^2}{x-1}$

$$= 3\left(\frac{x^2}{x-1}\right)^2 \times \frac{(x-1)\,2x - x^2}{(x-1)^2} = \frac{3\,x^5(x-2)}{(x-1)^4}.$$

and generally,

the d. c. of $u^3 = 3u^2 \times du/dx$, where u is any function of x.

Similarly,

the d. c. of $u^n = nu^{n-1} \times du/dx$, whatever the value of n.

This last result has already been obtained (Art. 31) from the rule for differentiating a product, in the case when n is a positive integer; we here obtain it in the more general case when n is positive or negative, integral or fractional.

Taking, as another example, different powers of the same expression,

the d. c. of $(1-2x^2)^4 = 4\,(1-2x^2)^3 \times$ d. c. of $1-2x^2 = 4\,(1-2x^2)^3 \times -4x$;

,, ,, $\quad \sqrt{(1-2x^2)} = \dfrac{1}{2\sqrt{(1-2x^2)}} \times$ d.c. of $1-2x^2 = \dfrac{1}{2\sqrt{(1-2x^2)}} \times -4x$;

,, ,, $\quad \dfrac{1}{(1-2x^2)^5} = -5\,(1-2x^2)^{-6} \times$ d.c. of $1-2x^2 = -5\,(1-2x^2)^{-6} \times -4x$;

and generally, the d. c. of any power

$$(1-2x^2)^n = n\,(1-2x^2)^{n-1} \times -4x.$$

Again, the d. c. of

$$\sqrt{(a^2+x^2)} = \frac{1}{2\sqrt{(a^2+x^2)}} \times 2x = \frac{x}{\sqrt{(a^2+x^2)}}.$$

The d. c. of $x\sqrt{(a^2+x^2)}$ (a product in which the second factor is a function of a function) is, using the result just obtained,

$$= x \times \frac{x}{\sqrt{(a^2+x^2)}} + \sqrt{(a^2+x^2)} \times 1 = \frac{a^2+2x^2}{\sqrt{(a^2+x^2)}}.$$

The d. c. of the quotient $\sqrt{(a^2+x^2)}/x$

$$= \frac{x\dfrac{x}{\sqrt{(a^2+x^2)}} - \sqrt{(a^2+x^2)}}{x^2}, \text{ which reduces to } \frac{-a^2}{x^2\sqrt{(a^2+x^2)}}.$$

35. Theorem (viii). The relation between differential coefficients of inverse functions.

If y is a continuous function of x, then x is generally a continuous function of y.

Let δx and δy be corresponding increments of x and y, then evidently

$$\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1,$$

and therefore, when δx and $\delta y \longrightarrow 0$,

$$\frac{dx}{dy} \times \frac{dy}{dx} = 1,$$

i.e. the differential coefficients of two inverse functions are reciprocals.

This result may be regarded as a particular case of the preceding result, from which it may be obtained by putting $y = x$; the rule for a function of a function then gives

$$1 = \frac{dx}{du} \times \frac{du}{dx}, \quad \text{i.e.} \quad \frac{dx}{du} = 1 \Big/ \frac{du}{dx}.$$

This theorem is also obvious geometrically. It has been seen (Art. 23) that dy/dx is the tangent of the angle ψ which the tangent to the graph of the function makes with the positive direction of the axis of x; similarly, dx/dy is the tangent of the angle ψ' which the tangent to the graph makes with the positive direction of the axis of y.

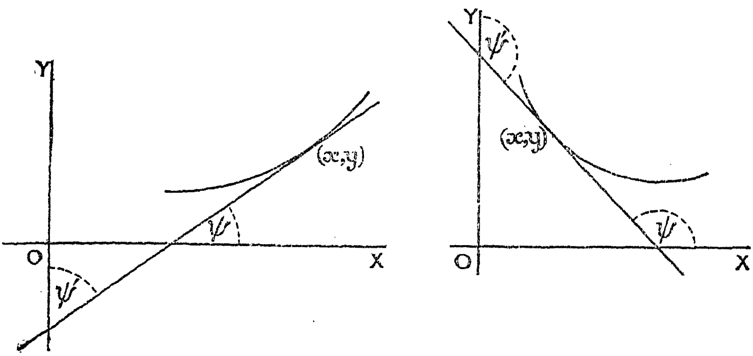


Fig. 42.

The sum of these angles is either $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ (Fig. 42), and in either case the tangent of one of them is equal to the cotangent of the other, i.e. $\tan\psi = 1/\tan\psi'$.

$$\therefore \quad \frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}.$$

Examples. If $y = \sqrt[5]{x}$, then $x = y^5$; $\therefore$ $dx/dy = 5y^4 = 5x^{4/5}$; $\therefore$ $\dfrac{dy}{dx} = \dfrac{1}{5x^{4/5}}$. In this case the d. c. of a root is deduced from that of the corresponding power.

Again, if $y^2+3y=x$, $dx/dy=2y+3$; $\therefore\ dy/dx=1/(2y+3)$, which gives dy/dx in terms of y. To find y in terms of x, and then differentiate, is a more troublesome operation and involves a much more complicated differentiation.

Examples XI.

Differentiate the following with respect to x:

1. $(4x-5)^6$; $\sqrt{(4x-5)}$; $\dfrac{1}{(4x-5)^2}$; $(4x-5)^n$; $\dfrac{1}{4x-5}$; $\dfrac{1}{\sqrt{(4x-5)}}$; $\dfrac{1}{\sqrt[n]{(4x-5)}}$.

2. $(3-7x)^6$; $\sqrt{(3-7x)}$; $\dfrac{1}{(3-7x)^2}$; $(3-7x)^n$; $\dfrac{1}{(3-7x)}$; $\dfrac{1}{\sqrt{(3-7x)}}$; $\dfrac{1}{\sqrt[n]{(3-7x)}}$.

3. $(x^2-1)^6$; $\sqrt{(x^2-1)}$; $\dfrac{1}{(x^2-1)^2}$; $(x^2-1)^n$; $\dfrac{1}{x^2-1}$; $\dfrac{1}{\sqrt{(x^2-1)}}$; $\dfrac{1}{\sqrt[n]{(x^2-1)}}$.

4. $(a-x)^6$; $\sqrt{(a-x)}$; $\dfrac{1}{(a-x)^2}$; $(a-x)^n$; $\dfrac{1}{a-x}$; $\dfrac{1}{\sqrt{(a-x)}}$; $\dfrac{1}{\sqrt[n]{(a-x)}}$.

5. $(x^n-a^n)^6$; $\sqrt{(x^n-a^n)}$; $\dfrac{1}{(x^n-a^n)^2}$; $(x^n-a^n)^n$; $\dfrac{1}{x^n-a^n}$; $\dfrac{1}{\sqrt{(x^n-a^n)}}$; $\dfrac{1}{\sqrt[n]{(x^n-a^n)}}$.

6. $(ax^2+bx+c)^6$; $\sqrt{(ax^2+bx+c)}$; $\dfrac{1}{(ax^2+bx+c)^2}$; $(ax^2+bx+c)^n$; $\dfrac{1}{ax^2+bx+c}$; $\dfrac{1}{\sqrt{(ax^2+bx+c)}}$; $\dfrac{1}{\sqrt[n]{(ax^2+bx+c)}}$.

7. $\dfrac{1}{(a^2-x^2)^n}$.

8. $\dfrac{1}{\sqrt[3]{(a^2-x^2)^2}}$.
9. $\dfrac{1}{\sqrt{(a^2-x^2)}}$.
10. $\dfrac{1}{\sqrt[5]{(a^2-x^2)}}$.

11. $\sqrt[3]{(a^3-x^3)}$.
12. $\sqrt[4]{(a^3-x^3)^3}$.
13. $\left(\dfrac{x-1}{x}\right)^5$.

14. $\sqrt[6]{\left(\dfrac{x-1}{x}\right)}$.
15. $\left(\dfrac{x}{x^2+1}\right)^n$.
16. $\sqrt{\left(\dfrac{x}{x^2+1}\right)}$.

17. $\sqrt{\left(\dfrac{x}{a-x}\right)}$.
18. $\left(\dfrac{x}{a-x}\right)^n$
19. $x\sqrt{(2x+1)}$.

20. $x^2\sqrt{(1-x)}$.
21. $\dfrac{1}{\sqrt{(2x+1)}}$.
22. $\dfrac{\sqrt{(2x+1)}}{x}$.

23. $\dfrac{x^2}{\sqrt{(1-x)}}$.
24. $\dfrac{\sqrt{(1-x)}}{x^2}$.
25. $x^2\sqrt{(a^2-x^2)}$.

26. $\dfrac{x^2}{\sqrt{(a^2-x^2)}}$.
27. $\dfrac{\sqrt{(x^2-a^2)}}{x^2}$.
28. $x(a-x)^n$.

29. $\dfrac{(a-x)^n}{x}$.
30. $\dfrac{x}{(a-x)^n}$.
31. $(a-x)^2(b-x)^3$.

32. $\dfrac{(a-x)^2}{(b-x)^3}$.
33. $(a-x)^n(b-x)^m$.
34. $\dfrac{(a-x)^n}{(b-x)^m}$.

Find dy/dx in the four examples following.

35. $y^3+3y^2=x+1$.
36. $(3y+2)/(3+2y)=x$.

37. $x(y+a)=y^2$.
38. $xy+ax+by+y^2=0$.

39. Prove that the d. c. of an even function of x is an odd function of x, and conversely that the d. c. of an odd function is even.

40. Illustrate Theorem vi, Art. 33, geometrically, by taking u as the area of a rectangle of which v is one side.

41. Find the slope of the curve $y^5+4xy^4=5$ at the point (1, 1).

42. Find the slope of the curve $y^3-3xy^2+a^2x=a^3$ at the point $(a, 3a)$.

36. Differentiation of Implicit Functions.

In the case of implicit functions (Art. 4), it is often difficult or impossible to find y explicitly in terms of x, and then by differentiating obtain the value of dy/dx in terms of x alone; but the value of dy/dx in terms of x and y can be obtained by differentiating, with respect to x, each term of the equation between x and y as it stands.

It has been proved (Arts. 31 and 34) that the d. c. of y^n with respect to x is $ny^{n-1}\,dy/dx$. The terms which occur most frequently in practice, when y is given implicitly as a function of x, are terms of the type $ax^m y^n$; and the d. c. of this term is equal to

$$a\left(x^m \,.\, ny^{n-1}\frac{dy}{dx}+y^n \,.\, mx^{m-1}\right),$$

by using the ordinary rule for the d. c. of a product.

One or two examples will now suffice to show how to find the value of dy/dx in such cases.

Examples:

(i) $x^2+y^2=a^2$. Since x^2+y^2 is constant, its d. c. is zero.

$$\therefore\ 2x+2y\,dy/dx=0,\ \text{whence}\ dy/dx=-x/y.$$

[In this simple case, y can be at once expressed explicitly in terms of x, for we have $y=\sqrt{(a^2-x^2)}$;

$$\therefore\ \frac{dy}{dx}=\frac{1}{2\sqrt{(a^2-x^2)}}\times -2x=-\frac{x}{\sqrt{(a^2-x^2)}},\ \text{which}=-\frac{x}{y}\ \text{as before.}]$$

(ii) $x^3+3axy+y^3=a^3$.

Differentiating each term with respect to x,

$$3x^2+3a\left(x\frac{dy}{dx}+y\right)+3y^2\frac{dy}{dx}=0,$$

whence
$$ax\frac{dy}{dx}+y^2\frac{dy}{dx}=-ay-x^2,$$

$$\text{and}\quad \frac{dy}{dx}=-\frac{ay+x^2}{ax+y^2}.$$

(iii) $x^2y^3+y^2x^3+a^5=0$.

Here
$$x^2 \,.\, 3y^2\frac{dy}{dx}+y^3 \,.\, 2x+y^2 \,.\, 3x^2+x^3 \,.\, 2y\frac{dy}{dx}=0,$$

whence
$$\frac{dy}{dx}=-\frac{3x^2y^2+2xy^3}{3x^2y^2+2x^3y}=-\frac{3xy+2y^2}{3xy+2x^2}.$$

(iv) If y is given in the form $\sqrt[n]{(a^n - x^n)}$, we may write $y^n = a^n - x^n$; then, differentiating with respect to x, we have

$$ny^{n-1}\frac{dy}{dx} = -nx^{n-1}. \therefore \frac{dy}{dx} = -\frac{x^{n-1}}{y^{n-1}}.$$

Examples XII.

Find dy/dx when x and y are connected by the following relations:

1. $x^3 + y^3 = a^3$.
2. $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
3. $x^n + y^n = a^n$.
4. $x^2 + xy + y^2 = 0$.
5. $x^3 + x^2y + xy^2 + y^3 = a^3$.
6. $1 + x^2y + xy^2 = 0$.
7. $x^m y^n = a^{m+n}$.
8. $a^m x^n = b^n y^m$.
9. $(x+y)^2 = ax^2 + by^2$.
10. $(x+y)^3 = 3axy$.
11. $(x^2+y^2)^2 = a^2(x^2 - y^2)$.
12. $ax^2 + 2bxy + cy^2 = 1$.
13. $ax^3 + bx^2y + cxy^2 + dy^3 = 1$.
14. $y = (a^{2/3} - x^{2/3})^{3/2}$.
15. $y = (a^{p/q} - x^{p/q})^{q/p}$.
16. $x^{2n} + x^n y^n + y^{2n} = a^{2n}$.
17. $x^3y^4 + x^4y^3 = a^7$.
18. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.
19. $1 + x + y + xy = a$.
20. $(a+x)(a+y) = x^2 + y^2$.
21. Find the value of dv/dp (i) when $pv = k$, (ii) when $pv^\gamma = k$, (iii) when $(p + av^{-2})(v - b) = k$.

37. Calculation of small corrections.

We have, in several numerical examples in Art. 26, shown how the result of Art. 24 can be used to find the approximate change in the value of a function due to a given small change in the value of its argument. Numerical results are frequently calculated from given data by aid of a mathematical formula. These data are often obtained by measurement or observation, and therefore cannot be found with absolute accuracy. An error in one or more of the data will produce an error in the value of any quantity calculated from them, and an important practical application of the differential coefficient is to determine the influence, upon the result of a calculation, of given small errors in the measurement of the quantities upon which it depends. At present, we shall confine ourselves to finding the effect of an error in one variable only. The general question of finding the aggregate effect of errors in several observations will be considered later (Chapter XXIII).

It is generally the relative or proportional error (i. e. the ratio of the error to the calculated value), or the percentage error, which is of importance, rather than the actual error.

Example 1. *A given quantity of metal is to be cast into the form of a right circular cylinder of radius* 5 *inches and height* 10 *inches; if the radius is made* $\frac{1}{20}$ *inch too large, what will be the difference in the height?*

Let r and h denote the radius and height respectively. Then

$$\pi r^2 h = \text{the volume} = \pi 25 \times 10, \text{ and therefore } h = 250/r^2.$$

We want to find the change in h due to a given change in r, therefore we need the d. c. of h with respect to r.

We have $dh/dr = 250 \times -2r^{-3} = -500/r^3$. Therefore if r be increased by $\frac{1}{20}$, h will decrease by (approximately) $\frac{1}{20} \times 500/5^3$, i.e. $\frac{1}{5}$ inch; hence the height will be 9·8 inches approximately.

Example 2. *The pressure* p *and volume* v *of a given mass of gas at constant temperature are connected by the relation* pv = k (*a constant*). *If the pressure of* 10 *cubic feet of the gas be* 14 *lb. per square inch, find the pressure when the volume is reduced to* 9·92 *cubic feet.*

Here we need the change in p due to a given small change in v; therefore we find the d. c. of p with respect to v.

$$p = \frac{k}{v}; \qquad \frac{dp}{dv} = -\frac{k}{v^2} = -\frac{pv}{v^2} = -\frac{p}{v},$$

i.e. if the volume be increased by a small amount, the pressure will decrease by nearly p/v as much.

In this case, the volume is decreased by ·08 cubic foot. Therefore the pressure will increase by $\frac{14}{10} \times {}$·08, i. e. ·112 lb. per square inch, i. e. the pressure will be 14·112 lb. per square inch.

Example 3. *The time of oscillation* t *of a simple pendulum of length* l *is given by the formula* $t = 2\pi\sqrt{(l/g)}$. *Find* (i) *the percentage change in the value of* t *if the length be increased* 1 *per cent.*

$$t = \frac{2\pi}{\sqrt{g}} \times \sqrt{l}; \qquad \therefore \quad \frac{dt}{dl} = \frac{2\pi}{\sqrt{g}} \times \frac{1}{2\sqrt{l}} = \frac{\pi}{\sqrt{(gl)}}.$$

by Art. 24, $\delta t = \dfrac{\pi}{\sqrt{(gl)}} \times \delta l$ approximately, and $\delta l = \dfrac{l}{100}$;

$$\delta t = \frac{\pi}{\sqrt{(gl)}} \times \frac{l}{100} = \frac{\pi}{100}\sqrt{\frac{l}{g}},$$

and the percentage error $= \dfrac{\delta t}{t} \times 100 = \dfrac{\pi}{100}\sqrt{\dfrac{l}{g}} \times \dfrac{100}{t} = \dfrac{1}{2}$.

Find (ii) *the percentage change in the value of* t, *if the pendulum be removed to a place where the value of* g *is diminished by* ·2 *per cent., the length being unaltered.*

In this case we need the d.c. of t with respect to g.

$$t = 2\pi\sqrt{l} \times g^{-1/2}; \qquad \frac{dt}{dg} = 2\pi\sqrt{l} \times -\frac{1}{2}g^{-3/2} = -\pi\sqrt{\frac{l}{g^3}};$$

approximately,

$$\delta t = -\pi\sqrt{\frac{l}{g^3}} \times \delta g = -\pi\sqrt{\frac{l}{g^3}} \times -\frac{g}{500} = +\frac{\pi}{500}\sqrt{\frac{l}{g}} = \frac{t}{1000};$$

$$\therefore \text{ the percentage change in } t = \frac{\delta t}{t} \times 100 = \frac{1}{10},$$

and the time of oscillation is increased by ·1 per cent.

Example 4. *If the preceding formula be used to calculate the value of* g *from observations of* t *and* l, *find* (i) *the possible error in the value of* g, *if the error in* l *may be* ·5 *per cent. either way,* t *remaining constant.*

We now need dg/dl.

We have $$g : \frac{4\pi^2}{} . l ; \qquad \frac{}{dl} \quad \frac{4\pi^2}{}$$

$\therefore$ approximately, the error in $g = \frac{4\pi^2}{t^2} \times$ the error in $l = \frac{g}{l} \times$

$$\frac{\delta g}{g} \quad \frac{\delta l}{l} = \pm {\cdot}005.$$

The proportional error in the value of g is equal to the proportional error in the value of l, as is obvious at once from the fact that g varies directly as l.

Find (ii) *the relative error in the value of* g *owing to the observed value of the time of oscillation being* ·1 *per cent. too much.*

Here we need dg/dt.

$$g : 4\pi^2 l \times \frac{1}{t^2}; \qquad \frac{dg}{dt} = 4\pi^2 l \times -2t^{-3} = -\frac{8\pi^2 l}{t^3}.$$

$$-\frac{8\pi^2 l}{t^3} \times \delta t \text{ approximately};$$

$$\therefore \quad \frac{\delta g}{g} = -\frac{8\pi^2 l}{t^3} \times \frac{t}{1000} \Big/ \frac{4\pi^2 l}{t^2} = -\frac{1}{500},$$

i.e. the value of g is too small by about $\frac{1}{500}$ of its calculated value.

38. Coefficients of expansion.

It is a well-known fact that most substances expand when heated, and that the amount of the expansion depends upon the rise in temperature; hence the dimensions of such a body are functions of its temperature. Consider a uniform bar which is of length l when its temperature is θ. Let the length be $l+\delta l$ when the temperature is raised to $\theta+\delta\theta$, so that δl is the increase of length due to the increase $\delta\theta$ in temperature. As $\delta\theta \to 0$, $\delta l \to 0$, but the ratio $\delta l/\delta\theta$ tends to a limiting value $dl/d\theta$. $dl/ld\theta$ is called the *coefficient of linear expansion*. This is frequently a small constant. Denoting it by α, we have $dl/d\theta = \alpha l$. If $\delta\theta$ be small, we have approximately $\delta l = l\alpha . \delta\theta$ and $l+\delta l = l(1+\alpha\delta\theta)$; $\delta l = \alpha$ if l and $\delta\theta$ are each unity, therefore α is approximately the increase per unit length of the bar for one degree rise in temperature.

Similarly, if A be the area of a lamina of the same material at temperature θ, and if δA be the increase of area when the temperature is raised by an amount $\delta\theta$, $dA/Ad\theta$ is the *coefficient of superficial expansion*. It is approximately the increase per unit area for one degree rise of temperature. If the lamina be a square of side l, $A = l^2$, and we have

$$\frac{1}{A}\frac{dA}{d\theta} = \frac{1}{l^2} . 2l\frac{dl}{d\theta} = \frac{2}{l}\frac{dl}{d\theta}.$$

Therefore the coefficient of superficial expansion is twice the coefficient of linear expansion.

Again, if V be the volume of a quantity of the material at temperature θ, and if δV be the increase of volume due to a rise $\delta\theta$ in temperature, $dV/Vd\theta$ is the *coefficient of cubical expansion*. It is approximately the increase per unit volume for one degree rise of temperature. If the volume be a cube of side l, $V=l^3$, and

$$\frac{1}{V}\frac{dV}{d\theta}=\frac{1}{l^3}\times 3l^2\frac{dl}{d\theta}=\frac{3}{l}\frac{dl}{d\theta}.$$

Therefore the coefficient of cubical expansion is three times the coefficient of linear expansion.

Example. A gramme of water (of which the volume at 4° C. is 1 c.c.) occupies at temperature θ° C. a volume $1+k(\theta-4)^2$ c.c. where k is a small numerical constant; find the coefficient of expansion at 0° C. and at 30° C.

Since $V=1+k(\theta-4)^2$, we have

$$\frac{dV}{d\theta}=2k(\theta-4) \quad\text{and}\quad \frac{1}{V}\frac{dV}{d\theta}=\frac{2k(\theta-4)}{1+k(\theta-4)^2}.$$

When $\theta=0$, this is equal to $-8k/(1+16k)$. When $\theta=10$, this is equal to $12k/(1+36k)$. Neglecting squares and higher powers of k, these results become $-8k$ and $12k$ respectively.

Coefficient of elasticity of volume of a fluid.

Again, suppose that a quantity of a fluid of unit mass changes so that the volume v is a definite function of the intensity of pressure p. An increase of pressure δp will produce a decrease of volume δv; $-\delta v/v$ is called the mean compression. The ratio of the increase of pressure to the mean compression, i.e. $-v\delta p/\delta v$, tends, as $\delta v\to 0$, to a limiting value $-vdp/dv$, and this is defined as the *elasticity of volume* of the fluid.

If a gas expands at constant temperature, then, by Boyle's Law, $pv=k$.

Differentiating this equation with respect to v, we obtain

$$p+v\frac{dp}{dv}=0, \text{ whence } -v\frac{dp}{dv}=p,$$

i.e. the elasticity is equal to the intensity of pressure p.

If a gas expands adiabatically, then $pv^\gamma=$ a constant k. Differentiating with respect to v, we have

$$p\gamma v^{\gamma-1}+v^\gamma\frac{dp}{dv}=0,$$

whence, dividing by $v^{\gamma-1}$, we obtain $-v\dfrac{dp}{dv}=\gamma p$,

i.e. the elasticity is equal to γp.

Examples XIII.

1. If the side of a square can be measured accurately to $\frac{1}{100}$ inch, find the possible error in the area of a square whose side is measured to be 15 inches.

2. If the diameter of a sphere can be measured to $\frac{1}{50}$ inch, find the possible error in (i) the volume, (ii) the superficial area, when the diameter is found to be 20 inches.

3. Find the possible error in the area of a circle whose circumference is measured and found to be 56 inches with a possible error of $\frac{1}{20}$ inch.

4. A given quantity of metal is to be cast into the form of a cylinder of radius 4 inches and height 15 inches; if the radius is made $\frac{1}{64}$ inch too small, what will be the difference in the height?

5. Twenty-seven cubic feet of material are to be put in the form of a cube; if there is ·1 cubic foot short, what will the length of the edge of the cube be?

6. A square plot of ground is to be measured out with an area of 900 square yards. What error in the length of the side will make the area 1 square yard too much?

7. The diameter of a sphere can be measured to $\frac{1}{100}$ inch; find the percentage error in (i) the volume, (ii) the area of the surface of the sphere.

8. Two sides and the included angle of a triangle are measured as 80 inches, 40 inches, and 60° respectively; if an error of $\frac{1}{2}$ inch is made in measuring the first side, what will be the resulting error in (i) the area of the triangle, (ii) the length of the third side, when calculated from those values?

9. The side c of a triangle is calculated from the formula
$$c^2 = a^2 + b^2 - 2\,ab\cos C;$$
find the percentage error in the value of c due to an error of 1 per cent. in the value of a.

10. Four rods, each 15 inches long, are joined together to form a square. If two opposite corners are pressed towards each other until their distance apart is just 21 inches, how far apart will the other two corners then be?

11. A ladder 50 feet long rests with its upper end against a vertical wall and its lower end on the ground 14 feet from the wall; if the lower end is pulled a distance of 3 inches further from the wall, how far will the upper end descend?

12. The pressure p and volume v of a given mass of gas at constant temperature are connected by the relation $pv = k$; if the pressure of 10 cubic feet of the gas be 14 lb. weight per square inch, find (i) the pressure when the volume is reduced to 9·9 cubic feet; (ii) what change of volume will increase the pressure to 14·2 lb. per square inch.

13. The distances x and x', from a lens of focal length f, of a point on the axis of the lens and of its image are connected by the relation $1/x + 1/x' = 1/f$; find the magnification of a small object in the direction of the axis if $x' = 1$ foot when $x = 4$ inches.

14. The value of g is calculated to be 32·2 from the formula $t = 2\pi\sqrt{(l/g)}$ where t is the time of oscillation of a pendulum of length l; if an error of 1 per cent. is made in measuring t, find (i) the actual error, (ii) the percentage error in the value of g.

15. Find the change in the time of oscillation of a pendulum if its length be increased 1 per cent. Find also how much it will lose per day, if it originally kept correct time.

16. Find the change in the time of oscillation, and the number of seconds gained or lost per week, if a pendulum, which keeps correct time in a place where $g = 32{\cdot}2$, is removed to a place where $g = 32{\cdot}1$.

17. A formula for the variation of electrical resistance R of a platinum wire with the temperature θ is $R = R_0(1 + a\theta + b\theta^2)$ where R_0, a and b are constants; find the increase of resistance due to a given small rise of temperature.

18. With the data of the example in Art. 38, find the coefficient of expansion of water at 9° C.

19. The coefficient of expansion of a bar of metal is ·00003; find the increase in the length of a bar originally 10 yards long, when its temperature is raised 1° C.

20. Twenty cubic feet of air at atmospheric pressure are compressed to a volume of 5 cubic feet; find the greatest cubical elasticity when the expansion follows (i) the law $pv = k$, (ii) the law $pv^{1{\cdot}4} = k$.

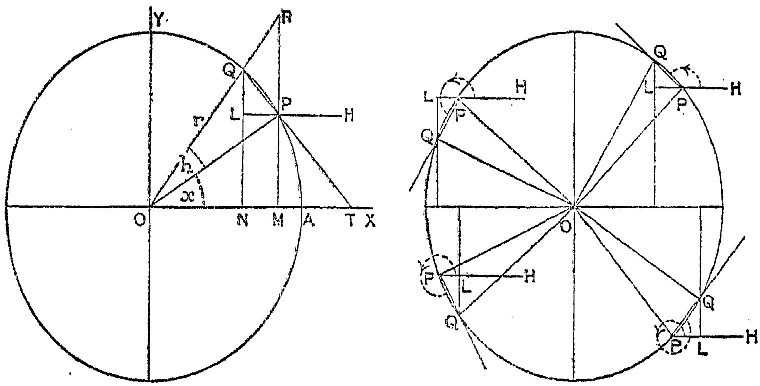
CHAPTER IV

DIFFERENTIATION OF SIMPLE TRIGONOMETRICAL FUNCTIONS

39. Differential coefficient of $\sin x$.

This can be obtained either analytically, by the method of Art. 26, which involves the use of either the 'addition formulae' or the 'product formulae' of trigonometry, or geometrically. The latter method involves merely the simplest ideas and properties of the trigonometrical ratios, and we will therefore consider it first.

(i) *Geometrically.* Let AOP (Fig. 43) be an angle of radian measure x at the centre O of a circle of radius r; and let AOQ be an angle $x+h$, so that POQ is the increase h in x. Let PM, QN



(a) Fig. 43. (b)

be drawn perpendicular to OA, PL perpendicular to QN, and PH parallel to the positive direction of the axis of x.

Then $\qquad \sin x = MP/r, \quad \sin(x+h) = NQ/r;$

$\therefore$ the increase in $\sin x = (NQ-MP)/r = LQ/r$.

$$\therefore \quad \frac{\text{the increase in } \sin x}{\text{the increase in } x} = \frac{LQ}{r \cdot h} = \frac{LQ}{\text{arc } PQ}$$

$$= \frac{LQ}{\text{chord } PQ} \times \frac{\text{chord } PQ}{\text{arc } PQ} = \sin HPQ \times \frac{\text{chord } PQ}{\text{arc } PQ}.$$

As $h \to 0$, Q moves indefinitely near to P; the limiting position of QP is the tangent at P, and the limit of the angle HPQ is the angle XTP which the tangent at P makes with the positive direction of the axis of x, i. e. $\frac{1}{2}\pi + x$.

Therefore $\sin HPQ$ tends to the limit $\sin(\frac{1}{2}\pi + x)$, and [Art. 13 (10)] the ratio chord PQ/arc PQ tends to the limit 1.

$$\therefore \quad \text{the d. c. of } \sin x = \mathrm{Lt}\frac{\text{increase in } \sin x}{\text{increase in } x}$$

$$= \sin(\tfrac{1}{2}\pi + x) \times 1 = \cos x.$$

In figure (a) the angle x is taken less than $\frac{1}{2}\pi$, but by drawing figures for the other cases (b), it is easily seen that, with the usual conventions of sign and supposing the angle between PH and PQ to be always measured in the positive direction from PH, this angle always tends to the limit $\frac{1}{2}\pi + x$ as $h \to 0$, and the above reasoning always holds.

(ii) *Analytically.* Let $y = \sin x$, and let x be measured in radians. If x is increased to $x+h$, y becomes $\sin(x+h)$.

$\therefore$ $\delta y = \sin(x+h) - \sin x = 2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h$ (Product formula).

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h} = \cos(x+\tfrac{1}{2}h) \times \frac{\sin\frac{1}{2}h}{\frac{1}{2}h};$$

as $h \to 0$, the first factor $\to \cos x$ and the second factor $\to 1$.

$$\frac{dy}{dx} = \mathop{\mathrm{Lt}}_{h\to 0}\frac{\delta y}{\delta x} = \cos x.$$

Or we may proceed as follows:

$\delta y = \sin(x+h) - \sin x = \sin x \cos h + \cos x \sin h - \sin x$ (Addition formula).

$$= \cos x \sin h - \sin x(1 - \cos h);$$

$$\therefore \quad \frac{\delta y}{\delta x} = \cos x\,\frac{\sin h}{h} - \sin x\,\frac{1-\cos h}{h};$$

$$\frac{dy}{dx} = \text{the limit of this when } h \to 0$$

$$= \cos x \times 1 - \sin x \times 0 \quad \text{[Art. 13 (10)]}$$

$$= \cos x.$$

The student must notice carefully that the d. c. of $\sin mx$ (where m is a constant) is not $\cos mx$, but, by the rule of Art. 34, $\cos mx \times m$,

i. e. *the d. c. of* $\sin$ mx *is* m $\cos$ mx,

e. g. the d. c. of $\sin 2x = 2\cos 2x$,

the d. c. of $\sin\frac{1}{3}x = \frac{1}{3}\cos x$.

40. Differential coefficient of $\cos x$

This may be found by exactly the same methods as the d. c. of $\sin x$. Since $\cos x$ decreases from $+1$ to -1 as x increases from 0 to π, it is evident that for such values of x, its d. c. will be negative (Art. 25).

(i) *Geometrically.* From Figure 43 we have

$$\cos x = OM/r, \quad \cos(x+h) = ON/r;$$

$\therefore$ the increase in $\cos x = (ON - OM)/r$

$$= -NM/r = -LP/r;$$

$$\frac{\text{the increase in } \cos x}{\text{the increase in } x} = -\frac{LP}{rh} = +\frac{PL}{\text{arc } PQ}, \text{ since } -LP = +PL,$$

$$\frac{PL}{\text{chord } PQ} \times \frac{\text{chord } PQ}{\text{arc } PQ}$$

$$= \cos HPQ \times \frac{\text{chord } PQ}{\text{arc } PQ}.$$

As before, when $h \to 0$, the angle $HPQ \to \frac{1}{2}\pi + x$, and the second factor $\to 1$.

$\therefore$ the d. c. of $\cos x = \cos(\frac{1}{2}\pi + x) \times 1 = -\sin x$.

(ii) *Analytically.*

Proceeding exactly as in the case of $\sin x$, we get

$$\frac{\delta y}{\delta x} = \sin(x + \tfrac{1}{2}h) \times -\frac{\sin \frac{1}{2}h}{\frac{1}{2}h}.$$

As $h \to 0$, the first factor tends to the limit $\sin x$ and the second to the limit -1; therefore $dy/dx = -\sin x$, and by the general rule of Art. 34, it follows that

the d. c. of $\cos mx = -m \sin mx$.

If the angle x be measured in degrees instead of in circular measure, these differential coefficients take a less simple form, an inconvenient numerical factor being introduced.

For, in that case, the radian measure of $h^\circ = \frac{1}{180}\pi h$.

$\therefore$ since $\quad \text{Lt} \sin\theta/\theta = 1$, as θ *measured in radians* $\to 0$,

we have $$\text{Lt} \frac{\sin h^\circ}{h^\circ} = \frac{\pi}{180} \text{Lt} \frac{\sin h^\circ}{\frac{1}{180}\pi h} = \frac{\pi}{180} \times 1,$$

and hence the differential coefficients of $\sin x^\circ$ and $\cos x^\circ$ are $\frac{1}{180}\pi \cos x^\circ$ and $-\frac{1}{180}\pi \sin x^\circ$, respectively.

41. Differential coefficient of $\tan x$.

(i) *Geometrically.* In Fig. 43, let OQ produced meet MP produced in R.

Then $\tan x = MP/OM$, and $\tan(x+h) = MR/OM$;

$\therefore$ the increase in $\tan x = (MR - MP)/OM = PR/OM = PR/r\cos x$.

$$\frac{\text{the increase in } \tan x}{\text{the increase in } x} = \frac{PR}{rh\cos x} = \frac{1}{\cos x} \times \frac{PR}{\text{arc } PQ}$$

$$= \sec x \times \frac{PR}{PQ} \times \frac{PQ}{\text{arc } PQ};$$

Now the angle $PQR \to \frac{1}{2}\pi$ as $h \to 0$; therefore the triangles PQR and LQP are ultimately similar, and $PR/PQ \to PQ/LQ$, i.e. cosec HPQ, which $\to \operatorname{cosec}(\frac{1}{2}\pi + x)$;

$$\therefore \text{ the d. c. of } \tan x = \sec x \times \operatorname{cosec}(\tfrac{1}{2}\pi + x) \times 1$$

$$= \sec x \times \sec x = \sec^2 x.$$

(ii) *Analytically.*

Let $y = \tan x$. If x is increased to $x+h$, y becomes $\tan(x+h)$;

$$\therefore \quad \delta y = \tan(x+h) - \tan x = \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x}$$

$$= \frac{\sin(x+h-x)}{\cos(x+h)\cos x} = \frac{\sin h}{\cos(x+h)\cos x};$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{\sin h}{h} \cdot \frac{1}{\cos(x+h)\cos x}.$$

As $h \to 0$, the first factor $\to 1$, and the second $\to 1/\cos x \,.\, \cos x$.

$$\therefore \text{ the d. c. of } \tan x = 1/\cos^2 x = \sec^2 x.$$

This is always $+$, whatever be the value of x; therefore $\tan x$ always increases as x increases (except as it passes through its points of discontinuity, and then $\delta y/\delta x$ does not tend to a finite limit) as is obvious from its graph. Geometrically, the tangent to the graph always makes an acute angle with the axis of x.

From Art. 34, *the d. c. of tan* mx = m *sec*^{2} mx.

42. Differential coefficients of other circular functions.

The differential coefficients of $\cot x$, $\sec x$, and $\operatorname{cosec} x$ can be obtained in a similar manner to those of $\sin x$, $\cos x$, and $\tan x$, by either of the preceding methods.

It should be noticed that, from the d. c. of $\sin x$, the d. c.'s of all the other circular functions can easily be deduced by the aid of the general rules of the last chapter:

We have

d. c. of $\sin x = \cos x$.

d. c. of $\cos x$, i.e. of $\sin(\frac{1}{2}\pi + x)$, $= \cos(\frac{1}{2}\pi + x) \times 1$ [Art. 34]

$$= -\sin x.$$

d. c. of $\tan x$, i.e. of $\dfrac{\sin x}{\cos x}$, $= \dfrac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x}$ [Art. 33]

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

d. c. of $\cot x$, i.e. of $\dfrac{\cos x}{\sin x}$, $= \dfrac{\sin x \times -\sin x - \cos x \times \cos x}{\sin^2 x}$ [Art. 33]

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

d. c. of $\sec x$, i.e. of $\dfrac{1}{\cos x}$, $= -\dfrac{1}{\cos^2 x} \times -\sin x$ [Art. 34] $= \dfrac{\sin x}{\cos^2 x}$

$$= \sec x \tan x.$$

d. c. of $\operatorname{cosec} x$, i.e. of $\dfrac{1}{\sin x}$, $= -\dfrac{1}{\sin^2 x} \times \cos x$ [Art. 34]

$$= -\frac{\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x.$$

43. Application to numerical examples.

We have now found the rates at which all the circular functions are changing for any value of x, and will apply them to numerical examples.

Ex. (i). *To find the value of* $\sec 60° 1'$.

From elementary geometry, $\sec 60° = 2$, and it has just been shown that the d. c. of $\sec x$ is $\sec x \tan x$. Therefore (Art. 24) if x is increased by a very small amount, $\sec x$ will increase by approximately $\sec x \tan x$ times as much; hence, if x be $60°$ and if it increase by $1'$, i. e. in radian measure $\pi/10800$, which is small, the secant will increase by $\sec 60° \tan 60° \times \pi/10800$, i. e. by $2 \times \sqrt{3} \times \pi/10800$, which works out to ·001008 nearly;

$$\therefore \quad \sec 60° 1' = 2{\cdot}001008 \text{ approximately.}$$

(ii) *To find the value of* $\cos 135° 1'$.

From geometry, $\cos 135° = -1/\sqrt{2}$. The d. c. of $\cos x = -\sin x$, and therefore if x increases by a very small amount, $\cos x$ will *decrease* by approximately $\sin x$ times as much. Hence if x be increased from $135°$ to $135° 1'$, $\cos x$ will decrease by $\sin 135° \times$ the radian measure of $1'$, i.e. by $1/\sqrt{2} \times \pi/10800$ nearly; this works out to ·0002057.

$$\therefore \quad \cos 135° 1' = -1/\sqrt{2} - {\cdot}0002057 = -{\cdot}7073125.$$

(iii) *The height of a tower is calculated from its observed elevation at a point which is a measured distance from its base in the horizontal plane upon which it stands. If this distance is 450 feet, and the elevation is observed as 35° 30′, find the approximate error in the height due to an error of 5′ in the angle of elevation.*

Taking first *general values* θ and h for the elevation and height, which are being varied, we have

$$h = 450 \tan\theta, \text{ and therefore } dh/d\theta = 450 \sec^2\theta;$$

hence a small increase in the value of θ produces an increase of approximately $450\sec^2\theta$ times as much in the value of h.

In the example given, $\theta = 35° \, 30'$, and the error in $\theta = 5' = \pi/2160$ in radian measure.

$\therefore$ the resulting error in the height of the tower

$$= \tfrac{1}{2160}\pi \times 450 \sec^2 35° \, 30' \text{ approximately.}$$

Evaluating this by logarithms, we get ·9875, i. e. the error in the height of the tower is nearly ·9875 of a foot.

44. Application of general rules to trigonometrical functions.

By the aid of the differential coefficients of $\sin x$, $\cos x$, and $\tan x$, together with the general rules for differentiating products, quotients, and functions of a function, many other differential coefficients can be at once written down. The following are typical examples:

The d. c. of $x^n \sin x = x^n \cos x + nx^{n-1} \sin x$ (Art. 30)

" " $\cos(\alpha - 2x) = -\sin(\alpha - 2x) \times$ d. c. of $\alpha - 2x$ (Art. 34) $= 2\sin(\alpha - 2x)$.

" " $\sin^4 x = 4\sin^3 x \times$ d. c. of $\sin x$ (Art. 34) $= 4\sin^3 x \cos x$.

" " $\tan^n x = n\tan^{n-1} x \times$ d. c. of $\tan x = n\tan^{n-1} x \sec^2 x$.

" " $\dfrac{\sin^2 x}{\cos x} = \dfrac{\cos x . 2\sin x \cos x - \sin^2 x(-\sin x)}{\cos^2 x}$ (Arts. 33, 34) $= \dfrac{\sin x(\cos^2 x + 1)}{\cos^2 x}$.

" " $\operatorname{cosec}^4 x$, i. e. of $(\sin x)^{-4}$, $= -4(\sin x)^{-5} \times \cos x$ (Art. 34) $= -4\cos x/\sin^5 x$.

" " $\sin^n mx = n\sin^{n-1} mx \times$ d. c. of $\sin mx$ (Art. 34) $= n\sin^{n-1} mx \times m\cos mx$.

Examples XIV.

Differentiate with respect to x:

1. $\sin 5x$, $\sin\frac{1}{3}x$, $\sin(nx - \alpha)$, $\cos ax$, $\cos(x/p)$, $\cos(\frac{1}{5}\pi - 2x)$.
2. $\tan 3x$, $\tan(x + \alpha)$.
3. $\cot mx$, $\cot(\alpha - 2x)$.
4. $\sec mx$, $\sec(\frac{1}{4}\pi + x)$.
5. $\operatorname{cosec} mx$, $\operatorname{cosec}(\beta - \frac{1}{2}x)$.
6. $\sin^3 x$, $\sin^n x$.
7. $\cos^5 x$, $\cos^m x$.
8. $\sqrt{\sin x}$.
9. $\operatorname{cosec}^2 x$.
10. $\sqrt{\operatorname{cosec} x}$.
11. $\sqrt[3]{\cos x}$.
12. $\sec^4 x$.
13. $\cot^n x$.
14. $\sin^2 2x$.

15. $\cos^3 ax$.
16. $\tan^n 3x$.
17. $\cot^2 \frac{1}{2}x$.
18. $x^4 \sin 3x$.
19. $x^n \cos x$.
20. $\sqrt{x} \, . \tan x$.
21. $(\sin 2x)/x^3$.
22. $\sin 3x \cos 4x$.
23. $\sin mx \cos nx$.
24. $\sin x \tan x$.
25. $\sin^2 x \tan x$.
26. $(a + b \sin x)^2$.
27. $\sqrt{(3 + 4 \cos x)}$.
28. $\sin x - \frac{1}{3} \sin^3 x$.
29. $\tan x + \frac{1}{3} \tan^3 x$.
30. $\dfrac{3 + 4 \sin x}{4 + 3 \sin x}$.
31. $\dfrac{a - b \cos x}{a + b \cos x}$.
32. $\dfrac{1 + \tan x}{1 - \tan x}$.
33. $\sin^2 x/(1 + \sin x)$.
34. $\sin 2x \cos^2 x$.
35. $\sin^2 x \cos^2 x$.
36. $\sin x/\cos^2 x$.
37. $\sin^m x \cos^n x$.
38. $\sin^m ax \cos^n bx$.
39. $x^n \tan^m ax$.

Find dy/dx in the following cases:

40. $\sin mx - \cos ny = c$.
41. $\sin^2 x + \cos^2 y = a^2$.
42. $\sin x \cos y = c$.
43. $y \tan y = x$.

Obtain, by the aid of the d. c.'s of the circular functions, the approximate values of:

44. $\cos 60° 1'$.
45. $\sin 120° 2'$.
46. $\tan 45° 1'$.
47. $\cot 135° 3'$.
48. $\sqrt{\sin 60° 5'}$.
49. $\sqrt[3]{\tan 135° 2'}$.
50. $\operatorname{cosec}^2 30° 2'$.
51. $\sin^2 29° 57'$.

52. The width of a river is calculated from the elevation, at a point on one bank, of a tree 50 ft. high on the opposite bank; find the approximate error in the width due to an error of 5′ in the angle, which is observed as 18°.
53. Two sides of a triangle are 20 ft. and 40 ft. and the included angle is 30°; if the angle be increased by 2′, find the resulting increase in the length of the third side.
54. In the preceding question, find the resulting increase in the area of the triangle.
55. The side a of a triangle is calculated from the values $b = 30$, $B = 70°$, $A = 42°$; find the error in a due to an error of 15′ in A.
56. The angle A of a triangle is calculated from the values $a = 70$, $b = 90$, $B = 65°$; find (i) the actual error, (ii) the percentage error due to an error of ·2° in B.
57. The area of a triangle is calculated from the observed values of b, c, A; find the relative error due to a known error δA in the value of A.
58. If $x°$ be the reading of a tangent galvanometer when a current y passes through it, $y = C \tan x$, where C is a constant; find (i) the error, (ii) the percentage error in the value of the current due to an error of $\frac{1}{2}°$ in the reading when $x = 45°$.
59. The distance of a boat at sea is calculated from its angle of depression 15°, observed at the top of a cliff 120 ft. high; find the error in the distance if the angle be $\frac{1}{4}°$ too small.
60. The height of a tower is calculated from its angles of elevation 35° and 28°, observed at two points 150 ft. apart in a horizontal straight line through its base. If the former measurement is found to be $\frac{1}{2}°$ out, what will be the resulting error in the calculated height?

Miscellaneous examples for practice in differentiation. XV.

Find the differential coefficients of the following functions of x:

$(x - 3)^5$.
2. $(7 - x)^8$.
3. $\sqrt{(1 - x^3)}$.
4. $x^2 (1 - x)^2$.
5. $1/\sqrt{(x^2 - 3x - 2)}$.
6. $1/(5 - 7x)^4$.

7. $1/\sqrt[3]{(x^2+1)}$.
8. $x\sqrt{(4-x^2)}$.
9. $\sqrt{\{x(4-x^2)\}}$.
10. $\dfrac{\sqrt{(4-x^2)}}{x}$.
11. $\dfrac{x}{\sqrt{(4-x^2)}}$.
12. $\sqrt{\dfrac{x}{4-x^2}}$.
13. $\sqrt{\{(4-x^2)/x\}}$.
14. $\sin^2(x-\alpha)$.
15. $\cos^n \tfrac{1}{2}x$.
16. $x-\tan x$.
17. $x\tan x$.
18. $x/\tan x$.
19. $(\tan x)/x$.
20. $\sin x\cos^2 x$.
21. $\sqrt{x}\,.\,\sin x$.
22. $\sqrt{x}/\sin x$.
23. $(\sin x)/\sqrt{x}$.
24. $\sqrt{\sin x}$.
25. $\sin\sqrt{x}$.
26. $\sqrt{(x\sin x)}$.
27. $x\sin\sqrt{x}$.
28. $\sqrt{(\sin x/x)}$.
29. $x/\sin\sqrt{x}$.
30. $\sqrt{(x/\sin x)}$.
31. $\sin\sqrt{x}\,.\,/x$.
32. $x\sqrt{\sin x}$.
33. $\sqrt{(\sin x)}/x$.
34. $x/\sqrt{\sin x}$.
35. $\sqrt{x}\,.\,\sin\sqrt{x}$.
36. $\sqrt{x}/\sin\sqrt{x}$.
37. $(\sin\sqrt{x})/\sqrt{x}$.
38. $2x\sqrt{(1-x)}$.
39. $(\sec x)/a$.
40. $\sec(x/a)$.
41. $\sec(a/x)$.
42. $x^m(a-x)^n$.
43. $x^m/(a-x)^n$.
44. $(a-x)^n/x^m$.
45. $\sqrt[n]{(a-x)}$.
46. $\sqrt{(a^n-x^n)}$.
47. $\sqrt{(a-x)^n}$.
48. $\sin^3(\alpha-x)$.
49. $\sin 3(\alpha-x)$.
50. $\sin(\alpha-x)^3$.
51. $x^2\cos 2x$.
52. $(x\cos 2x)^2$.
53. $x^2/\cos 2x$.
54. $(\cos^2 2x)/x$.
55. $(\cos^2 2x)/x^2$.
56. $x\cos^2 2x$.
57. $(\cos 2x)/x^2$.
58. $x^2/\cos^2 2x$.
59. $x/\cos^2 2x$.
60. $\sqrt{(1+\sin^2 2x)}$.
61. $(1-\cos 2x)^n$.
62. $(a+b\sin^2 x)^n$
63. $\sqrt[n]{(1+\cos nx)}$.
64. $\dfrac{1}{1+\sin^n x}$.
65. $\left(a+b\sin\dfrac{x}{b}\right)$.
66. $\dfrac{1+\sin 2x}{1-\sin 2x}$
67. $\dfrac{1+\sin^2 x}{1-\sin^2 x}$.
68. $\dfrac{1+\sin^2 2x}{1-\sin^2 2x}$.
69. $\dfrac{1}{x+\sqrt{(a^2+x^2)}}$.
70. $\sqrt{\dfrac{1+\sin x}{1-\sin x}}$
71. $\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}$.
72. $\dfrac{x^2-3x+5}{x^2+5x-3}$.
73. $\dfrac{a-x}{\sqrt{(a^2-x^2)}}$.
74. $\dfrac{(x-3)(x+2)}{(x+3)(x-2)}$.
75. $\dfrac{x}{\sqrt{(2ax-x^2)}}$.
76. $\sin^3 x\cos 3x$.
77. $\sin 3x\cos^3 x$.
78. $\sin 3x\cos 3x$.
79. $\sin^3 x\cos^3 x$.
80. $\dfrac{}{\cos 3x}$.
81. $\dfrac{\cos 3x}{\sin^3 x}$.
82. $\dfrac{\sin 3x}{\cos^3 x}$.
83. $\dfrac{\cos^3 x}{\sin 3x}$.
84. $\dfrac{\sin 3x}{\cos 3x}$
85. $\dfrac{\cos 3x}{\sin 3x}$.
86. $\dfrac{\sin^3 x}{\cos^3 x}$
87. $\dfrac{\cos^3 x}{\sin^3 x}$
88. $\sin 3x\cos^3 3x$.
89. $\cos 3x\sin^3 3x$.
90. $\sin^3 3x\cos^3 3x$.
91. $\dfrac{\sin 3x}{\cos^3 3x}$.
92. $\dfrac{\cos^3 3x}{\sin 3x}$.
93. $\dfrac{\cos 3x}{\sin^3 3x}$.
94. $\dfrac{\sin^3 3x}{\cos 3x}$.
95. $\dfrac{\sin^3 3x}{\cos^3 3x}$.
96. $\dfrac{\cos^3 3x}{\sin^3 3x}$.
97. $\sin^3 x\cos^3 3x$.
98. $\dfrac{\sin^3 x}{\cos^3 3x}$.
99. $\dfrac{\cos^3 3x}{\sin^3 x}$.
100. $\cos^3 x\sin^3 3x$.
101. $\dfrac{\cos^3 x}{\sin^3 3x}$.
102. $\dfrac{\sin^3 3x}{\cos^3 x}$.
103. $\sin^3 x\sin 3x$.
104. $\dfrac{}{\sin 3x}$.
105. $\dfrac{\sin 3x}{\sin^3 x}$.

106. $\cos^3 x \cos 3x$. 107. $\dfrac{\cos^3 x}{\cos 3x}$. 108. $\dfrac{\cos 3x}{\cos^3 x}$.

109. $\sin x \sin^3 3x$. 110. $\dfrac{\sin x}{\sin^3 3x}$. 111. $\dfrac{\sin^3 3x}{\sin x}$.

112. $\cos x \cos^3 3x$. 113. $\dfrac{\cos x}{\cos^3 3x}$. 114. $\dfrac{\cos^3 3x}{\cos x}$.

115. $\sin x \cos^3 3x$. 116. $\dfrac{\sin x}{\cos^3 3x}$. 117. $\dfrac{\cos^3 3x}{\sin x}$.

118. $\cos x \sin^3 3x$. 119. $\dfrac{\cos x}{\sin^3 3x}$. 120. $\dfrac{\sin^3 3x}{\cos x}$.

121. $x^2\sqrt{(a^2-x^2)}$. 122. $\sqrt{(a^2-x^2)}/x^2$. 123. $x^2/\sqrt{(a^2-x^2)}$.

124. $x^3\sqrt{(a^2-x^2)}$. 125. $x^3/\sqrt{(a^2-x^2)}$. 126. $x^3(a^2-x^2)^n$.

127. $x^n(a^2-x^2)^n$. 128. $\dfrac{(1+x)^2}{1+2x}$. 129. $\left(\dfrac{1+x}{1+2x}\right)^2$.

130. $\dfrac{(1+x)^2}{(1+2x)^3}$. 131. $\dfrac{(1-x)^4}{(1+x)^3}$. 132. $\dfrac{(1-x)^3}{(2-x)^2}$.

133. $\dfrac{1-2x}{(1+3x)^2}$. 134. $\dfrac{(a-x)^3}{(a+x)^4}$. 135. $\dfrac{1+x^2}{(1-x^2)^2}$.

136. $\dfrac{(1+x^2)^2}{1-x^2}$. 137. $\dfrac{a^2-x^2}{(a^2+x^2)^2}$. 138. $\left(\dfrac{a^2-x^2}{a^2+x^2}\right)^2$.

139. $\left(\dfrac{a-x}{b-x}\right)^n$. 140. $(a-x)^n(b-x)^n$. 141. $(a^2-x^2)^n(b^2-x^2)^n$.

142. $x\sqrt{(3-4x+2x^2)}$. 143. $x(3-4x+2x^2)^{3/2}$. 144. $\sqrt{(3-4x+2x^2)}/x$.

145. $(1+x)/\sqrt{(2x+x^2)}$. 146. $x^2\sqrt{(3-4x+2x^2)}$. 147. $x^2/\sqrt{(3-4x+2x^2)}$.

148. $x^n \sin^n x$. 149. $x^n \sin^2 nx$. 150. $x^n \sin^n nx$.

CHAPTER V

GEOMETRICAL APPLICATIONS OF THE DIFFERENTIAL COEFFICIENT

45. Direction of tangent.

It has been shown (Art. 23) that if the tangent at any point (x, y) of a curve, whose equation is $y = f(x)$, makes an angle ψ with the positive direction of the axis of x, then the value of dy/dx at that point is equal to $\tan\psi$. This is the starting-point of many applications of the calculus to geometry.

Examples:

(i) *Find the inclination to the axis of* x *of the tangent at the point* (2, 4) *to the curve* $y = x/(1+x^2)$.

$$\frac{dy}{dx} = \frac{(1+x^2) - x \cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2};$$

at the point (2, 4), this $\quad = -3/5^2 = -{\cdot}12$;

$$\therefore \quad \tan\psi = -{\cdot}12, \quad \text{and} \quad \psi = 173^\circ\, 9'.$$

The tangent makes an angle of $173^\circ\, 9'$ with the positive direction of the axis of x.

(ii) *Find the direction of the tangent at* (3, 2) *to the curve* $x^3 + y^3 = 35$.

In this case, differentiating the equation as it stands with respect to x (Art. 36), we have $\quad 3x^2 + 3y^2\, dy/dx = 0$;

$$\therefore \quad dy/dx = -x^2/y^2, \text{ which at the point (3, 2) becomes } -\tfrac{9}{4}.$$

$$\therefore \quad \tan\psi = -\tfrac{9}{4} = -2{\cdot}25, \quad \text{and} \quad \psi = 113^\circ\, 58'.$$

The tangent makes an angle of $113^\circ\, 58'$ with the positive direction of the axis of x.

If a curve passes through the origin, the value of dy/dx there gives the form of the curve at the origin.

For instance, in example (i), when $x = 0$, $dy/dx = 1$; $\therefore$ $\tan\psi = 1$ and $\psi = 45^\circ$; the tangent to the curve at the origin bisects the angle between the axes.

In the curve $\quad y = x^2/(1+x^2)$,

$$\frac{dy}{dx} = \frac{(1+x^2)\, 2x - x^2 \cdot 2x}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}.$$

$\therefore$ at the origin, $dy/dx = 0$ and $\psi = 0$; the curve touches the axis of x at the origin.

In the curve $y = x^{2/3}/(1+x^2)$,

$$\frac{dy}{dx} = \frac{(1+x^2)\frac{2}{3}x^{-1/3} - x^{2/3}.2x}{(1+x^2)^2}$$

$$= \frac{\frac{2}{3}(1+x^2) - 2x^2}{x^{1/3}(1+x^2)^2}$$

$$= \frac{2-4x^2}{3x^{1/3}(1+x^2)^2}.$$

As $x \to 0$, this $\to \infty$ and hence $\psi \to 90°$; the curve touches the axis of y at the origin.

46. Equation of tangent to a curve at any point.

The fact that $dy/dx = \tan\psi$ enables us to find at once the equation of the tangent to a given curve at a given point.

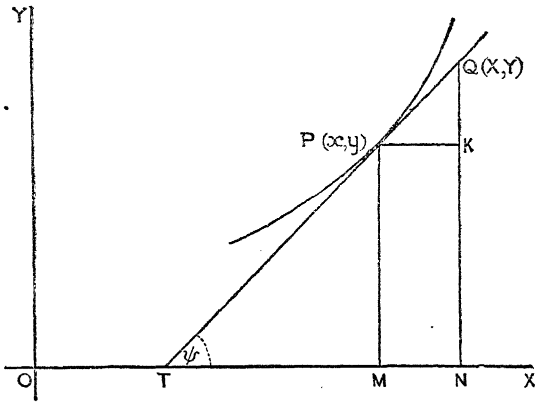


Fig. 44.

Let the tangent at the point $P(x, y)$ of a curve cut the axis of x in T (Fig. 44), and let (X, Y) be the coordinates of any other point Q on the tangent. Draw the ordinates PM and QN, and draw PK perpendicular to QN.

Then $KQ/PK = \tan KPQ = \tan NTP = \tan\psi = dy/dx$

and $KQ = Y - y,\ PK = X - x$;

$$\frac{Y-y}{X-x} = \frac{dy}{dx}, \quad \text{i.e.} \quad Y - y = (X-x)\frac{dy}{dx}.$$

This equation is quite general, and gives the equation of the tangent at any point to any curve whose equation is known; (x, y) are the coordinates of the point of contact, and the value of dy/dx is obtained by differentiating the equation of the curve and substituting in the result the values of x and y.

Examples:

(i) *Find the equation of the tangent to* $y^3 = x^2$ *at the point* (8, 4).

Differentiating, we have $3y^2 dy/dx = 2x$;

$\therefore$ at the point (8, 4) $\quad \dfrac{dy}{dx} = \dfrac{2x}{3y^2} = \dfrac{16}{48} = \dfrac{1}{3}$;

$\therefore$ the equation of the tangent is

$$Y - 4 = (X - 8)\tfrac{1}{3},$$

i.e.

$$X - 3Y + 4 = 0;$$

or, using the ordinary notation, since x and y are no longer required for the point of contact, $\quad x - 3y + 4 = 0.$

(ii) *Find the equation of the tangent to the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *at any point* (x, y) *on the curve.*

Differentiating, $\quad \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0;$

$$\therefore \quad dy/dx = -b^2x/a^2y,$$

and the equation of the tangent is

$$Y - y = -(X - x)\frac{b^2x}{a^2y}.$$

i.e.

$$a^2yY - a^2y^2 = -b^2xX + b^2x^2$$
$$b^2xX + a^2yY = b^2x^2 + a^2y^2;$$

$\therefore$ dividing by a^2b^2, $\quad \dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

This is the required equation in its simplest form,

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1.$$

(iii) *Find the equation of the tangent to the circle*

$$x^2 + y^2 - 3x + 4y - 31 = 0$$

at the point $(-2, 3)$.

Differentiating the equation as it stands with respect to x,

$$2x + 2y\frac{dy}{dx} - 3 + 4\frac{dy}{dx} = 0.$$

$\therefore$ at the point $(-2, 3)$,

$$-4 + 6\frac{dy}{dx} - 3 + 4\frac{dy}{dx} = 0, \quad \text{whence } 10\frac{dy}{dx} = 7.$$

Hence the equation of the tangent is

$$y - 3 = (x + 2)\tfrac{7}{10},$$

i.e.

$$7x - 10y + 44 = 0.$$

The next two examples show how geometrical properties of a curve may be deduced.

(iv) *Find the equation of the tangent to the parabola* $y^2 = 4ax$ (p. 17) *at any point on the curve, and prove that if the tangent at* P (Fig. 45) *meets the axis in* T, *and* PN *be the ordinate of* P, *then* T *and* N *are equidistant from the vertex* A *of the parabola, i.e.* AT = AN.

Differentiating the equation $y^2 = 4ax$ with respect to x, we have

$$2y\,dy/dx = 4a, \text{ i. e. } dy/dx = 2a/y;$$

hence the equation of the tangent is

$$Y - y = (X - x)\,2a/y,$$

i.e. $$Yy - y^2 = 2aX - 2ax;$$

$$\therefore\ Yy = 2aX - 2ax + y^2 = 2aX - 2ax + 4ax = 2a(X + x).$$

This is the equation of the tangent PT.

Where this cuts the axis of x, $Y = 0$; $\quad\therefore\ 0 = 2a(X + x)$;

$$\therefore\ X, \text{ i. e. } AT = -x = -AN.$$

Hence A is the middle point of TN always.

(iv) *Find the equation of the tangent to the hyperbola* $xy = c^2$ (p. 21) *at any point on the curve, and show*

(*a*) *that the portion of the tangent between the asymptotes is bisected at the point of contact;*

(*b*) *that the tangent cuts off from the asymptotes a triangle of constant area.*

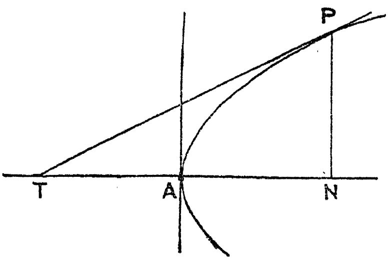


Fig. 45.

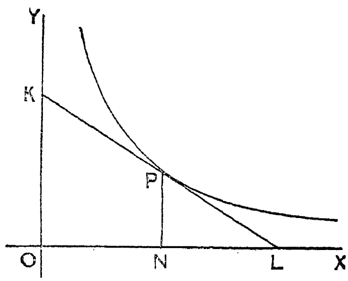


Fig. 46.

Differentiating the equation $xy = c^2$ with respect to x, we have

$$x\,dy/dx + y = 0, \text{ i. e. } dy/dx = -y/x.$$

$\therefore$ the equation of the tangent is

$$Y - y = -(X - x)\,y/x.$$

$$\therefore\ Xy + Yx = 2xy.$$

$\therefore$ dividing by xy, $\qquad X/x + Y/y = 2.$

Let the tangent at P (Fig. 46) cut the axes in L and K, and let PN be the ordinate of P.

Where the tangent cuts the axis of x, $Y = 0$; $\quad\therefore\ X/x = 2$,

i.e. $\quad X$ (which is OL) $= 2x$ (which is ON), so that $OL = 2\,ON$.

$\therefore\ KL = 2KP$, and P is the middle point of KL.

Again, where the tangent cuts the axis of y, $X=0$;

$$\therefore \quad Y/y=2, \text{ i.e. } Y \text{ (which is } OK) = 2y.$$

Now area of triangle $KOL = \frac{1}{2}\,OK \, . \, OL = \frac{1}{2} \, . \, 2y \, . \, 2x = 2xy = 2c^2$, which is constant for all positions of the point P.

47. Equation of normal to a curve at any point.

The normal at a point is the perpendicular to the tangent through the point of contact; its equation can be found in the same way as the equation of the tangent.

Let the normal at $P(x, y)$ meet the axis of x in G (Fig. 47), and let (X, Y) be the coordinates of any point Q on the normal. Draw PK perpendicular to the ordinate of Q.

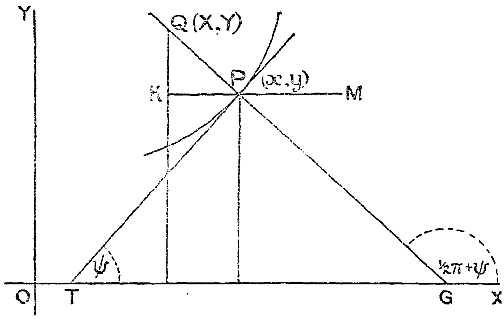


Fig. 47.

$$\text{Then} \quad \frac{Y-y}{X-x} = \frac{KQ}{PK} = \tan MPQ = \tan XGP$$

$$= \tan\left(\tfrac{1}{2}\pi+\psi\right) = -\cot\psi = -1 \Big/ \frac{dy}{dx};$$

$$\therefore \quad (Y-y)\frac{dy}{dx} = -(X-x).$$

Hence the equation of the normal at (x, y) is

$$X-x+(Y-y)\frac{dy}{dx}=0.$$

Examples:

(i) *Find the equation of the normal to the curve* $9x^2-4y^2=108$ *at the point* $(4, 3)$.

Differentiating with respect to x, $18x-8y\,dy/dx=0$;

$$dy/dx = 18x/8y = \text{(at the given point) } 72/24 = 3;$$

$\therefore$ the equation of the normal is $\qquad X-4+(Y-3)\,3=0$;

or, using the ordinary letters, now that x and y are no longer required to denote the coordinates of the point of contact,

$$x+3y=13.$$

There is of course no need to use the general formulae for the equations of the tangent and normal; in any particular example, the numerical value of dy/dx at the given point can be obtained as in Art. 45, and then by drawing a figure as in this or the preceding article, the required equation can be written down at once.

(ii) *Find the equation of the normal at any point of the ellipse* $x^2/a^2+y^2/b^2=1$, *and prove that if the normal at* P (Fig. 48) *meets the axis* CA *in* G, *and* PN *be the ordinate of* P, *then* $CG=e^2CN$, *where* e *is the eccentricity of the ellipse* (p. 19).

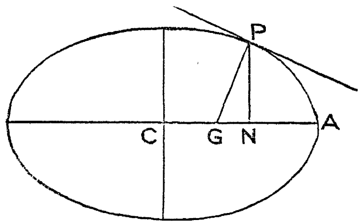


Fig. 48.

From Art. 46, Ex. (ii) $\quad dy/dx=-b^2x/a^2y$,

$\therefore$ equation of normal is

$$X-x-(Y-y)\frac{b^2x}{a^2y}=0,$$

i.e. $$Xa^2y-a^2xy-Yb^2x+b^2xy=0;$$

dividing by xy, $$\frac{a^2X}{x}-\frac{b^2Y}{y}=a^2-b^2.$$

This is the equation of the normal at any point (x, y).

Where this cuts the axis of x, i.e. at G, $Y=0$ and $X=CG$;

$$\therefore \quad a^2X/x=a^2-b^2,$$

and $$X=(a^2-b^2)\frac{x}{a^2}=\left(1-\frac{b^2}{a^2}\right)x=e^2x;$$

i.e. $$CG=e^2CN.$$

Examples XVI.

Find the inclinations to the axis of x of the tangents to the following curves:

1. $2y+7=x^3$ at $(3, 10)$. 2. $y=6x/(x^2-1)$ at $(2, 4)$.
3. $x^4+y^4=17$ at $(-2, 1)$. 4. $y=\sin^2 x$ at $(\frac{1}{3}\pi, \frac{3}{4})$.

Find the equations of the tangents and normals to the following curves:

5. $y = 2x^2 - 4x + 5$ at $(3, 11)$.
6. $y = 5x^2/(1+x^2)$ at $(2, 4)$.
7. $x^2 + y^2 = 20$ at $(-4, -2)$.
8. $\sqrt{x} + \sqrt{y} = 5$ at $(9, 4)$.
9. $2x^2 - xy + 3y^2 = 18$ at $(3, 1)$.
10. $x^2 + y^2 - 4x - 2y + 1 = 0$ at $(2, -1)$.
11. Find the equation of the tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at any point (x, y) on the curve.
12. Find the equation of the tangent to $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x, y) on the curve.
13. Find the points where the tangent to $y = x^3 - 12x + 4$ is parallel to the axis of x.
14. Find the points where the tangent to $y = a^2x/(a^2 + x^2)$ is parallel to the axis of x.
15. At what point of $y^2 + a^2 = ax$ will the tangent be inclined at 45° to the axis of x?
16. At what points of the circle $x^2 + y^2 = 25$ is the tangent parallel to the straight line $4x = 3y$?
17. Prove that the curves $y = x^2$ and $6y = 7 - x^3$ intersect at right angles at the point $(1, 1)$.
18. Find the angle of intersection of the curves $xy = 6$, $x^2y = 12$.
19. At what angle do the parabolas $y^2 = 8x$, $x^2 = 4y - 12$ intersect?
20. Find the angle at which the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 6x$ intersect.
21. Show that the ellipse $\frac{1}{25}x^2 + \frac{1}{9}y^2 = 1$ and the hyperbola $x^2 - y^2 = 8$ intersect at right angles.
22. Find the equation of the tangent at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$, and show that the portion of the tangent intercepted between the axes is of constant length.
23. Prove that the tangent at any point of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ makes on the axes two intercepts whose sum is constant.
24. Show that at not more than $n-1$ points can tangents to
$$y = ax^n + bx^{n-1} + \dots + k$$
be parallel to a given direction.
25. The tangent at any point P of the curve $y = x^3$ cuts the axis of x in T, and PN is the ordinate of P, prove that $OT = 2\,TN$.

    Find the corresponding result for the curve $y = x^n$.
26. Find the equation of the tangent at any point to $x^m y^n = a^{m+n}$, and prove that the portion of it intercepted between the axes is divided in the ratio $m : n$ at the point of contact.
27. Prove that the length of the tangent to the hyperbola $xy = c^2$ intercepted between the axes is twice the distance of the point of contact from the origin.
28. Find the equation of the tangent to the conic
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
at any point.
29. Find the equation of the tangent at any point to the curve
$$y(x^2 + y^2) = ax^2.$$
30. Find the forms of the following curves near the origin: $y = x^2/(1-x^2)$, $y = x/(1-x^2)$, $y^3 = x^2/(1-x^2)^2$.
31. Prove that, at the origin, the curve $y^2 = x^3$ touches the axis of x, $y^2 = x(x-1)(x-2)$ touches the axis of y, and $y^2 = x^2(1-x^2)$ bisects the angle between the axes.

32. Prove that the curve $y^m = x^n/(1+x)$ touches the axis of x or the axis of y at the origin according as $m <$ or $> n$. What happens if $m = n$?

33. If the tangent at a point P of an ellipse meet the axes CA and CB in T and t, and if PN, PM be perpendiculars to these axes respectively, show that $CN \,.\, CT = a^2$; $CM \,.\, Ct = b^2$.

34. Find, in terms of x, y, and dy/dx, the inclination of the tangent at any point P of a curve to the straight line joining P to the origin.

35. The tangent at any point P of a curve meets the axes of x and y in T and T', and the normal at P meets them in N and N' respectively; prove that $T'N/TN' = dy/dx$.

48. Lengths of tangent, normal, subtangent, and subnormal.

If the tangent and normal at a point P (Fig. 49) of a curve meet the axis of x in T and G respectively, and if PN be the ordinate of P, then NT and NG are called the *subtangent* and *subnormal*

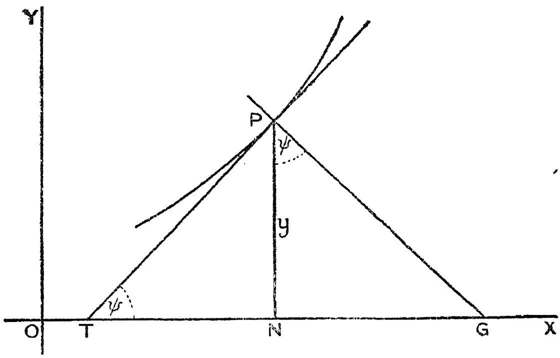


Fig. 49.

respectively, and the lengths of PT and PG are called the lengths of the tangent and normal respectively.

All these lengths can, on drawing a figure, be at once written down in terms of y and dy/dx. For $\angle GPN = \angle PTG = \psi$, and $NP = y$;

hence the subnormal $NG = y \tan\psi = y\, dy/dx$;

$$\text{the subtangent } NT = y \cot\psi = y\Big/\frac{dy}{dx};$$

$$\text{the normal } PG \quad = y \sec\psi = y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}};$$

$$\text{the tangent } PT \quad = y \operatorname{cosec}\psi = y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\Big/\frac{dy}{dx}.$$

The student should not attempt to remember these results, but should draw a figure, and obtain from it as above the particular results he requires.

Examples:

(i) *Prove that, in the parabola, the subnormal is constant.*

The simplest form of the equation of a parabola is $y^2 = 4ax$; differentiating with respect to x, $\quad 2y\,dy/dx = 4a$;

$$\therefore \text{ the subnormal } y\,dy/dx = 2a,$$

i.e. if in Fig. 10 the normal at P be drawn to meet the axis in G,

$$NG = 2a = 2AS = \tfrac{1}{2} \text{ the latus rectum (see Ex. II, 20).}$$

(ii) *The tangent at any point* P *of the curve* $y = x^n$ *cuts the axis of* x *in* T, *and* PN *is the ordinate of* P (Fig. 50); *prove that* OT = (n − 1) TN.

Here the subtangent

$$TN = y\cot\psi = y\Big/\frac{dy}{dx} = \frac{x^n}{nx^{n-1}} = \frac{x}{n} = \frac{1}{n}\,.\,ON,$$

whence
$$OT = \frac{n-1}{n}\,.\,ON = (n-1)\,TN.$$

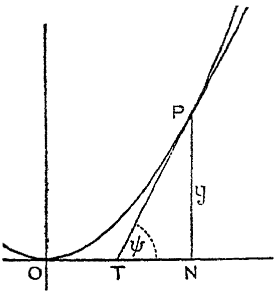


Fig. 50.

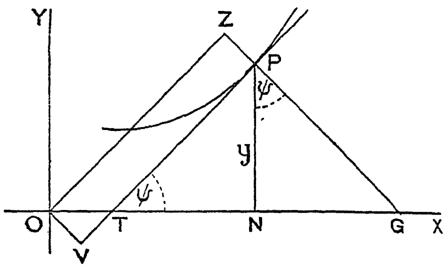


Fig. 51.

49. Further properties of curves.

The lengths of many other lines connected with a curve can be obtained in a similar manner. First the length of the line is obtained from the figure in terms of x, y, and ψ as in the preceding article, and then from the fact that $\tan\psi = dy/dx$, the value of any other ratio of ψ can be obtained in terms of y and dy/dx by elementary trigonometry.

For instance, suppose the lengths of the perpendiculars from the origin to the tangent and normal are required. Let OV, OZ (Fig. 51) be perpendiculars from the origin O to the tangent PT and normal PG.

Then $\quad OV = OT\sin\psi = (ON - TN)\sin\psi = (x - y\cot\psi)\sin\psi$

$$= \left(x - \frac{y}{\dfrac{dy}{dx}}\right)\frac{\dfrac{dy}{dx}}{\sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}} \quad \frac{x\dfrac{dy}{dx} - y}{\sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}}$$

And $\quad OZ = OG\cos\psi = (ON+NG)\cos\psi$

$$= (x+y\tan\psi)\cos\psi = \frac{x+y\dfrac{dy}{dx}}{\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}}$$

In particular cases, it is best not to use these general formulae, but to draw the curve roughly and work out each case from the figure.

Two examples of a rather more difficult nature than those already given are here worked out:

Ex. (i) *In the curve* $x^{2/3}+y^{2/3}=a^{2/3}$, *find the lengths of the perpendiculars from the origin to the tangent and normal, and if* V *be the foot of the perpendicular from the origin* O *to the tangent at* P, *prove that the locus of the middle point* Q *of* PV *is a circle.*

This curve is a very well known one, and on account of its shape, is named the 'astroid'.

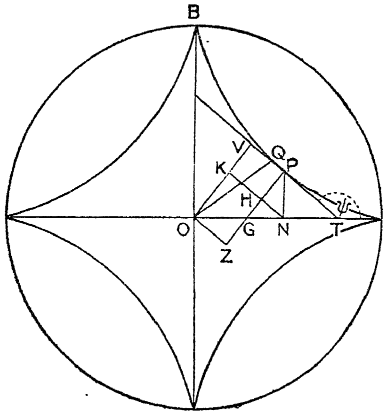


Fig. 52.

Differentiating its equation with respect to x,

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0 \quad\text{or}\quad \frac{dy}{dx}=-\frac{y^{1/3}}{x^{1/3}}.$$

This is $\tan\psi$, $\quad\therefore\quad \tan PTN = -\tan\psi = y^{1/3}/x^{1/3}$.

$$\therefore\quad \sin PTN = \frac{y^{1/3}}{\sqrt{(x^{2/3}+y^{2/3})}} = \frac{y^{1/3}}{a^{1/3}},\quad\text{and}\quad \cos PTN = \frac{x^{1/3}}{a^{1/3}}.$$

Draw NK perpendicular to OV, cutting PG in H.

Then $\quad OV = OK+HP = ON\sin ONK+PN\cos HPN$

$$= x\,.\frac{y^{1/3}}{a^{1/3}}+y\frac{x^{1/3}}{a^{1/3}}$$

$$= \frac{x^{1/3}y^{1/3}}{a^{1/3}}(x^{2/3}+y^{2/3}) = \frac{x^{1/3}y^{1/3}}{a^{1/3}}\,.\,a^{2/3} = (axy)^{1/3}$$

Similarly $\quad OZ = KN - HN = ON \cos ONK - PN \sin NPH$

$$= x \cdot \frac{x^{1/3}}{a^{1/3}} - y \cdot \frac{y^{1/3}}{a^{1/3}}$$

$$= \frac{x^{4/3} - y^{4/3}}{a^{1/3}} = \frac{(x^{2/3} - y^{2/3})(x^{2/3} + y^{2/3})}{a^{1/3}}$$

$$= a^{1/3}(x^{2/3} - y^{2/3}).$$

These are the lengths of the perpendiculars from O to the tangent and normal at P in terms of the coordinates of P.

Next, if the locus of Q is a circle, it is evident from symmetry that O must be its centre. Therefore, finding the length of OQ,

$$OQ^2 = OV^2 + VQ^2 = OV^2 + \tfrac{1}{4}OZ^2$$

$$= (axy)^{2/3} + \tfrac{1}{4}a^{2/3}(x^{4/3} + y^{4/3} - 2x^{2/3}y^{2/3})$$

$$= \tfrac{1}{4}a^{2/3}[4x^{2/3}y^{2/3} + x^{4/3} + y^{4/3} - 2x^{2/3}y^{2/3}]$$

$$= \tfrac{1}{4}a^{2/3}[x^{2/3} + y^{2/3}]^2 = \tfrac{1}{4}a^{2/3} \times a^{4/3} = \tfrac{1}{4}a^2.$$

Hence $OQ = \frac{1}{2}a$, which is constant, so that the locus of Q is a circle, centre O and radius $\frac{1}{2}a$.

Ex. (ii) *Find the condition that the curves*

$$x^2/a^2 + y^2/b^2 = 1 \quad \textit{and} \quad x^2/a'^2 + y^2/b'^2 = 1$$

may cut at right angles.

The value of dy/dx for the first curve is given by the equation

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0, \quad \text{i.e.} \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y},$$

similarly for the second curve

$$\frac{dy}{dx} = -\frac{b'^2x}{a'^2y}.$$

The curves cut orthogonally, i. e. the tangents at their points of intersection are at right angles, therefore the angles which these tangents make with the axis of x differ by 90°, and the tangent of one = − the cotangent of the other;

$$\frac{b^2x}{a^2y} = -\frac{a'^2y}{b'^2x}, \quad \text{i.e.} \quad x^2 \qquad y^2 \qquad \text{(i)}$$

At the points of intersection, both equations are satisfied;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{}{\ ^2} + \frac{y^2}{b'^2}$$

$$\left(\frac{1}{a^2} - \frac{1}{a'^2}\right) = -y^2\left(\frac{1}{b^2} - \frac{1}{b'^2}\right)$$

$$-\frac{y^2(b'^2 - b^2)}{b^2b'^2};$$

∴ substituting the result of equation (i), $a'^2 - a^2 = b'^2 - b^2$.

Hence the required condition is $a^2 - b^2 = a'^2 - b'^2$.

50. Expression of coordinates x and y in terms of a third variable. The Cycloid.

In many cases, instead of finding the equation of a curve as an algebraical relation between x and y, it is more convenient to express both x and y as functions of some third variable; the equation connecting x and y can then be obtained, if required, by eliminating this third variable from the two equations given.

For instance $x = a\cos\theta$, $y = b\sin\theta$ are the coordinates of any point of an ellipse whose semi-axes are of lengths a and b. Whatever value be assigned to θ, the point $(a\cos\theta, b\sin\theta)$ is always on the ellipse, and the ordinary equation of the ellipse is found by eliminating θ; for $x/a = \cos\theta$, $y/b = \sin\theta$, and therefore squaring and adding, $x^2/a^2 + y^2/b^2 = 1$.

As a particular case, $x = a\cos\theta$, $y = a\sin\theta$, are general expressions for the coordinates of any point on a circle, radius a and centre the origin.

Similarly, $x = am^2$, $y = 2am$, where m is variable, denote the coordinates of any point on the parabola $y^2 = 4ax$; for, eliminating m, we have

$$x/a = m^2 = y^2/4a^2; \quad \therefore y^2 = 4ax;$$

so that, whatever the value of m, the point is on the parabola.

It is often of advantage to use these forms of the coordinates in investigating properties of conics.*

Again, in the 'astroid' mentioned in the preceding article, if $x = a\cos^3\phi$, we obtain, on substituting this in the equation $x^{2/3} + y^{2/3} = a^{2/3}$, $y = a\sin^3\phi$. Hence the coordinates of any point on this curve are given by the equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

In these examples, the equation between x and y is quite simple, but in some cases, although the equations which give x and y in terms of the third variable are simple, the equation between x and y obtained by elimination is very complicated and most inconvenient to work with.

A good example of this is furnished by the well-known curve called the 'cycloid'.

The cycloid. A cycloid is the locus of a point on the circumference of a circle which rolls (without sliding) along a fixed straight line; its equations are obtained at once from a figure.

* For the hyperbola, see Ex. XVII. 18.

Let a circle, centre C (Fig. 53) and radius a, roll along a straight line OX; let P be the position of the tracing point when the radius CP has turned through an angle θ, starting from the position in which P coincides with O. Therefore the arc NP = the straight line NO.

If (x, y) denote the coordinates of P, referred to O as origin and OX as axis of x, then

$$x = ON - PM = \text{arc } PN - PC \sin\theta$$
$$= a\theta - a\sin\theta = a(\theta - \sin\theta);$$
$$y = NC - MC = a - a\cos\theta = a(1 - \cos\theta).$$

These two equations constitute the most convenient form of the equation of a cycloid.*

In cases such as this, since x and y are both continuous functions of θ, a small increase $\delta\theta$ in θ will produce small increases δx and δy in x and y.

It is evident that $$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta\theta} \Big/ \frac{\delta x}{\delta\theta}.$$

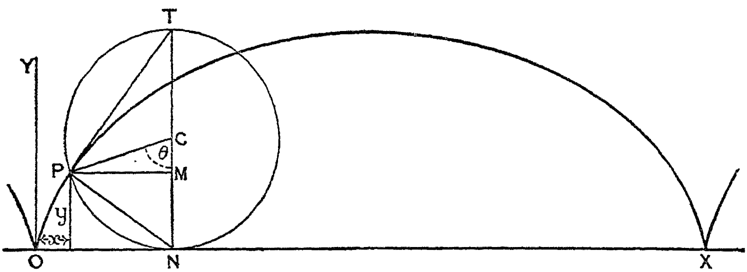


Fig. 53.

Hence, by Art. 15 (iii), when $\delta\theta$ and therefore also δx and $\delta y \to 0$, we have

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}.$$

In the case of the cycloid, this gives

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \cot\tfrac{1}{2}\theta.$$

* The student should eliminate θ and obtain the Cartesian equation in order to see how complicated and inconvenient an equation it is.

Referring to Fig. 53, $\frac{1}{2}\theta = \frac{1}{2}\angle PCN = \angle PTN$;

$\therefore$ $dy/dx = \cot PTN =$ tangent of angle which PT makes with the axis of x, from which it follows that PT is the tangent to the cycloid at P, and PN, being perpendicular to it, is the normal at P.

This follows at once from the definition of the curve, for, as P traces out the curve, its motion is for an instant one of rotation about N, i. e. in direction perpendicular to NP, i.e. along PT, since the angle NPT in a semicircle is a right angle. Hence PT is the tangent at P, and PN the normal at P.

Examples XVII.

Find the lengths of the tangent, normal, subtangent, and subnormal in the following cases:

1. $y^2 = 4(x+5)$ at $(4, 6)$.
2. $y = a\sin(x/b)$ at $(\frac{1}{6}\pi b, \frac{1}{2}a)$.
3. $\frac{1}{16}x^2 + \frac{1}{9}y^2 = 8$ at $(8, 6)$.
4. $x^2+y^2-6x-2y+5 = 0$ at $(2, -1)$.
5. Prove that the subnormal at any point of the curve $x^2-y^2 = a^2$ is equal to the abscissa.
6. In the curve $xy = c^2$, prove that the subnormal varies as the cube of the ordinate.
7. Show that, in the parabola $y^2 = 4ax$, the subtangent varies as the square of the ordinate.
8. Prove that, in the curve $y^{n+1} = a^n x$, the subtangent varies as the abscissa, and find the subnormal.
9. Show that, in the curve $ay^2 = (x+b)^3$, the subnormal varies as the square of the subtangent.
10. Prove that, in the curve $ax^2+by^2 = c$, the subnormal bears a constant ratio to the abscissa.
11. Find the subtangent, at the point where $x = a$, in the curve
$$ay^2 = (a+x)^2(3a-x).$$
12. Find the subtangent and subnormal at any point of the ellipse
$$x^2/a^2 + y^2/b^2 = 1,$$
and prove that the subtangent is the same (at the point with the same abscissa) as in the circle on the major axis of the ellipse as diameter.
13. In a certain well-known curve (called the tractrix), the slope at any point (x, y) on the curve is equal to $-y/\sqrt{(a^2-y^2)}$; prove that the length of the tangent is constant.
14. Prove that $x = a\cos^3\theta$, $y = a\sin^3\theta$ are the coordinates of a point on the astroid $x^{2/3}+y^{2/3} = a^{2/3}$, and find, in terms of θ, the equation of the tangent at any point.
15. Find, in terms of θ, the lengths of the tangent, normal, subtangent, and subnormal at any point of the astroid.
16. Find the equation of the tangent to the cycloid (i) when $\theta = \frac{1}{2}\pi$, (ii) for any value of θ.
17. Find the lengths of the subtangent and subnormal at the point on a cycloid where $\theta = \frac{1}{2}\pi$.
18. Prove that $x = a\sec\theta$, $y = b\tan\theta$ are the coordinates of a point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, and find the value of dy/dx in terms of θ.

19. Find the equation of the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a \cos \theta, b \sin \theta)$.

20. If the coordinates of a point on the parabola $y^2 = 4ax$ be taken as $(am^2, 2am)$, what is the geometrical meaning of m?

21. Find the lengths of the subtangent and subnormal at any point of the cardioid, given by $x = a(2 \cos \theta + \cos 2\theta)$, $y = a(2 \sin \theta + \sin 2\theta)$.

22. Find, in terms of y and dy/dx, the lengths of the perpendiculars from the foot of the ordinate to the tangent and normal at any point of a curve.

23. Find the length of the perpendicular OY from the origin to the tangent at a point P of the hyperbola $xy = c^2$, and show that the rectangle $OY \cdot OP$ is constant.

24. Prove that, if a gas obeys Boyle's law $pv = k$, the cubical elasticity (Art. 38) is represented by TM, where T is the point in which the tangent to the curve $pv = k$ at the point P cuts the axis of p, and PM is perpendicular to that axis.

CHAPTER VI

MAXIMA AND MINIMA

51. Definition of maxima and minima.

We shall now show how to find the maximum and minimum values of a function of one variable, confining ourselves to cases where the function and its differential coefficient are continuous.

If a continuous function increases up to a certain value and then begins to decrease, that value is called a *maximum* value of the function; similarly, if the function decreases to a certain value and then begins to increase, that value is called a *minimum* value of the function; in other words, a maximum value is one which

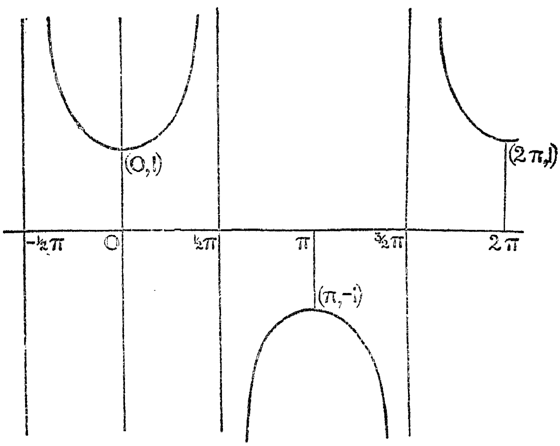


Fig. 54.

is greater and a minimum value is one which is less than all other values in the immediate neighbourhood on either side.

According to this definition, a function may have any number of maxima and minima; and a maximum value is not necessarily the greatest nor a minimum value the least of all the values of the function; in fact it is quite possible for some or even all of the maxima to be less than some or all of the minima.

This is illustrated by the function $\sec x$. As x increases from $-\frac{1}{2}\pi$ to 0, $\sec x$ decreases from ∞ to 1; as x increases from 0 to $\frac{1}{2}\pi$, $\sec x$ increases from 1 to ∞. Therefore $\sec x$ has the minimum value 1

when $x = 0$. When $x = \frac{1}{2}\pi$, $\sec x$ is discontinuous. As x increases from $\frac{1}{2}\pi$ to π, $\sec x$ increases from $-\infty$ to -1; as x increases from π to $\frac{3}{2}\pi$, $\sec x$ decreases from -1 to $-\infty$. Therefore $\sec x$ has the maximum value -1 when $x = \pi$. When $x = \frac{3}{2}\pi$, $\sec x$ is discontinuous.

These variations are repeated an indefinite number of times, and the variations begin to recur after x has increased by 2π or any multiple of 2π. (This is expressed by the statement that $\sec x$ is a *periodic function of* x, and its *period* is 2π.) Therefore $\sec x$ has an infinite number of minima, each $+1$, and an infinite number of maxima, each -1, and the minima are greater than the maxima (Fig. 54).

52. **Alternate maxima and minima.**

It is evident that, in a function which is always continuous, maxima and minima must occur alternately; because after any maximum the function is decreasing, and before the next maximum it is increasing, therefore, if it is continuous, there must be some intermediate point where the function ceases to decrease and begins to increase; such a point is a minimum. Hence between any two consecutive maxima there is a minimum, and similarly between any two consecutive minima there is a maximum.

The *circular functions* furnish good illustrations of these definitions and ideas. $\operatorname{Sec} x$ has been considered in the preceding article, and $\operatorname{cosec} x$ may be used to illustrate the same points.

$\operatorname{Sin} x$ and $\cos x$ are always continuous; both have an infinite number of maxima, each $+1$, and an infinite number of minima, each -1, occurring alternately at intervals of π in the value of x. ($\operatorname{Sin} x$ and $\cos x$ are periodic functions whose period is 2π.)

$\operatorname{Tan} x$ and $\cot x$ have no maxima or minima. As x increases from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, $\tan x$ increases from $-\infty$ to $+\infty$; when $x = \frac{1}{2}\pi$, $\tan x$ is discontinuous; and as x increases from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, $\tan x$ again increases from $-\infty$ to $+\infty$, and so on. There is therefore no value of x at which $\tan x$ ceases to increase and begins to decrease. Similarly for $\cot x$. (The variations in the values of $\tan x$ and $\cot x$ begin to recur after intervals of π; therefore $\tan x$ and $\cot x$ are periodic functions whose period is π, not 2π, as in the case of the other circular functions.)

53. **Conditions for a maximum or minimum.**

It has been pointed out (Art. 25) that the differential coefficient of a function $f(x)$ is $+$ or $-$ according as the function increases or decreases as x increases.

Just before a max.,	$f(x)$ is increasing as x increases,	∴ its d. c. is +.
„ after „ „	„ „ decreasing „ „ „	∴ „ „ „ −.
„ before a min.,	„ „ decreasing „ „	„ „ −.
„ after „ „	„ „ increasing „ „ „	∴ „ „ +.

Hence, in passing through a maximum or a minimum value, the d. c. of the function must change sign, and therefore, at the maximum or minimum, the d. c., if continuous, must equal zero (Art. 17 (4)).

Hence *a value of* y *is a maximum or a minimum value when* dy/dx *is equal to zero and changes sign* as y passes through that value.

If dy/dx *changes from* + *to* −, *the value is a maximum.*
If dy/dx *changes from* − *to* +, *the value is a minimum.*

Notice that the condition $dy/dx = 0$ alone is not a sufficient condition for a maximum or minimum; y may increase up to a certain value ($dy/dx +$), remain constant for an instant ($dy/dx = 0$), and then begin to increase again (dy/dx again +); dy/dx in this case does not change sign, and the value for which $dy/dx = 0$ is not a maximum.

54. Geometrical treatment of maxima and minima.

All these results follow at once from geometrical considerations.

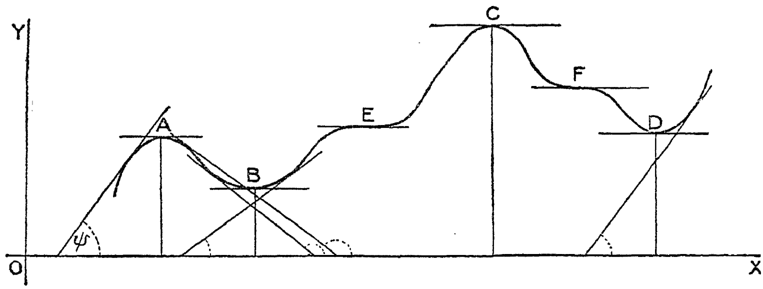


Fig. 55.

In the curve shown in Fig. 55, the ordinates at A and C represent maximum values of the function, and the ordinates at B and D represent minimum values. If the tangent at a point (x, y) of the curve make an angle ψ with the positive direction of the axis of x, $dy/dx = \tan\psi$. At A, B, C, D the tangents are clearly parallel to the axis of x; therefore $\psi = 0$ and $\tan\psi = 0$, i. e. $dy/dx = 0$.

Just before A or C,	ψ is acute,	∴ $\tan\psi$ is +,	i.e. dy/dx is +.
„ after „ „ „	„ „ obtuse,	„ „ „ „ −,	„ „ „ −.
„ before B or D,	„ „ obtuse,	„ „ „ „ −,	„ „ −.
„ after „ „ „	„ „ acute,	„ „ „ „ +,	„ „ +.

Hence in passing through a maximum, dy/dx changes from + to −, and in passing through a minimum, from − to +.

But A, B, C, D are not the only points where the tangent is parallel to the axis of x; at such points as E and F, the tangent is parallel to OX and therefore $dy/dx = 0$, but in passing through these points dy/dx does not change sign.

Just before and after E, ψ is acute, dy/dx is + in both cases.
Just before and after F, ψ is obtuse, dy/dx is − in both cases.

The points E and F are called *points of inflexion*, and such points will be considered more fully later on (Art. 59).

All points where $dy/dx = 0$ are included in the term *stationary points*, because the rate of change of the function at such points is zero. They include, as we have just seen, maxima, minima, and those points of inflexion at which the tangent is parallel to the axis of x. A curve may have points of inflexion where the tangent is not parallel to the axis of x; at such points, of course dy/dx is not zero.

It is possible for a function to have maxima and minima of a different nature from those indicated above, e. g. at points such as A, B, C, in Fig. 56. The ordinates at A and C are maxima, and the ordinate at B is a minimum according to the definition of Art. 51. At such points as these, y is continuous, but dy/dx is discontinuous; at A, it is infinite, the tangent being perpendicular to the axis of x, and therefore $\tan\psi = \infty$; at B and C, dy/dx suddenly changes by a finite amount as the tangent passes from one side of the point to the other. [In these cases, the condition that dy/dx changes sign in passing through the point is fulfilled; in passing through A and C, dy/dx change from + to −, and in passing through B, from − to +.] Such points do not occur in the functions which are encountered in elementary examples.

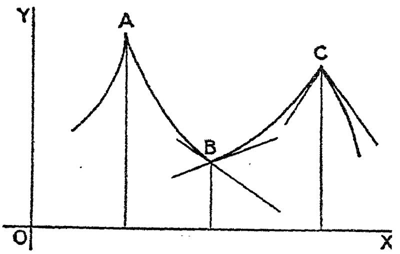


Fig. 56.

It is evident that the determination of the maximum and minimum values of a function, the 'turning-values' as they are often called, is of great assistance in drawing the graph of the function.

55. Examples.

We will now apply these principles to a few algebraical and trigonometrical examples.

(i) *Find the maximum and minimum values of* x^3-9x^2+15x, *and draw roughly the graph of the function.*

Here $\qquad dy/dx = 3x^2-18x+15 = 3(x-1)(x-5)$;

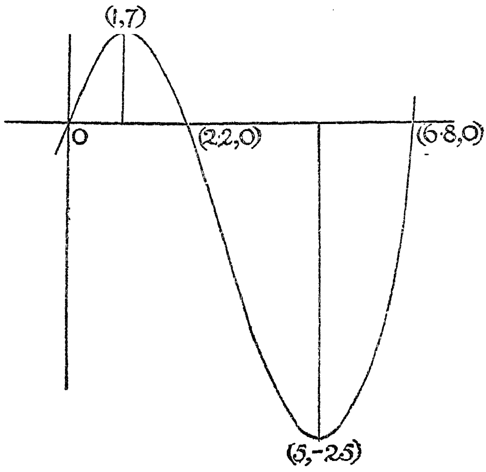
$\therefore \quad dy/dx = 0$ when $x=1$ and when $x=5$.

To find whether and how dy/dx changes sign as x passes through these values, it is best to start below the smallest value and trace the changes in the sign of dy/dx as x increases through each value in turn.

If x is slightly <1,	the first factor is $-$,	and the second $-$,		dy/dx is $+$
If x is slightly >1,	,, ,, $+$,	,, ,, $-$,	$\therefore$	dy/dx is $-$
If x is slightly <5,	,, ,, $+$,	,, ,, $-$,	$\therefore$	dy/dx is $-$
If x is slightly >5,	,, ,, $+$,	,, ,, $+$,	$\therefore$	dy/dx is $+$

Therefore dy/dx changes from $+$ to $-$ as x increases through the value 1, and from $-$ to $+$ as x increases through the value 5; hence y is a maximum when $x=1$, and is then equal to 7, and a minimum when $x=5$, and is then equal to -25.

Moreover, the graph goes through the origin since $y=0$ when $x=0$, and it cuts the axis of x where $y=0$. $\therefore\ x^3-9x^2+15x=0$, i.e. $x(x^2-9x+15)=0$, whence $x=0$, $2{\cdot}2$, $6{\cdot}8$ nearly.



g. 57.

Therefore the graph is roughly as shown in Fig. 57. Clearly no finite value of x can make y infinite, and after passing the point $(5, -25)$, y must continually increase and the graph rise; for if it ever descended again, there would be another maximum, since the function is always continuous. Similarly, it must continually ascend from $-\infty$ to the point $(1, 7)$.

(ii) $y = x^4 - 6x^2 + 8x + 10.$

$$dy/dx = 4x^3 - 12x + 8 = 4(x-1)^2(x+2),$$

$$\therefore \quad dy/dx = 0 \text{ when } x = 1 \text{ and when } x = -2.$$

To find the change of sign, starting below the smaller value,

if x is slightly < -2,	the signs of the factors are	$+, -,$	$\therefore$ dy/dx is $-$
if x is slightly > -2,	,, ,, ,,	$+, +,$	$\therefore$ dy/dx is $+$
if x is slightly < 1,	,, ,, ,,	$+, +,$	dy/dx is $+$
if x is slightly > 1,	,, ,, ,,	$+, +,$	dy/dx is $+$

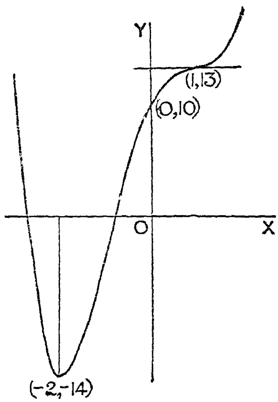


Fig. 58.

Therefore dy/dx changes from $-$ to $+$ as x increases through -2, and does not change sign as x increases through $+1$; hence y is a minimum when $x = -2$, and is then equal to -14; and there is a point of inflexion when $x = 1$, and y is then equal to 13.

The curve cuts the axis of y where $x = 0$, and therefore $y = 10$; it is shown roughly in Fig. 58. It must continually descend from ∞ to $(-2, -14)$, and continually ascend from $(1, 13)$ to ∞. There is no maximum.

(iii) $y = \dfrac{x^2 - 4x + 9}{x^2 + 4x + 9}.$

$$\frac{dy}{dx} = \frac{(x^2+4x+9)(2x-4) - (x^2-4x+9)(2x+4)}{(x^2+4x+9)^2},$$

which reduces to $$\frac{8(x^2-9)}{(x^2+4x+9)^2};$$

$$\therefore \quad dy/dx = 0 \text{ when } x = \pm 3.$$

If x is slightly < -3,	the num. is $+$,	and the denom. is	$+$	dy/dx is $+$
If x is slightly > -3,	,, ,, $-$,	,, ,,	$+$	dy/dx is $-$
If x is slightly < 3,	,, ,, $-$,	,, ,,	$+$	dy/dx is $-$
If x is slightly > 3,	,, ,, $+$,	,, ,,	$+$,	$\therefore$ dy/dx is $+$

Therefore y is a maximum when $x = -3$, and a minimum when $x = +3$.

When $x = -3$, $y = \frac{30}{6} = 5$, and when $x = +3$, $y = \frac{6}{30} = \frac{1}{5}$.

When $x = 0$, $y = 1$; and by writing the equation in the form $y = \dfrac{1 - 12/x + 9/x^2}{1 + 12/x + 9/x^2}$ (see Art. 13 (7)), we see that, as $x \to \infty$, y approaches the limit 1. Therefore $y = 1$ is an asymptote.

The general trend of the graph is therefore as shown in Fig. 59.

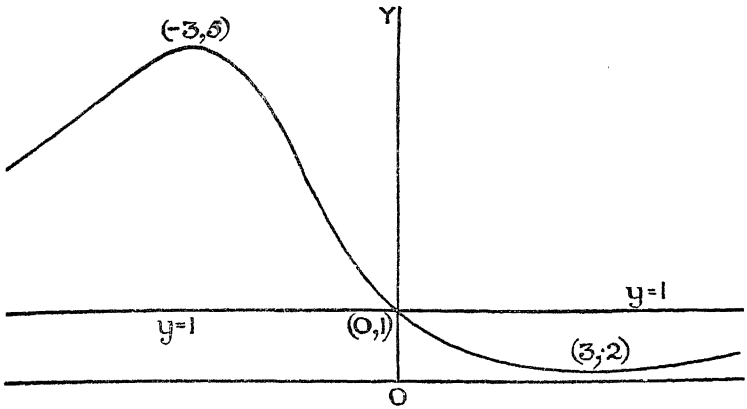


Fig. 59.

(iv) $y = a \sin\theta + b\cos\theta$.

$$dy/d\theta = a\cos\theta - b\sin\theta;$$

$\therefore$ $dy/d\theta = 0$ when $b\sin\theta = a\cos\theta$, i.e. when $\tan\theta = a/b$, and then $\sin\theta = \pm a/\sqrt{(a^2+b^2)}$, $\cos\theta = \pm b/\sqrt{(a^2+b^2)}$, both signs being + or both −, since $\tan\theta$ is +.

Therefore the maximum and minimum values of y are

$$a \times \frac{\pm a}{\sqrt{(a^2+b^2)}} + b \times \frac{\pm b}{\sqrt{(a^2+b^2)}}, \text{ i.e. } \pm\sqrt{(a^2+b^2)}.$$

Since y is always continuous the greater value is the maximum; therefore y has an infinite number of maxima, each $+\sqrt{(a^2+b^2)}$,

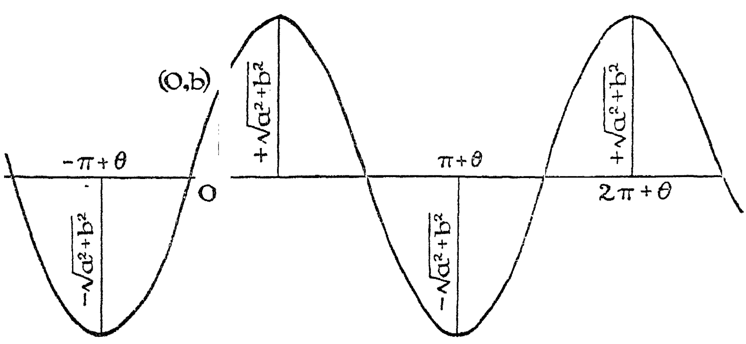


Fig. 60.

and an infinite number of minima, each $-\sqrt{(a^2+b^2)}$, occurring alternately at points where $\tan\theta = a/b$. Since

$$\tan(n\pi+\theta) = \tan\theta = a/b,$$

the turning-points occur at intervals of π in the value of θ (Fig. 60).

Examples XVIII.

Find the stationary points of the following functions 1–36, and discriminate between them. Also, draw roughly the graphs of the functions 1–20.

1. x^2-6x+8.
2. $16-6x-3x^2$.
3. $x^3-12x+5$.
4. $2x^3-15x^2+36x$.
5. $x^3+3x^2+20x-10$.
6. $x^3-9x^2+15x+11$.
7. x^3-3x^2+3x-1.
8. $x^4-8x^3+10x^2+40$.
9. $x^4-8x^3+22x^2-24x+12$.
10. $x^5-5x^4+5x^3-1$.
11. $x^4-2x^3+2x^2-6x+3$.
12. $(x-1)^2(x-2)$.
13. $(x-1)^3(x-2)^2$.
14. $(3x-3)^2/(x+1)^3$.
15. $\dfrac{x^2-2x+4}{x^2+2x+4}$.
16. $\dfrac{-x^2}{9+x^2}$.
17. $\dfrac{(4-x)^3}{2-x}$.
18. $\dfrac{}{(x+8)(x+2)}$.
19. $(x-2)(6-x)/x^2$.
20. $(x+a)(x+b)/x$.
21. $(x-3)^{1/3}(x-6)^{2/3}$.
22. $x\sqrt{(ax-x^2)}$.
23. $\dfrac{a^2}{x}+\dfrac{a^2}{4(a-x)}$.
24. $\sqrt{(x/a+a/x)}$.
25. $(x-3)\sqrt{(1+x^2)}$.
26. $\sin x+\cos x$.
27. $a\sin^2 x+b\cos^2 x$.
28. $\sin 2x-x$.
29. $4x+\tan 3x$.
30. $\cos 2x+\sin x$.
31. $\tan^2 x-2\tan x$.
32. $a\cot x+b\tan x$.
33. $\sin^3 x\cos x$.
34. $\sin(x-\alpha)\cos(x-\beta)$.
35. $\sin x/(1+\tan x)$.
36. $\tan x-8\sin x$.
37. Prove that $(x-a_1)^2+(x-a_2)^2+\ldots+(x-a_n)^2$ is a minimum when x is the arithmetic mean of $a_1, a_2, \ldots a_n$.
38. The bending moment of a beam of length l, at a distance x from one end, is equal to $\frac{1}{2}wlx-\frac{1}{2}wx^2$, where w is the (uniform) load per unit length; prove that the maximum bending moment is at the centre.
39. The force exerted by a circular electric current of radius a on a small magnet whose axis coincides with the axis of the circle varies as $x/(a^2+x^2)^{5/2}$, where x is the distance from the plane of the circuit. Find when the force is a maximum.
40. The total waste per mile in an electric conductor is equal to C^2r+A/r, where C is the current in amperes, r the resistance in ohms per mile, and A a constant; for what value of r will the waste be a minimum?
41. Prove that $\sqrt{\{(q^2-n^2)^2+4f^2n^2\}}$ (where q and f are constants) is least when $n^2=q^2-2f^2$.
42. Find the minimum value of C^2R+289/R [C constant].
43. The velocity of certain chemical reactions follows the law
$$v=k(b+x)(a-x);$$
when is the velocity a maximum?
44. Find where the width of the loop of the curve in Art. 9, Ex. vii is greatest. [Find when y^2 (not y) is a maximum.]
45. The curve $y^2=x^2(a^2-x^2)$ consists of two loops; find where their width perpendicular to the axis of x is greatest.
46. Find the maximum ordinate of the curve $y=(x-1)^2(5-2x)$.

47. When is the ratio of an integer to the square of the integer next above it a maximum or minimum?

48. Find when $x^{2/\gamma}-x^{1+1/\gamma}$ is a maximum. What is the maximum value if $\gamma = 1\cdot 4$?

49. The current sent through a resistance R by a battery consisting of a fixed number n of cells, each of voltage E and internal resistance r, arranged with x cells in series and n/x rows in parallel, is

$$nxE/(x^2r+nR) \text{ amperes.}$$

How many cells must be in series in order to give the maximum current?

50. If $y/R = (l-x)/x$, find the percentage error in y due to a given small error α in the value of x. For what value of x will the percentage error be least?

56. Problems on maxima and minima.

A large number of very interesting problems on maxima and minima can be solved by the aid of the foregoing principles. A few typical examples will be worked out.

In the first place, it frequently happens that the quantity whose maximum or minimum is required appears, when first expressed in symbols, as a function of more than one variable. It must be carefully borne in mind that the next step is to express it as a function of *one* of these variables only. By means of geometrical or other given relations between the variables, all but one of these variables must be eliminated. Having thus expressed the quantity as a function of a single variable, we proceed exactly as in the algebraical examples just considered. We differentiate with respect to the variable; and the values which make the differential coefficient vanish include the values which make the quantity a maximum or minimum. In many cases it is not necessary to examine the change of sign as was done in the preceding examples; it is often easy to see at once whether the solution be a maximum or minimum, as will be indicated in some of the examples which follow.

Examples:

(i) *Find the rectangle of given area which has the shortest diagonal.*

If x and y be the lengths of the sides, the length of the diagonal is $\sqrt{(x^2+y^2)}$.

It will evidently serve to find when the square on the diagonal is a minimum; the differentiation is then simpler.

x and y are connected by the relation $xy = A$, the given area; and therefore, eliminating y, the square on the diagonal $= x^2+A^2/x^2$.

The d. c. of this is $2x-2A^2/x^3$, which is equal to zero when $2x = 2A^2/x^3$, i. e. when $x^4 = A^2$.

$\therefore\ x^2 = A$ (since x^2 is necessarily $+$) $= xy$.

$\therefore\ x = y$ (since $x = 0$ is not admissible) and the figure is a square.

If $x^2 < A$, then $x < A^2/x^3$ and the d. c. is $-$;
if $x^2 > A$, then $x > A^2/x^3$ and the d. c. is $+$.

Therefore the solution is a minimum, as is evident geometrically, because a rectangle of area A with x either very small (y would then have to be very large) or very large (y would then be very small) would evidently have a very long diagonal.

(ii) *A figure consists of a semicircle with a rectangle constructed on its diameter; given that the perimeter of the figure is* 20 *feet, find its dimensions in order that its area may be a maximum.*

Let r be the radius of the semicircle, and $2r$ and x the lengths of the sides of the rectangle.

Then the perimeter $\qquad \pi r + 2x + 2r = 20. \qquad$ (i)

The area $\qquad A = \frac{1}{2}\pi r^2 + 2rx.$

We begin by eliminating one of the variables; x is the more convenient to eliminate.

From (i) $\qquad 2x = 20 - \pi r - 2r;$

$\therefore$ substituting in the expression for A,

$$A = \tfrac{1}{2}\pi r^2 + r(20 - \pi r - 2r)$$
$$= 20r - \tfrac{1}{2}\pi r^2 - 2r^2;$$

and $\qquad dA/dr = 20 - \pi r - 4r.$

This vanishes when $\qquad \pi r + 4r = 20$

$$= \pi r + 2x + 2r \text{ from (i)},$$

i.e. when $\qquad r = x.$

The side of the rectangle is therefore equal to the radius of the semicircle;

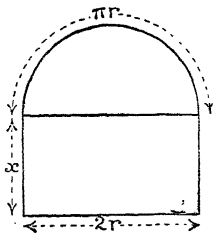


Fig. 61.

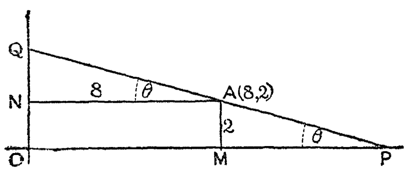


Fig. 62.

this gives the shape Fig. 61. The actual dimensions are given by the equation above, $\pi r + 4r = 20$,

i.e. $\qquad r = 20/(4+\pi) = 20/7{\cdot}1416 = 2{\cdot}8$ feet approximately.

If $\pi r + 4r < 20$, dA/dr is $+$; if $\pi r + 4r > 20$, dA/dr is $-$.

Therefore the solution is a maximum.

(iii) *A straight line drawn through the point* (8, 2) *cuts the axes of coordinates on the positive side of the origin in* P *and* Q (Fig. 62); *find when* OP + OQ *is a minimum.*

In questions of this type, an angle is generally the most convenient variable to use.

Denoting the angle OPQ by θ, and $OP+OQ$ by u, we have

$$u = OM+MP+ON+NQ = 8+2\cot\theta+2+8\tan\theta,$$

$$du/d\theta = -2\operatorname{cosec}^2\theta+8\sec^2\theta.$$

This is equal to 0 when $2\operatorname{cosec}^2\theta = 8\sec^2\theta$, i. e. when $\tan\theta = \pm\frac{1}{2}$.

From the conditions of the question, $\tan\theta$ must be acute; therefore taking $\tan\theta = \frac{1}{2}$, we have

$$u = 8+2\times 2+2+8\times\tfrac{1}{2} = 18.$$

This is obviously a minimum, for it is clear that u will increase indefinitely as θ approaches either of the values 0 or $\frac{1}{2}\pi$. In the first case OP, in the second case OQ, becomes very large.

(iv) *The increase in consumption of an article is proportional to the decrease in the tax upon it; if the consumption be* a *lb. when there is no tax, and* b *lb. when the tax is* n *pence per lb., find the amount of tax most profitable to the exchequer.*

Let z lb. be the amount consumed when the tax is x pence per lb.; then y, the yield to the exchequer, is equal to xz pence, and this is to be a maximum. One of the two quantities x and z must now be eliminated.

The consumption increases from z to a when the tax decreases from x to 0, and from b to a when the tax decreases from n to 0.

Since the increase in the consumption is proportional to the decrease in the tax, it follows that

$$\frac{a-z}{a-b} = \frac{x}{n} \quad\text{and}\quad \therefore\quad x = \frac{n(a-z)}{a-b}.$$

Eliminating x, $$y = xz = \frac{n(az-z^2)}{a-b}.$$

Differentiating, $$\frac{dy}{dz} = \frac{n(a-2z)}{a-b};$$

which is equal to 0, when $z = \frac{1}{2}a$.

This makes y a maximum, since dy/dz is + if $z<\frac{1}{2}a$, and − if $z>\frac{1}{2}a$; and then

$$x = \frac{n(a-z)}{a-b} = \frac{n\,.\,\frac{1}{2}a}{a-b} = \frac{na}{2(a-b)} \text{ pence per lb.}$$

This is the tax which yields the maximum revenue.

(v) *If* v_1 *and* v_2 *be the velocities of light in two different media, find the path by which light can travel in the shortest time*

(*a*) *between two fixed points* A *and* B *in the same medium, by reflexion at the surface separating the two media;*

(*b*) *between two fixed points* A *and* C, *one in each medium.*

(*a*) Let MPN (Fig. 63) be the boundary between the two media, APB the path of the ray of light when reflected at MN. Since it is confined to the one medium, the distance $AP+PB$ is to be a minimum.

Let $AM=a$, $BN=b$, $MN=c$, $MP=x$; therefore $PN=c-x$.

Then $$y = AP + PB = \sqrt{(a^2+x^2)} + \sqrt{\{b^2+(c-x)^2\}},$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{2\sqrt{(a^2+x^2)}} + \frac{-2(c-x)}{2\sqrt{\{b^2+(c-x)^2\}}}.$$

This is equal to 0 when $$\frac{x}{\sqrt{(a^2+x^2)}} = \frac{c-x}{\sqrt{\{b^2+(c-x)^2\}}},$$

i. e. geometrically, when $MP/PA = PN/PB$, and therefore the angles APM and BPN are equal.

It is obvious that this solution is a minimum.

Hence the minimum path is that in which AP and PB are equally inclined to MPN.

This is the ordinary law of reflexion of light.

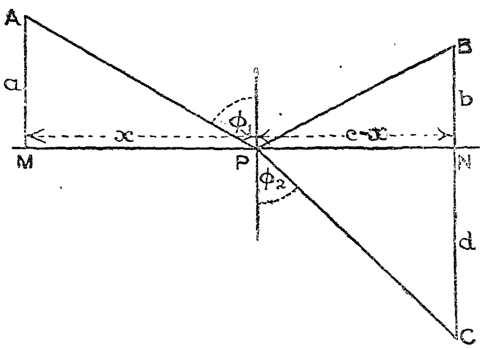


Fig. 63.

(*b*) Let APC be the path when the light is refracted into the second medium.

Let $NC = d$, and let ϕ_1, ϕ_2 be the angles which AP and PC respectively make with the normal at P.

Then the time t along $$APC = AP/v_1 + PC/v_2$$
$$= \sqrt{(a^2+x^2)}/v_1 + \sqrt{\{d^2+(c-x)^2\}}/v_2.$$

$$\therefore \quad \frac{dt}{dx} = \frac{1}{v_1}\cdot\frac{x}{\sqrt{(a^2+x^2)}} + \frac{1}{v_2}\cdot\frac{-(c-x)}{\sqrt{\{d^2+(c-x)^2\}}},$$

and this is equal to 0 when $$\frac{1}{v_1}\cdot\frac{x}{\sqrt{(a^2+x^2)}} = \frac{1}{v_2}\cdot\frac{c-x}{\sqrt{\{d^2+(c-x)^2\}}},$$

which may be written in the form $(\sin\phi_1)/v_1 = (\sin\phi_2)/v_2$.

This again obviously gives a minimum solution.

Hence the path of the ray which leads from A to C in the shortest time is such that

$$\sin\phi_1/\sin\phi_2 = v_1/v_2,$$

where ϕ_1, ϕ_2 are the inclinations of the incident and refracted rays to the normal to the surface separating the two media, and v_1/v_2 is a constant (called the *refractive index* from the one medium to the other) depending upon the nature of the two media and the kind of light.

This is the ordinary law of refraction of light.

(vi) *Two straight roads intersect at right angles; a motor-car, travelling at 20 miles per hour along one of the roads, passes the crossing at the instant when another motor-car, travelling at 15 miles per hour along the other road towards the crossing, is 10 miles distant from it; find when the two cars are at the least distance apart.*

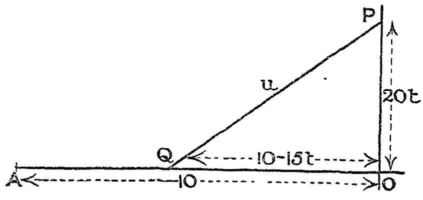


Fig 64.

After time t (measured in hours) the first car is $20\,t$ miles from the crossing, and the second, having travelled $15\,t$ miles, is $10-15\,t$ miles from it. Therefore, if u be the distance between them at that instant,

$$u^2 = (20\,t)^2 + (10-15\,t)^2 = 625\,t^2 - 300\,t + 100.$$

It is most convenient to find when u^2 is least.

Its d. c. with respect to t is $1250\,t-300$, which is equal to 0 when $t = \frac{300}{1250} = \frac{6}{25} = \cdot 24$ hours, i.e. 14·4 minutes, and u^2 is then equal to $625 \times \frac{36}{625} - 300 \times \frac{6}{25} + 100 = 36 - 72 + 100 = 64.$

Therefore $\quad u = 8$ miles.

The solution is a minimum, since the d. c. is $-$ if $t < \cdot 24$, and $+$ if $t > \cdot 24$. Therefore the cars are at the least distance, 8 miles apart, 14 minutes 24 seconds after the first car has passed the crossing.

This problem can easily be solved algebraically (see below), or by elementary mechanics.

It should be noticed that any quadratic expression, such as the one which occurs in the preceding example, has one, and only one, maximum or minimum, which can easily be found algebraically by completing the square, thus:

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}.$$

The last term is constant, and the minimum value of $(x+b/2a)^2$ is zero, since, being a perfect square, it cannot be $-$. Hence, if a be $+$, the expression is least (since the least value of the variable term is then added) when $x = -b/2a$; and, if a be $-$, the expression is greatest (since the least value of the variable term is then subtracted) when $x = -b/2a$.

Therefore ax^2+bx+c is a maximum or a minimum when $x = -b/2a$, according as a is $-$ or $+$. [Cf. with p. 18, where it was shown that the graph of $y = ax^2+bx+c$ is a parabola with axis vertical, and vertex at the highest or lowest point of the curve according as a is $-$ or $+$.]

In the example of the preceding article, we have

$$u^2 = 625\,t^2 - 300\,t + 100 = 625\,(t^2 - \tfrac{12}{25}t) + 100 = 625\,(t-\tfrac{6}{25})^2 - 625 \times (\tfrac{6}{25})^2 + 100$$
$$= 625\,(t-\tfrac{6}{25})^2 + 64,$$

which is obviously least, and then equal to 64, i.e. $u = 8$, when $t = 6/25$.

By the method of the calculus, in the general case,

$$dy/dx = 2ax + b = 2a(x + b/2a),$$

and this vanishes when $x = -b/2a$.

If $x < -b/2a$, i.e. if $x + b/2a$ is $-$, dy/dx is $+$ or $-$, according as a is $-$ or $+$;

and, if $x > -b/2a$, i.e. if $x + b/2a$ is $+$, dy/dx is $-$ or $+$, according as a is $-$ or $+$.

Therefore, as x increases through the value $-b/2a$, dy/dx changes from $+$ to $-$ if a be $-$, and from $-$ to $+$ if a be $+$.

Hence $x = -b/2a$ gives a maximum or minimum value of y according as a is $-$ or $+$, which agrees with the algebraical result.

The maximum or minimum value of y is $(4ac - b^2)/4a$.

Examples XIX.

1. The sum of two numbers is 40; find when the sum of their squares is a minimum.
2. The difference of two numbers is 100; when does the square of the larger exceed five times the square of the smaller by the maximum amount?
3. The sum of two numbers is a; when will three times the square of one together with twice the square of the other be least?
4. When will the sum of a number and its reciprocal be a minimum, and when a maximum? Illustrate this graphically.
5. The denominator of a fraction exceeds the square of its numerator by 16; find the maximum and minimum values of the fraction. Illustrate graphically.
6. Find when the sum of the squares of the reciprocals of two numbers which differ by 1 is least.
7. A rectangle has an area of 25 square feet; find when (i) its perimeter, (ii) the length of its diagonal is least.
8. Prove that the rectangle of a given perimeter which has the shortest diagonal is a square.
9. A rectangle is inscribed in a given circle of radius a; find when its perimeter is a maximum or minimum.
10. Find the rectangle of maximum area whose sides pass through the angular points of a given rectangle with sides of lengths a and b.
11. Find the dimensions of the cylinder of maximum volume which can be inscribed in a given sphere. Prove that its volume is $\cdot 5773\ldots$ of that of the sphere.
12. The total area of the surface (i.e. curved surface and both ends) of a cylinder is 150π square feet; find when the volume is a maximum.
13. An open cylindrical vessel is to be made of thin material to hold 100 gallons; find the dimensions in order that the amount of material used may be a minimum. [Take 1 gallon = $\cdot 1605$ cubic feet.]
14. Find when the curved surface of a cylinder inscribed in a given sphere is a maximum.

15. A rectangle is inscribed in a given right-angled triangle with one angle coincident with the right angle; find when its area is a maximum. Show that its perimeter has no maximum or minimum. How do you explain this latter fact?

16. A cylinder is inscribed in a given right circular cone. (i) When is its volume a maximum? (ii) When is its curved surface a maximum? (iii) When is its total surface a maximum? Show that in the last case there is no solution if the semi-vertical angle of the cone exceeds a certain value, and find this value.

17. A cone is circumscribed about a given sphere; find when its volume is a minimum.

18. A rectangle is inscribed in a given triangle; find its maximum area.

19. When is the area of an isosceles triangle inscribed in a given circle a maximum?

20. Find the dimensions of the cone of maximum volume which can be inscribed in a given sphere. Prove that the cone has also a greater curved surface than any other cone inscribed in the sphere.

21. Prove that a conical tent which is to have a given volume will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

22. A sector is cut out of a circular sheet of paper, and the two straight edges of the remainder are put together so that a cone is formed; prove that the volume of this cone is a maximum when the angle of the sector removed is about 66°. Draw a graph to show how the volume of the cone depends on the angle of the sector.

23. The regulations of the Parcel Post state that a parcel must not exceed 6 feet in length and girth combined; find the dimensions of the cylinder of maximum volume which can be sent.

24. A cylinder is inscribed in a sphere of radius r; find its height when the area of its entire surface is a maximum.

25. A right circular cone is inscribed in a given right circular cone so that the vertex of the inside cone is at the centre of the base of the other; find when its volume is a maximum.

26. Through a point whose coordinates referred to rectangular axes are (a, b), a straight line is drawn making positive intercepts OP, OQ on the axes; find the minimum area of the triangle OPQ.

27. In the preceding case, find also the minimum value of $OP+OQ$.

28. Find also the minimum length of PQ.

29. Find also the minimum value of the rectangle $OP \, . \, OQ$.

30. Given the perimeter of a circular sector, find when its area is a maximum.

31. Given the area of a right-angled triangle, find when its perimeter is a minimum.

32. If the stiffness of a rectangular beam varies directly on the breadth and as the cube of the depth, find the breadth of the stiffest beam that can be cut from a cylindrical log of diameter 2 feet.

✓33. A rectangular sheet of tin is 5 feet long and 28 inches wide; four equal squares are removed from the corners and the sides are then turned up so as to form an open rectangular box; find the size of the pieces that must be cut out in order that the box may have the greatest volume.

34. A rectangular sheep-pen is to be made alongside of a hedge which serves as one of the sides of the pen, and is to enclose an area of 200 square yards; find the least number of hurdles, each 6 feet long, required for the other three sides.

35. A statue 10 feet high stands on the top of a column 35 feet high; at what distance from the column in the horizontal plane through its foot should a man stand in order to get the best view of the statue, i.e. in order that the statue may subtend the greatest angle at his eye, which is supposed to be 5 feet above the ground?

36. The sides of a wooden trough are each 1 foot wide, and are equally inclined to the bottom of the trough which is 9 inches wide; what must be the width across the top in order that the volume may be a maximum.

37. If the power required to propel a steamer through the water varies directly as the cube of the velocity, find the most economical rate of steaming against a current which runs at a miles per hour.

38. Two straight roads across a moor intersect at right angles; a man on one road, three-quarters of a mile from the crossing, wishes to strike across the moor in order to get to a place 2 miles from the crossing along the other road; if he can walk 5 miles per hour along the roads, but only 4 miles per hour across the moor, where should he strike the second road in order to reach his destination in the shortest possible time? How much time will he save by going this way instead of by the shortest way? Prove that the point at which he should strike the road is the same whatever be the distance of his destination from the crossing, provided it is more than a mile.

39. An electric light is to be placed vertically over the centre of a circular enclosure 30 yards in diameter; at what height should it be placed in order that a path round the enclosure may be illuminated as brightly as possible? (The brightness of a surface varies inversely as the square of the distance from the light and directly as the cosine of the angle which the rays make with the normal to the surface.)

40. At what point on the line joining two sources of light will the brightness be least, if the intensity of one is 8 times that of the other?

41. Find the greatest rectangle which can be inscribed in the segment of a parabola cut off by the latus-rectum.

42. Prove that the least intercept made by the axes on a tangent to an ellipse is equal to the sum of the semi-axes of the ellipse.

43. One corner of a rectangular sheet of paper of width 1 foot is folded over so as to reach the opposite edge of the sheet; find the minimum length of the crease.

44. In the preceding question, find the minimum area of the part folded over.

45. A rectangular sheet of metal is bent into the form of part of the curved surface of a right circular cylinder; if it is then closed at the ends, prove that the volume of the trough thereby formed is greatest when the trough is exactly half a cylinder.

46. The segment of a parabola, bounded by the latus-rectum, rotates about the axis, thereby forming a solid known as a paraboloid of revolution; find the maximum cylinder which can be inscribed in this solid.

47. Find the maximum area of the triangle formed by joining the ends of a chord of a given circle to one extremity of the diameter which bisects the chord.

48. A straight line is drawn through the angular point C of a triangle ABC inclined at an angle θ to BC; find when the sum of the projections of the sides AC and BC upon it is a maximum.

49. The section of a dormer window consists of a rectangle surmounted by an equilateral triangle; if the perimeter be given as 16 feet, find the width of the window in order that the maximum amount of light may be admitted.

50. Find the area of the greatest rectangle which can be inscribed in the ellipse $\frac{1}{16}x^2+\frac{1}{9}y^2=1$.

51. Find the area of the greatest isosceles triangle which can be inscribed in the same ellipse, with its vertex at one end of (i) the major axis, (ii) the minor axis.

52. Find the minimum distance between the straight line $x-2y+10=0$ and the parabola $y^2=8x$.

53. Show that the sum of the squares of the distances of a point from the angular points of a triangle is least when the point is the centroid of the triangle.

54. Two straight roads intersect at an angle of 60°. A motor-car, travelling at 30 miles an hour along one road, passes the crossing at the instant when another motor-car, travelling at 20 miles an hour along the other road towards the crossing, is 2 miles away; find when the distance between the cars is least and what this least distance is.

55. Find a point on a given straight line such that the sum of the squares of its distances from two given points (not on the line) is a minimum.

56. The perimeter of an isosceles triangle is given; what vertical angle will give the maximum area?

57. The strength of a rectangular beam of given length varies as the breadth into the square of the depth; find the dimensions of the strongest rectangular beam which can be cut from a cylindrical log 1 foot in diameter.

58. A given mass m raises another mass m' by means of a string passing vertically over a pulley; find m' in order that the momentum acquired by it in a given time may be a maximum.

59. A mass M is drawn up a smooth incline of given height by a mass m attached to it by a string passing over a pulley at the top of the incline and hanging vertically. Find the angle of the incline in order that the time of ascent may be a minimum.

60. How much water should be put into a closed right circular cylinder, standing on a horizontal plane, in order to bring the centre of gravity as low as possible, the weight of the cylinder being $\frac{9}{16}$ of the weight of all the water it can contain?

61. A wall 9 feet high is 21 feet 4 inches from a house; find the length of the shortest ladder which will reach the house when the lower end is on the (horizontal) ground on the other side of the wall.

62. A piece of wire of length l is to be cut into two pieces, one of which is to be bent into the form of a square, and the other into the form of a circle; find when the sum of the areas of the circle and square is least.

63. A heavy lever (weight w per unit length) with the fulcrum at one end, is used to raise a weight W at a given distance a from that end; find the length of the lever in order that the weight may be lifted with the least effort.

64. Two ships are sailing with velocities u and v along courses which are inclined at an angle θ; if at a certain instant they are at distances a and b from the intersection of their courses, find their minimum distance apart.

65. A man is to get as much land as he can compass in a given time; he is to move in a circle, and if he does not get back by the end of the given time, he gets the segment whose arc he has traced. Prove that his best plan is to describe a semicircle.

66. Find the rectangle of maximum area which can be inscribed in the curve $(x/a)^{2/3}+(y/b)^{2/3}=1$.

67. Find the volume of the greatest cone which can be constructed with its vertex at the centre of a given sphere and the circumference of its base on the surface of the sphere.

68. A circular cylinder has a hemisphere hollowed out from each end. Given the total surface, find when the volume is a maximum.

69. The normal at a point $(am^2, 2am)$ of the parabola $y^2=4ax$ cuts the parabola again in Q. Find the minimum length of PQ.

70. O is a fixed point outside a circle, A one end of the diameter through O, and OPP' a chord of the circle; prove that the area of the triangle PAP' is greatest when PP' subtends a right angle at the centre.

71. A steamer travelling due west at 20 knots is sighted by another steamer going at 16 knots. What course must the latter steer in order to cross the track of the former at the least possible distance from her?

72. Find the area of the ground-plan of the greatest rectangular building which can be erected on a plot of ground in the form of a segment of a circle with a base of 120 yards and height 20 yards.

CHAPTER VII

SUCCESSIVE DIFFERENTIATION AND POINTS OF INFLEXION

57. Differential coefficients of higher order.

We have seen how various kinds of functions of x can be differentiated with respect to x; the resulting differential coefficient is also a function of x (except when the original function is a linear function of x, $ax+b$, in which case the differential coefficient is the constant a), and therefore it can be differentiated again with respect to x.

The result of this second differentiation is called the second differential coefficient of y with respect to x, and is denoted by the symbol

$$\frac{d^2y}{dx^2}.$$

This again can usually be differentiated with respect to x; the result is called the third differential coefficient of y with respect to x, and is denoted by the symbol

$$\frac{d^3y}{dx^3};$$

and so on.

Generally, the result of differentiating y n times in succession with respect to x is called the n^{th} differential coefficient of y with respect to x, and is denoted by

$$\frac{d^ny}{dx^n}.$$

If the original function is represented by the symbol $f(x)$, then the results of differentiating it 1, 2, 3, ... n times with respect to x are called the first, second, third, ... n^{th} derived functions, and are denoted by

$$f'(x),\ f''(x),\ f'''(x),\ \ldots f^{(n)}(x) \text{ respectively.}$$

The second differential coefficient is of very great importance in mechanics. The higher differential coefficients are of less frequent occurrence.

In the case of some of the simplest functions, if the first few

differential coefficients be written down, the law of formation of the successive differential coefficients can be seen by inspection, and the n^{th} d. c. written down at once.

Examples:

(i) $y = x^n$.

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{d^2y}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3};$$

. . . .

clearly, if n be a + integer, $\dfrac{d^ny}{dx^n} = n!$, a constant, and all the higher d. c.'s are zero.

(ii) $y = 1/x = x^{-1}$.

$$\frac{dy}{dx} = -1 \, . \, x^{-2}, \quad \frac{d^2y}{dx^2} = -1 \, . \, -2x^{-3} = \frac{1 \, . \, 2}{x^3},$$

$$\frac{d^3y}{dx^3} = -1 \, . \, -2 \, . \, -3x^{-4} = -\frac{3!}{x^4}.$$

. . . .

Hence
$$\frac{d^ny}{dx^n} = (-1)^n \frac{n!}{x^{n+1}}.$$

(iii) $y = \sin x$.

$$\frac{dy}{dx} = \cos x = \sin(\tfrac{1}{2}\pi + x), \quad \frac{d^2y}{dx^2} = -\sin x = \sin(\pi + x),$$

$$\frac{d^3y}{dx^3} = -\cos x = \sin(\tfrac{3}{2}\pi + x).$$

. . . . . .

Each differentiation merely increases the argument by $\frac{1}{2}\pi$.

Hence
$$\frac{d^ny}{dx^n} = \sin(\tfrac{1}{2}n\pi + x).$$

Examples XX.

Write down the 1st, 2nd, 3rd, and n^{th} differential coefficients of

1. x^{10}. 2. $a + b/x$. 3. $1/x^3$. 4. $1/\sqrt{x}$.
5. $\sqrt{x}$. 6. $(ax+b)^{10}$. 7. $1/(2x+1)$. 8. $1/(1-x)$.
9. $\sin(2x+\alpha)$. 10. $\cos x$. 11. $\sin^2 x$. 12. $\cos^2 2x$.

Write down the first 3 differential coefficients of

13. $x \sin x$. 14. $x^2 \cos x$. 15. $\tan x$. 16. $x^3 \sin 3x$.
17. $x^2/(1+x)$. 18. $x^n \cos nx$. 19. $\sec x$. 20. $\sqrt{(a^2+x^2)}$.

58. Application of the second differential coefficient to maxima and minima.

We have seen that y is a maximum when dy/dx vanishes and changes sign from + to −, and a minimum when dy/dx vanishes

and changes sign from $-$ to $+$. Since, as y passes through a maximum, dy/dx changes from $+$ to $-$, therefore it is decreasing as x increases, and its d. c. is $-$ (Art. 25), i.e. d^2y/dx^2 is $-$ at a maximum. Similarly, as y passes through a minimum, dy/dx changes from $-$ to $+$, therefore it is increasing as x increases, and its d. c. is $+$, i.e. d^2y/dx^2 is $+$ at a minimum.

Hence the *conditions for a maximum are* $\dfrac{dy}{dx} = 0, \quad \dfrac{d^2y}{dx^2}\ -$.

and *for a minimum* $\dfrac{dy}{dx} = 0, \quad \dfrac{d^2y}{dx^2}\ +$.

Sometimes it is more convenient to find the sign of the second d. c. than to find how the sign of the first d. c. changes.

E.g. in Ex. (i) worked out in Art. 55, $dy/dx = 0$ when $x = 1$ or 5. $d^2y/dx^2 = 6x - 18$, which is $-$ when $x = 1$, and $+$ when $x = 5$. Therefore $x = 1$ makes y a maximum, and $x = 5$ makes y a minimum.

In Ex. (ii) of the same article, $dy/dx = 0$ when $x = 1$ or -2, and $d^2y/dx^2 = 12x^2 - 12$, which is 0 when $x = 1$, and $+$ when $x = -2$. Therefore $x = -2$ makes y a minimum, and $x = 1$ gives neither a maximum nor a minimum.

In Ex. (iii) a troublesome differentiation is required to find the value of d^2y/dx^2, and it is much easier to find the change of sign of dy/dx.

In Ex. (iv) on the contrary, it is easier to use the second d. c., $d^2y/d\theta^2 = -a\sin\theta - b\cos\theta$, which is $-$ when the positive values of $\sin\theta$ and $\cos\theta$ are taken, and $+$ when their negative values are taken; hence the former give maxima and the latter minima.

59. Geometrical meaning of the second differential coefficient.

If, in the neighbourhood of a point P on a curve, the curve is above the tangent at P [as is the case at a point between A and B

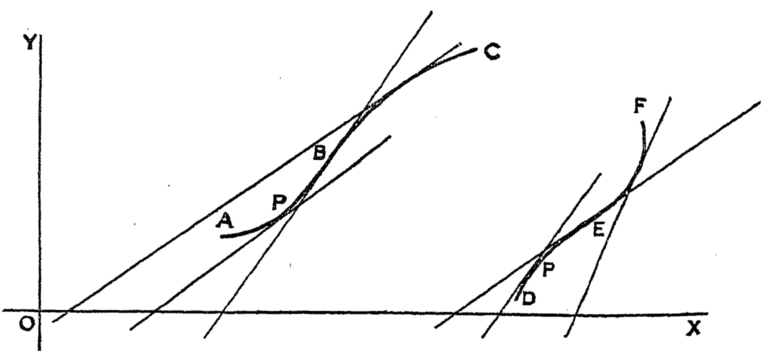


Fig. 65.

or between E and F in Fig. 65], it is said to be *concave upwards*; if the curve is below the tangent [as is the case at a point between

B and C or between D and E], it is said to be *concave downwards.* A point such as B or E, where the concavity changes from upwards to downwards or vice versa, is called a *point of inflexion.* The tangent to the curve at such a point crosses the curve; on opposite sides of the point of contact the curve is on opposite sides of the tangent.

If, at all points in the neighbourhood of a point P on the curve, the curve is concave upwards, then as x increases, the slope of the curve, i.e. dy/dx, increases. Therefore (Art. 25) its d. c. is positive, i.e. d^2y/dx^2 is $+$. Similarly, if at all points in the neighbourhood of P the curve is concave downwards, then the slope, dy/dx, decreases as x increases. Therefore its d. c., d^2y/dx^2, is $-$.

Taking the case of a circle, we have:—

in 1st quadrant,	$dy/dx-$,	d^2y/dx^2-,
in 2nd "	$dy/dx+$,	d^2y/dx^2-,
in 3rd "	$dy/dx-$,	d^2y/dx^2+,
in 4th "	$dy/dx+$,	d^2y/dx^2+.

Also, at a minimum the graph is concave upwards, and d^2y/dx^2 is $+$; at a maximum the graph is concave downwards, and d^2y/dx^2 is $-$, as in the preceding article.

Hence a curve is concave upwards or downwards at a point P according as the value of d^2y/dx^2 at the point is $+$ or $-$. It follows that in passing through a point of inflexion, where the concavity changes, d^2y/dx^2 changes sign, and therefore, if continuous at the point of inflexion, it is zero.

This may also be seen as follows: In Fig. 65, as the point P moves along the curve from A through B to C, the slope of the curve increases until the point B is reached, after passing which point the slope begins to decrease; therefore at the point of inflexion B, the slope dy/dx is a maximum; hence its d. c. $d^2y/dx^2 = 0$, and changes sign from $+$ to $-$; therefore also d^2y/dx^2 is decreasing as x increases, and *its* d. c. d^3y/dx^3 is $-$.

Similarly, as the point P moves along the curve from D to F through E, the slope decreases until the point E is reached, after which it increases again; therefore at the point of inflexion E, the slope dy/dx is a minimum; hence its d. c. $d^2y/dx^2 = 0$, and changes sign from $-$ to $+$; therefore also d^2y/dx^2 is increasing as x increases, and *its* d. c. d^3y/dx^3 is $+$.

Hence *the conditions for a point of inflexion are that* d^2y/dx^2 *must vanish at the point, and change sign in passing through it, or* $d^2y/dx^2 = 0$, $d^3y/dx^3 \neq 0$.

The value of dy/dx at the point of inflexion of course gives the direction of the tangent at the point. It will be zero if the tangent

at the point of inflexion is parallel to the axis of x as we have already seen in Art. 54, but this will not be the case in general.

It is obvious that in the case of a continuous function, a point of inflexion must occur between a maximum and a minimum.

60. Tangent at a point of inflexion.

It has been seen (Art. 14 (i)) that the tangent at a point P is the limiting position of a chord PQ when Q moves indefinitely near to P, i.e. the tangent passes through two 'consecutive points' on the curve. It should be noticed that the tangent at a point of inflexion passes through *three* 'consecutive points' on the curve. This is seen from Fig. 66.

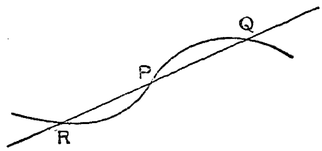


Fig. 66.

A straight line through a point of inflexion P will cut the curve again in two points Q and R. When Q is made to approach indefinitely near to P, R will approach and become indefinitely near to P on the other side, and the tangent at P is the limiting position of QPR when Q and R are both indefinitely near to P.

61. Recapitulation.

Let us now sum up the information as to the nature of a curve at a point, which can be gathered from the signs of the values of the first two differential coefficients at the point. This information is clearly of great assistance in drawing the curve. The results can be conveniently expressed in a tabular form as follows:

	$\frac{d^2y}{dx^2}$ +	$\frac{d^2y}{dx^2}$	$\frac{d^2y}{dx^2}=0, \frac{d^3y}{dx^3}\neq 0$ + −
	Curve rising and concave upwards	Curve rising and concave downwards	Point of inflexion on rising curve
	Curve falling and concave upwards	Curve falling and concave downwards	Point of inflexion on falling curve
$\frac{dy}{dx}=0$	Minimum	Maximum	Point of inflexion with slope zero

In each of the first figures in the last column, the curve passes from below the tangent to above it, i.e. d^2y/dx^2 changes from $-$ to $+$; therefore it is increasing and its d. c. d^3y/dx^3 is $+$. Similarly, in the second figures, d^3y/dx^3 is $-$.

Examples:

(i) $y = x^3 - 6x^2 + 8$.

In this case $dy/dx = 3x^2 - 12x = 3x(x-4)$

and $d^2y/dx^2 = 6x - 12 = 6(x-2)$.

Hence $dy/dx = 0$ when $x = 0$ or 4.

When $x = 0$, $d^2y/dx^2 = -12$ and $y = 8$; and when $x = 4$, $d^2y/dx^2 = 24$, and $y = -24$.

Also $d^2y/dx^2 = 0$ when $x = 2$; and $d^3y/dx^3 = 6$.

Therefore y is a maximum (8) when $x = 0$, a minimum (-24) when $x = 4$, and the graph has a point of inflexion when $x = 2$. The value of dy/dx when $x = 2$ is -12, and the value of $y = -8$; therefore the tangent at the point of inflexion $(2, -8)$ is inclined to the axis of x at an angle $\tan^{-1}(-12)$. If $x < 2$, d^2y/dx^2 is $-$, and the curve is concave downwards; if $x > 2$, d^2y/dx^2 is + and the curve is concave upwards.

Fig. 67 shows roughly the graph of the function.

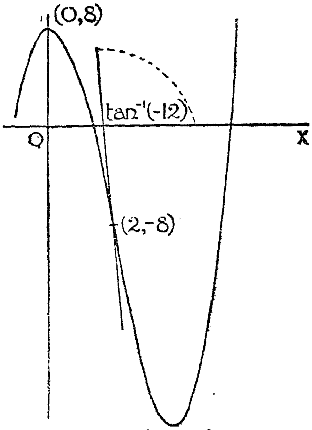


Fig. 67.

(ii) $$y = \frac{}{a^2 + x^2}.$$

Here $$\frac{dy}{dx} = a^2 \cdot \frac{a^2 + x^2 - x \cdot 2x}{(a^2+x^2)^2} = \frac{a^2(a^2 - x^2)}{(a^2+x^2)^2},$$

and $$\frac{d^2y}{dx^2} = a^2 \cdot \frac{(a^2+x^2)^2(-2x) - (a^2-x^2)\,2(a^2+x^2) \cdot 2x}{(a^2+x^2)^4}$$

$$= a^2 \cdot \frac{-(a^2+x^2)\,2x - 4x(a^2-x^2)}{(a^2+x^2)^3} = \frac{2a^2x(x^2-3a^2)}{(a^2+x^2)^3}.$$

$dy/dx = 0$ when $x = \pm a$; when $x = +a$, d^2y/dx^2 is $-$; $\therefore$ $x = +a$ makes y a maximum and equal to $\frac{1}{2}a$.

When $x = -a$, d^2y/dx^2 is +; $\therefore$ $x = -a$ makes y a minimum and equal to $-\frac{1}{2}a$.

$d^2y/dx^2 = 0$, when $x = 0$ and when $x^2 = 3a^2$; i.e. $x = \pm a\sqrt{3}$, and d^2y/dx^2 changes sign when x passes through each of these values. Hence there are 3 points of inflexion.

When $x = 0$, $y = 0$, and $dy/dx = 1$; therefore the origin is a point of inflexion, and the tangent there is inclined at 45° to the axis of x.

When $x = \pm a\sqrt{3}$, $y = \pm\frac{1}{4}a\sqrt{3}$, and $dy/dx = a^2(-2a^2)/(4a^2)^2 = -\frac{1}{8}$; therefore the tangents at the two points of inflexion $(a\sqrt{3}, \frac{1}{4}a\sqrt{3})$ and $(-a\sqrt{3}, -\frac{1}{4}a\sqrt{3})$ are inclined to the axis of x at an angle $\tan^{-1}(-\frac{1}{8})$, i.e. at about 173°.

Noticing that as $x \to \infty$, $y \to 0$, it follows that the form of the curve is as shown in Fig. 68.

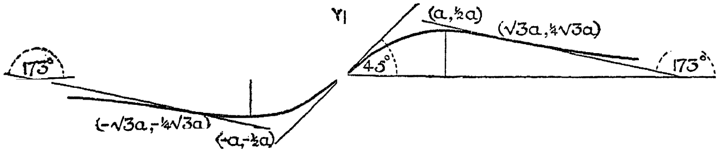


Fig. 68.

Examples XXI.

Find whether the concavity is upwards or downwards in the following cases 1–4:

1. At the point $(2, -4)$ of the curve $y = 10 - 3x - x^3$.
2. At the point $(3, 25)$ of the curve $y = x^2 + 7x - 5$.
3. At the points $(0, 0)$ and $(-2, 22)$ of the curve $y = x^3 + 3x^2 - 9x$.
4. At the points $(-1, -\frac{1}{2})$ and $(3, 2{\cdot}7)$ of the curve $y = x^3/(1 + x^2)$.
5. Prove that the curve $y = ax^2 + bx + c$ is everywhere concave upwards or downwards according as a is $+$ or $-$. (Cf. p. 18.)
6. Show that the curve $y = a + bx - x^2 - x^4$ is everywhere concave downwards.
7. Prove that the curve $y = a \sin x + b \cos x$ is concave downwards at all points above the axis of x, and concave upwards at all points below.

For what values of x are the following curves concave upwards, and when are they concave downwards?

8. $y = 2x^3 - 5x$.
9. $y = 4x^2 - x^3$.
10. $y = x^4 - 8x^3 + 10x - 6$.
11. $y = x/(1 - x^2)$.
12. Find the points of inflexion of the graph of $y = \cos x$.
13. Also of $y = \tan x$.
14. Prove that the curve $y = ax^3 + bx^2 + cx + d$ can have but one point of inflexion.

Find the points of inflexion (also the maxima and minima, and sketch the curve roughly) in the following cases 15–23:

15. $y = x^3 - 4x$.
16.* $y = x^2/(1 + x^2)$.
17. $y = \sin x - \cos x$.
18. $y = x^3/(x^2 + 12)$.
19. $y = x^2(4 - x^2)$.
20. $y = 9x/(x - 1)^2$.
21. $y = 4/(x^2 + 3)$.
22. $y^2 = x^2(4 - x^2)$.
23. $y = a + (b - cx)^3$.
24. Have the curves $y = x^4$, $y = x^5$ any points of inflexion?
25. What is the greatest possible number of points of inflexion for the curve $y = ax^n + bx^{n-1} + \ldots + k$ [n a positive integer]?
26. Draw curves at every point of which

(i) x is $-$, y $+$, dy/dx $+$, d^2y/dx^2 $+$;
(ii) x is $-$, y $-$, dy/dx $+$, d^2y/dx^2 $-$;
(iii) x is $+$, y $-$, dy/dx $-$, d^2y/dx^2 $+$;
(iv) x is $+$, y $+$, dy/dx $-$, d^2y/dx^2 $-$.

* See Art. 9, Ex. v.

27. Find the points of inflexion of the curve $xy^2 = a^2(a-x)$.

28. Prove that the curve $x^3 - y^3 = a^3$ cuts the axis of y at right angles at a point of inflexion.

29. Find the points of inflexion of the curve $y = a\sin^2 x + b\cos^2 x$.

In the following curves, find their intersections with the axes, their maxima and minima, where they are concave upwards and where downwards, their points of inflexion, and the equations of the tangents at those points. Draw the curves.

30. $y = x^4 - 10x^2 + 9$. 31. $y = (1-x^2)^2$. 32. $4 - y = (2-x^2)^2$.

CHAPTER VIII

APPLICATIONS TO MECHANICS

62. Velocity and acceleration.

We have seen that dy/dx is a measure of the rate of increase of y with respect to x. Now the distance of a moving point from a fixed point in its path is a function of the time; its velocity is the rate of increase of this distance, and its acceleration the rate of increase of its velocity with respect to the time. In other words, the velocity is the d. c. of the distance with respect to the time, and the acceleration is the d. c. of the velocity with respect to the time.

More precisely, let s be the distance, measured along the path, of a moving point P from a fixed point A of the path at the end of time t. After a little longer time $t+\delta t$, let the moving point be at Q, a distance $s+\delta s$ from A. Fig. 69 is drawn so that s increases with t.

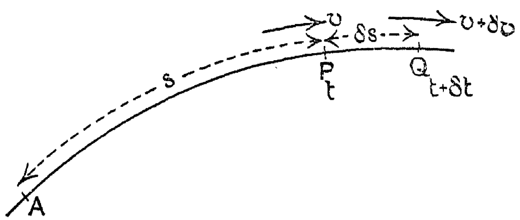


Fig. 69.

Then, in the interval of time δt, the point has travelled the distance δs, therefore the average velocity during the interval δt is $\delta s/\delta t$; if δt is diminished indefinitely, this average velocity $\delta s/\delta t$ tends to a definite limit. This limit of $\delta s/\delta t$ as $\delta t \to 0$, i.e. ds/dt, is called *the velocity at time* t. It will be + if s increases with t as in Fig. 69, and − if s decreases as t increases, i.e. if P is moving towards A.

Similarly, if v be the velocity of P at the end of time t, and if APQ be a straight line, so that the velocity is in a constant direction, $\delta v/\delta t$ is the average acceleration during the interval δt, and its limit dv/dt is called *the acceleration at time* t.

Since $v = ds/dt$, it follows that the acceleration dv/dt may be expressed in the form d^2s/dt^2.

The acceleration may also be written in a third form, which is independent of the time; for the velocity v is evidently a function of the distance s, therefore

$$\frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} \text{ (Art. 34)} = \frac{dv}{ds} \times v = v\frac{dv}{ds}.$$

Hence the acceleration may be expressed in any one of the three forms

$$\frac{dv}{dt}, \quad \frac{d^2 s}{dt^2}, \quad v\frac{dv}{ds}.$$

It is usual in mechanics to denote differential coefficients with respect to the time by dots placed above the dependent variables, thus dv/dt, ds/dt, $d^2 s/dt^2$, $d^2 x/dt^2$ are denoted by $\dot{v}$, $\dot{s}$, $\ddot{s}$, and $\ddot{x}$ respectively.

63. Particular cases.

As examples on the use of these expressions, consider the following cases:

(i) Let s be given by the equation $s = ut + \frac{1}{2}at^2$, where u and a are constants.

Then the velocity

$$v = ds/dt = u + \tfrac{1}{2}a \,.\, 2t = u + at,$$

which gives the velocity at time t, and from which it follows that u is the value obtained for v by putting $t = 0$, i.e. u is the initial velocity.

Also the acceleration $= dv/dt = a$, hence the point moves with constant acceleration a.

Again, if v be given by the equation $v^2 = u^2 + 2as$, we have, on differentiating with respect to s, $2v\,dv/ds = 2a$, i.e. the acceleration $v\,dv/ds = a$ as before.

This is the well-known case of uniformly accelerated motion.

(ii) Let s be given by the equation $s = a\cos nt$, where a and n are constants (a is the value obtained for s by putting $t = 0$, i.e. it is the initial distance from the origin).

Then the velocity $v = ds/dt = -an\sin nt$,
and the acceleration $= dv/dt = -an^2\cos nt = -n^2 s$.

Again, eliminating t between the values of v and s, we have

$$v^2 = a^2 n^2 \sin^2 nt = n^2(a^2 - a^2\cos^2 nt) = n^2(a^2 - s^2),$$

which gives v in terms of s.

Differentiating this with respect to s, $2v\,dv/ds = n^2(-2s)$, i.e. the acceleration $v\,dv/ds = -n^2 s$ as before; so that the acceleration is towards the origin and varies as the distance from the origin.

This is the well-known case of simple harmonic motion.

In these two cases the order of procedure adopted in elementary mechanics is reversed; there we begin by assuming an acceleration of a certain type, and then proceed to find the velocity in terms of the time and the position, and the distance travelled in terms of the time. This, as will be seen later (Art. 78), is also the method followed in the Integral Calculus.

64. Additional examples.

Given any relation between s and t, or between v and s, we can at once find the velocity and acceleration at any instant, as we have done in the two well-known cases just considered. Two more examples are appended.

(i) *Suppose a point to move in a straight line so that its distance* s *in feet from a fixed point* O *in the line at the end of* t *seconds is given by the equation* $s = 10+27t-t^3$; *to find the various circumstances of the motion.*

The velocity at the end of t seconds is given by

$$v = ds/dt = 27-3t^2,$$

whence the initial velocity $= 27$ ft. secs., the velocity after 2 secs. $= 27-12 = 15$ ft. secs., and the velocity is zero when $27 = 3t^2$, i.e. when $t = 3$; greater values of t make $v-$, and when $t = 3$,

$$s = 10+81-27 = 64 \text{ feet}.$$

Hence the particle starts with a velocity 27 ft. secs. at a point 10 feet from O (obtained by putting $t = 0$ in the value of s), moves to a distance of 64 feet from O, and then turns back towards O.

The acceleration at the end of t seconds $= dv/dt = -6t$.

Therefore the particle is subject to a retardation which is proportional to the time it has been in motion.

After 3 seconds the velocity is $-$, and the acceleration is $-$ and constantly increasing in absolute value; hence the particle continues to move in the negative direction with numerically increasing velocity. It will pass through O on the return journey, when $s = 0$, i.e. when $t^3-27t-10 = 0$, an equation which has a root a little less than 5·4.

(ii) *If* v^2 *is a quadratic function of* s, *i.e. if* $v^2 = as^2+bs+c$, *where* a, b, c *are constants, prove that the acceleration varies as the distance from a fixed point in the line of motion.*

Differentiating with respect to s,

$$2v\,dv/ds = 2as+b,$$

i.e., the acceleration $\quad v\,dv/ds = as+\frac{1}{2}b = a(s+b/2a).$

s is the distance of the moving point P from the origin; therefore, if a point A be taken in the line of motion at distance $b/2a$ from the origin and on the negative side of it, $AP = s+b/2a$. Hence the acceleration $= a.AP$, i.e. it varies as the distance of P from the fixed point A.

Examples XXII.

Find the velocity and acceleration (i) at the end of t seconds, (ii) at the end of 2 seconds, (iii) initially, in each of the cases 1–4.

1. $s = 12 + 20t - 2t^2$.
2. $s = t^3 - 2t^2 + 4$.
3. $s = 10 \cos \frac{1}{3}\pi t$.
4. $s = 6t + 8/(t+1)$.
5. Find the initial velocity and acceleration when $s = t^4 + 3t^2 + 4$.
6. " " " " $s = 8 \cos 2t + 4 \sin 2t$.
7. If $s = 2t^3 + 3t^2 + 4t - 20$, make a table giving the position, velocity, and acceleration initially, and after 1, 2, 5, 10 seconds.
8. Similarly if $s = 10 \cos \frac{1}{4}\pi t + 20 \sin \frac{1}{4}\pi t$.
9. If the distance travelled varies as the square root of the time, prove that the acceleration varies as the cube of the velocity.
10. If the velocity varies as the square of the distance travelled, prove that the acceleration varies as the cube of that distance.
11. A point moves so that $s = 16 + 48t - t^3$; when and where will it stop and reverse its direction of motion?
12. When and where if $s = a \sin \pi t$?
13. If the velocity varies inversely as the square root of the distance, prove that the acceleration varies as the fourth power of the velocity.
14. If s^2 be a quadratic function of t, prove that the acceleration varies inversely as s^3.
15. A point moves in a straight line so that its distance s from the origin at time t is given by the equation $s = 10 + 8 \sin 2t + 6 \cos 2t$; prove that its acceleration varies as its distance from a fixed point in the line of motion, that its motion is oscillatory, and that the origin is at one extremity of its path.

65. Force expressed as a differential coefficient.

(i) *Force and acceleration.* Newton's second law of motion states that the force in any direction is proportional to the rate of change of momentum in that direction. By suitably choosing the units, we have the force *equal* to the rate of change of momentum, i.e. (if the mass acted upon by the force remain constant)

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = m\frac{d^2 s}{dt^2},$$

the product of the mass into the acceleration.

Ex. Find the maximum force upon a particle of mass 2 oz. which moves so that its distance from the origin at time t is given by the equation $s = 10 \cos 2t$.

In this case $m = \frac{1}{8}$ lb., $ds/dt = -20 \sin 2t$, $d^2s/dt^2 = -40 \cos 2t$. Therefore the force on the particle at time $t = md^2s/dt^2 = -5 \cos 2t$ (in absolute units or poundals).

The negative sign indicates that the force acts towards the origin. The greatest value of $\cos 2t$ is 1, therefore the maximum force is 5 poundals or $\frac{5}{32}$ lb. wt.

(ii) *Force, work, and energy.* If a constant force F (Fig. 70) act at a point A, and if A be displaced to B and BM be drawn perpendicular to the line of action of F, $F.AM$ is called the *work* done by the force during the displacement; it is positive if AM is in the direction of the force as in (i), and negative if in the opposite direction as in (ii). No work is done if the displacement be perpendicular to F.

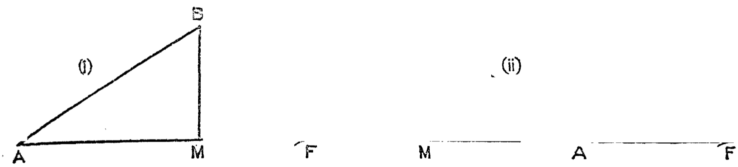


Fig. 70.

Let a variable force F act on a particle at P, and let x be the distance (measured parallel to F) of P from a fixed point A; let P be displaced to Q, a distance δx in the same direction, and let $F+\delta F$ be the magnitude of the force at Q. Then, if W be the work done by the force in moving the particle from some standard position to P, and δW the work done in the displacement δx, we have*

$$\delta W > F\delta x \quad \text{and} \quad < (F+\delta F)\,\delta x.$$

Therefore $\delta W/\delta x$ is between F and $F+\delta F$. In the limit, when δx (and therefore also δW and δF) $\to 0$, $F+\delta F \to F$, so that $\delta W/\delta x$, which is between them, also $\to F$, i.e. $dW/dx = F$; therefore the force is equal to the space-rate of change of the work.

Since $F =$ mass $\times$ acceleration, we have, taking the acceleration in the form $v\,dv/dx$,

$$F = mv\frac{dv}{dx} = m\frac{d}{dx}(\tfrac{1}{2}v^2) = \frac{d}{dx}(\tfrac{1}{2}mv^2);$$

i.e. the force is the space-rate of change of the kinetic energy.

Also the power of a working agent = its rate of doing work per second

$$= \frac{dW}{dt} = \frac{dW}{dx} \times \frac{dx}{dt} = Fv,$$

i.e. the force $\times$ the velocity.

Examples XXIII.

1. Given that the work done in stretching an elastic string varies as the square of the extension, prove that the tension of the string is proportional to the extension.

* If δF is $-$, the inequality signs will be reversed.

2. The distance of a moving point from a fixed point in its line of motion is given by the equation $s = 4+5t+t^2$; prove that the force causing the motion is constant.

3. The kinetic energy of a moving particle varies inversely as its distance from the origin; find the force acting on the particle.

4. A particle moves in a straight line so that its distance s from a fixed point of the line at the end of t secs. is given by the equation
$$v^2 = \mu(a^2-s^2).$$
Find the maximum force upon it during the motion.

5. A curve is plotted to show the velocity of a moving point as a function of the distance travelled. Show that the subnormal represents the acceleration.

6. A particle of mass $\frac{1}{4}$ lb. moves in a straight line so that its distance s from the origin at the end of time t is given by the equation
$$s = 5+4\sin 3t.$$
Find the maximum force upon it during the motion.

66. Relation between velocities in different directions.

If a point be moving in a plane, its coordinates (x, y) are functions of the time t; if $(x+\delta x,\ y+\delta y)$ be its coordinates at time $t+\delta t$, then
$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta t}\Big/\frac{\delta x}{\delta t}.$$
Hence when δt, and therefore δx and $\delta y \longrightarrow 0$, we have
$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{\dot{y}}{\dot{x}},$$
i.e. the d. c. of y with respect to x is equal to the ratio of the rate of increase of y to the rate of increase of x, both taken with respect to the time.

The relation between the time-rates of increase of two variables x and y can also be obtained directly by differentiating, with respect to the time, the equation which connects x and y. The method is illustrated in the following examples:

(i) *Two straight roads* OA, OB *intersect at right angles, and a house at* B *is* 4 *miles distant from* O*; a man walks towards* O *along the other road* AO *at the rate of* 4 *miles an hour; find the rate at which he is approaching the house, at the instant when he is* 3 *miles from* O.

If x and u denote his distances from O and B respectively, we have $u^2 = x^2+16$.

Differentiating with respect to t,
$$2u\frac{du}{dt} = 2x\frac{dx}{dt}, \quad \text{or} \quad \frac{du}{dt} = \frac{x}{u}\frac{dx}{dt}.$$
At the given instant, $x = 3$, $u = 5$, and dx/dt, his velocity along the road, $= -4$ [$-$, since x is decreasing]. Therefore
$$du/dt = \tfrac{3}{5}\times -4 = -2{\cdot}4 \text{ miles per hour},$$
i.e. u is diminishing, and he is approaching the house at the rate of 2·4 miles per hour at that particular instant.

(ii) *Suppose that the roads in the preceding example are inclined at* 60°.

In this case, by elementary trigonometry $(a^2 = b^2 + c^2 - 2\,bc\cos A)$, we have

$$u^2 = x^2 + 16 - 2\,.\,4\,x\cos 60^\circ = x^2 + 16 - 4\,x.$$

Differentiating with respect to t,

$$2\,u\frac{du}{dt} = 2\,x\frac{dx}{dt} - 4\frac{dx}{dt} = 2\,(x-2)\frac{dx}{dt},$$

$$\therefore\quad \frac{du}{dt} = \frac{x-2}{u}\cdot\frac{dx}{dt}.$$

When $x = 3$, $u^2 = 13$ and $u = \sqrt{13}$. Therefore

$$du/dt = 1/\sqrt{13} \times dx/dt = 1/\sqrt{13} \times -4 = -1\tfrac{1}{9} \text{ nearly.}$$

The man is approaching the house at the rate of $1\frac{1}{9}$ miles per hour.

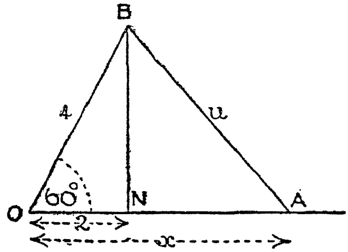


Fig. 71.

When he is 2 miles from O, i.e. when $x = 2$, we have $du/dt = 0$. In this case, it is evident geometrically that the man is at N, the foot of the perpendicular from B to OA, and hence at this particular instant he is not approaching B. Notice that in this case, u, his distance from B, is a minimum, and hence (Art. 53) it follows that $du/dt = 0$ at this point, and changes from $-$ to $+$. Before reaching N, du/dt is $-$, and he is approaching the house; after passing N, du/dt is $+$, and he is receding from the house.

[The student of mechanics will observe that in both cases the velocity of approach is simply the component of the man's actual velocity resolved along AB.]

67. Velocity along the arc of a curve.

It will be shown in Art. 82 that if s be the length of the arc of a curve, measured from a fixed point on the curve to a point P whose coordinates are (x, y), then, ultimately, when δx, δy, and δs are very small,

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2.$$

If the point P be supposed to move along the curve, s, x, and y are all functions of t. Dividing by $(\delta t)^2$, δt being the time taken to traverse the arc δs,

$$\left(\frac{\delta s}{\delta t}\right)^2 = \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2.$$

Therefore in the limit when δt and therefore also δs, δx, $\delta y \rightarrow 0$, we have

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

ds/dt is the rate at which the point is moving along the curve; dx/dt and dy/dt are the time-rates of increase of the abscissa and ordinate of the point, i.e. the velocities parallel to the axes of coordinates. This is the well-known result that the square of the resultant velocity of a point is equal to the sum of the squares of its component velocities in two directions at right angles.

Examples:

(i) *The coordinates of a moving point at the end of time* t *are given by the equations* $x = a \cos nt$, $y = a \sin nt$; *prove that the point describes a circle with velocity of constant magnitude.*

The path of the moving point is obtained by merely eliminating t from the given equations. Squaring and adding, we have at once $x^2 + y^2 = a^2$, which is the equation of a circle.

To find the velocity, we have, on differentiating with respect to t,

$$\dot{x} = -an \sin nt, \quad \dot{y} = an \cos nt.$$

Therefore, again squaring and adding,

$$\dot{x}^2 + \dot{y}^2 = a^2 n^2, \quad \text{i. e.} \quad \dot{s}^2 = a^2 n^2 \quad \text{and} \quad \dot{s} = \pm an,$$

which gives the resultant velocity $\dot{s}$ of constant magnitude.

The sign will depend upon the position of the point from which s is measured.

(ii) *A point moves in a parabola so that its velocity parallel to the axis of the curve is constant; find its velocity along the curve.*

Differentiating the equation of the parabola, $y^2 = 4ax$, with respect to t, we have
$$2y\, dy/dt = 4a\, dx/dt.$$

dx/dt is given to be constant; if it be equal to u, then $y\, dy/dt = 2au$.

$$\therefore \quad \dot{s}^2 = \dot{x}^2 + \dot{y}^2 = u^2 + 4a^2u^2/y^2$$
$$= u^2(1 + 4a^2/y^2) = u^2(1 + a/x);$$

$\therefore$ the velocity along the curve

$$ds/dt = \pm u\sqrt{(1 + a/x)}.$$

(iii) *A point is moving in an ellipse of eccentricity* $\frac{3}{4}$ *with a velocity of* 40 *ft. secs.; find its component velocities parallel to the axes of the ellipse when it is at an extremity of a latus rectum.*

Here $\dot{s} = 40$, and the values of $\dot{x}$ and $\dot{y}$ are required.

Taking the equation $x^2/a^2 + y^2/b^2 = 1$, and differentiating with respect to t, we have

$$\frac{2x}{a^2}\frac{dx}{dt} + \frac{2y}{b^2}\frac{dy}{dt} = 0, \quad \text{or} \quad \frac{dy}{dt} = -\frac{b^2x}{a^2y}\frac{dx}{dt}.$$

At the end of a latus rectum, $x = \pm ae$, $y = \pm b^2/a$ [pp. 19, 24];

$$\therefore \quad \frac{dy}{dt} = -\frac{b^2(\pm ae)}{a^2(\pm b^2/a)} \cdot \frac{dx}{dt} = \pm e \frac{dx}{dt} = \pm \frac{3}{4} \frac{dx}{dt}.$$

Hence $$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2 + \tfrac{9}{16}\dot{x}^2 = \tfrac{25}{16}\dot{x}^2,$$

$$\therefore \quad \dot{x} = \pm \tfrac{4}{5} \times \dot{s} = \pm \tfrac{4}{5} \times 40 = \pm 32 \text{ ft. secs.},$$

and $$\dot{y} = \pm \tfrac{3}{4}\dot{x} = \pm 24 \text{ ft. secs.}$$

There are four possible arrangements of signs which will be associated with the four extremities of the latera recta.

Examples XXIV.

1. The rectangular coordinates of a moving point at the end of time t are given by the equations $x = 30t$, $y = 40t - 16t^2$; find the resultant velocity at the end of $2\frac{1}{2}$ secs.

2. The coordinates of a moving point at the end of time t are given by the equations $x = a + c\cos t$, $y = b + c\sin t$; prove that the resultant velocity and acceleration are of constant magnitude.

3. A man walks at 4 miles an hour on a horizontal plane towards a column 100 ft. in height; at what rate is he approaching the top of the column, at the instant when he is 75 ft. from it?

4. Two straight lines of railway are inclined at an angle of 120°, and a train on one line is travelling towards the junction A at 40 miles per hour. At what rate is it approaching a station on the other line 2 miles from A, at the instant when it is also 2 miles from A.

5. A man 6 ft. high walks at the rate of 6 ft. per sec. along a horizontal pavement lighted by a lamp 10 ft. vertically above it; find the rate at which the length of his shadow on the pavement changes.

6. Find the rate, in the preceding question, if the pavement is an incline of 1 in 10, and the man is walking up it towards the lamp.

7. A ladder, 34 ft. long, rests in a vertical plane with one end on a horizontal road and the other against a vertical wall. If the lower end is pulled away from the wall with a velocity of 10 ft. per min., find the rate at which the upper end is descending at the instant when the foot of the ladder is 16 ft. from the wall.
   When will the ends be moving with equal velocities?
   When will the upper end be descending at the rate of 20 ft. per min.?

8. A truck is drawn along a straight horizontal road by a rope which passes round a windlass 14 ft. vertically above a point A of the road. If the rope is wound in at the rate of 20 ft. per min., find the velocity of the truck when it is 48 ft. from A.

9. A man standing on a quay draws a boat towards him by means of a rope which he is pulling in at the rate of $1\frac{1}{2}$ ft. per sec. At what rate is the boat moving when there are still 25 ft. of rope out, the man's hands being 7 ft. above the level of the water?

10. Two rings P and Q, connected by a rod 20 ft. long, slide along two fixed wires OA, OB at right angles. If P be made to move along OA at the rate of 5 ft. per min., when is Q moving along BO at the rate of 2 ft. per min.?

11. If in the preceding question OA and OB are inclined at an angle $\cos^{-1}\frac{1}{8}$, find the velocity of Q along BO when it is 15 ft. from O.

12. Two trains start from the same station at the same time. One goes due north at 30 miles per hour, and the other due north-east at 20 miles per hour. At what rate is the distance between them increasing after $\frac{1}{4}$ hour?

13. If the slower train starts at 1.50 p.m., and the other at 2 p.m., at what rate is the distance between them increasing at 2.10 p.m.?

14. Two straight roads intersect at right angles. A man walking along one of the roads at 3 miles per hour passes the crossing at 2 o'clock, and another man walking along the other road at 4 miles per hour passes it at half-past two. Find the rate at which the distance between the men is changing (i) at a quarter to 2, (ii) at 3 o'clock.

15. A straight road passes over a river which runs at right angles to it by means of a bridge 50 ft. high. A boat travelling at the rate of 4 ft. per sec. passes under the middle of the bridge one minute before a man walking at 6 ft. per sec. along the road reaches the middle of the bridge. Find the rate at which the distance between the boat and the man is changing (i) 1 minute before the boat reaches the bridge, (ii) $\frac{1}{2}$ minute after the man has passed the middle of the bridge.

16. A, B, C are three villages. The distance from A to B is 5 miles, from B to C 4 miles, and from C to A 3 miles. A man walks from A to B, then on to C, and back directly from C to A at 4 miles per hour; find, when half-way between each pair of villages, the rate at which he is approaching or receding from the third village.

17. A man walks round a circular track with constant velocity, and his shadow, cast by the sun, always intersects the diameter of the track perpendicular to it. Prove that the rate at which his shadow moves along this diameter varies as his distance from it.

18. A lighthouse is one mile from the nearest point A of a straight line of shore, and the light revolves twice per minute; how fast is the light travelling along the shore (i) at A, (ii) $\frac{1}{2}$ mile from A, (iii) 1 mile from A?

19. A man walks round a circle of radius 20 yds. at the rate of 4 ft. per sec., and a light at the centre of the circle throws his shadow on a straight wall built along a tangent line to the circle. Find the velocity with which his shadow moves along the wall (i) when he is 5 yds. from it, (ii) when he is 8 yds. from it.

20. A right circular cone is filled with water and placed with its axis vertical and vertex downwards. If the water flows out at a constant rate of 5 c. in. per sec. through a hole at the vertex, at what rate is the surface of the water descending at the instant when the radius of the surface is 6 inches?

21. The ends of the water reservoir of a town are vertical, the sides slope at an angle of 45°, and the bottom is a horizontal rectangle 200 yds. by 80 yds. If the water-level is sinking at the rate of 5 ft. per day (and no more water is running in), at what rate per day is water being supplied to the town at the instant when the water is 20 ft. deep?

22. A piston slides freely in a circular cylinder of radius 9 inches; at what rate is the piston moving when steam is being admitted into the cylinder at the rate of 22 c. ft. per sec.?

23. The diameter of a sphere increases from 4 in. to 12 in. in 10 minutes, equal volumes being added in equal intervals of time. Find the radius after 5 minutes. At what rate is the radius then increasing?

24. The area of a circle increases from 16 sq. in. to 100 sq. in. in 20 secs., equal areas being added in equal intervals of time. Find the radius after 15 seconds. At what rate is the radius then increasing?

25. A trough, whose cross-section is an isosceles triangle, is a ft. long and b ft. broad at the top. Water is poured into it at the rate of k c. ft. per sec. At what rate is the water rising in the trough when it is one quarter full?

26. A point moves in the parabola $y^2 = 12x$, so that its velocity parallel to the axis of the curve is everywhere 10 ft. per sec. Find its velocity perpendicular to this axis and the resultant velocity, when at the point (3, 6).

27. The path of a moving point is the curve $y = 10 \sin 2x$. If the velocity parallel to the axis of x is constant and equal to 5 ft. per sec., find the resultant velocity at the point whose abscissa is $\frac{5}{12}\pi$.

28. In the preceding question, prove that the acceleration at any point is proportional to the ordinate of the point.

29. A point moves in the ellipse $9x^2 + 16y^2 = 288$; if, at the point (4, 3), the velocity parallel to the minor axis be 10 ft. per sec., find the velocity parallel to the major axis, and the resultant velocity.

30. If, in the cycloid (Art. 50), the angle θ is increasing at the rate of 1 radian in 10 seconds, find the velocity of the point P along the arc at the instants (i) when $\theta = \frac{1}{2}\pi$, (ii) when $\theta = \frac{2}{3}\pi$, the radius of the generating circle being 20 inches.

31. A point is moving in the parabola $y^2 = 12x$ at the rate of 10 ft. per sec.; find its component velocities parallel to the axes when it is at the point (3, 6).

32. A point is moving in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with constant velocity u ft. secs.; find its velocities parallel to the axes at any point.

33. A man walks at 4 miles per hour along a road in the form of a parabola whose equation is $x^2 = 25y$, and whose axis is due north and south. Find his velocities due N. and due E., when he is 1 mile N. of the vertex.

68. Angular velocity and acceleration about a point. Motion in a circle.

If θ be the inclination to a fixed straight line OA of the line joining a moving point P to a fixed point O, the time-rate of increase of the angle θ, i.e. $d\theta/dt$ or $\dot{\theta}$, is called the *angular velocity* of the point P about the point O, or the angular velocity of the straight line OP.

Similarly, if ω denote the angular velocity of P about O, the time-rate of increase of ω, i.e. $d\omega/dt$ or $\dot{\omega}$, is called the *angular acceleration* of P about O. Since $\omega = \dot{\theta}$, the angular acceleration of P about O may be written $d^2\theta/dt^2$ or $\ddot{\theta}$.

Let a particle describe a circle of radius r about a point O (Fig. 72). Let P be the position of the particle at any instant when the radius OP makes an angle θ with a fixed radius OA, and Q its position after time δt, during which the radius turns through an angle $\delta\theta$.

If s be the length of the arc AP, the velocity v of P in the direction of the tangent PT is $\dot{s}$ or $r\dot{\theta}$, since $s = r\theta$, and r is constant.

The velocity of P in the direction of the normal PO is zero, for the particle is moving in the direction perpendicular to PO.

If $v+\delta v$ be the velocity at Q, this may be resolved into $(v+\delta v)\cos\delta\theta$ and $(v+\delta v)\sin\delta\theta$, parallel to PT and PO respectively.

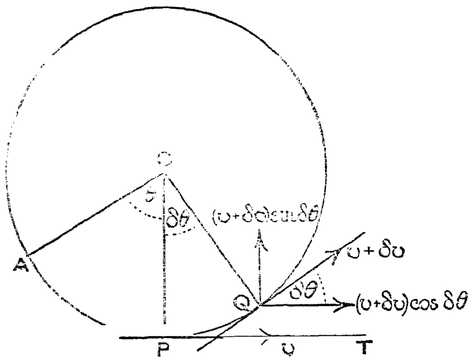


Fig. 72.

The acceleration in any direction is the time-rate of change of velocity in that direction. Therefore acceleration at P along the tangent PT

$$=\underset{\delta t\to 0}{\mathrm{Lt}}\frac{(v+\delta v)\cos\delta\theta-v}{\delta t}=\underset{\delta t\to 0}{\mathrm{Lt}}\left[\frac{\delta v}{\delta t}\cos\delta\theta-v\frac{(1-\cos\delta\theta)}{\delta t}\right].$$

Now $$\frac{1-\cos\delta\theta}{\delta t}\quad\frac{1-\cos\delta\theta}{}\times\frac{\delta\theta}{\delta t}$$

which, as $\delta t\to 0$, tends to the limit $0\times\dot\theta$, i.e. zero [Art. 13 (10).]

$$\therefore\ \text{the acceleration along } PT=\mathrm{Lt}\frac{\delta v}{\delta t}\cos\delta\theta=\frac{dv}{dt}.$$

This may also be written d^2s/dt^2 or $r\,d^2\theta/dt^2$, i.e. $\ddot s$ or $r\ddot\theta$.

Since $dv/dt=dv/ds\times ds/dt$, it may also be put in the form $v\,dv/ds$, just as in the case when the particle is moving in a straight line.

The acceleration at P in the direction PO

$$=\underset{\delta t\to 0}{\mathrm{Lt}}\frac{(v+\delta v)\sin\delta\theta}{\delta t}$$

$$=\mathrm{Lt}\,(v+\delta v)\cdot\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta t},\ \text{as } \delta t \text{ and therefore } \delta\theta\to 0,$$

$$=v\times 1\times d\theta/dt$$

$$=r\dot\theta^2 \text{ or } v^2/r.$$

Hence, if θ denote the angle which the radius through the particle makes with a fixed radius, the components of the velocity and acceleration of the particle along the tangent and normal are $r\dot{\theta}$, 0 and $r\ddot{\theta}$, $r\dot{\theta}^2$ respectively, and, if m be the mass of the particle, the forces acting on it are equivalent to $mr\ddot{\theta}$, $mr\dot{\theta}^2$ along the tangent and normal respectively.

In the particular case when the particle is moving *uniformly* in the circle, $\dot{\theta}$ is constant. Therefore $\ddot{\theta}$ is zero, and the resultant acceleration is $r\dot{\theta}^2$ along the radius; in this case the force on the particle necessary to keep it moving uniformly in its circular path is $m\omega^2 r$ or mv^2/r towards the centre of the path.

69. Crank and connecting-rod.

The relations between the motions of connected parts of a machine can often be obtained by the use of these principles as illustrated in the following example, which is of rather greater difficulty than those hitherto considered.

A crank OQ (Fig. 73) *rotates about* O *with constant angular velocity* ω, *and a connecting-rod* QP *is hinged to it at one end* Q, *while the other end* P *moves along a fixed straight line* OX; *determine the motion of* P. *If a perpendicular from* O *to* OX *meet* PQ *produced in* R, *prove that* OR *and* QR *represent in magnitude the velocity of* P *and the angular velocity of* PQ *respectively. Find also the acceleration of* P.

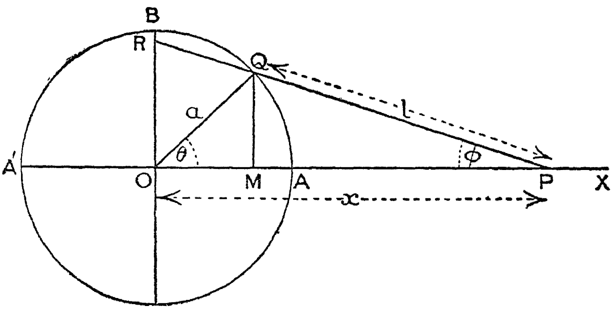


Fig. 73.

Let a and l be the lengths of OQ and PQ, and θ and ϕ the angles QOX, QPO respectively. Draw QM perpendicular to OX.

As Q moves round the circle, P moves backwards and forwards along the line OX. If x denote the distance OP, we have

$$x = OM + MP = a\cos\theta + l\cos\phi.$$

Also $$l\sin\phi = QM = a\sin\theta.$$

Differentiating both equations with respect to t,

$$\dot{x} = -a\sin\theta\,.\,\dot{\theta} - l\sin\phi\,.\,\dot{\phi}.$$

$$l\cos\phi\,.\,\dot{\phi} = a\cos\theta\,.\,\dot{\theta}.$$

Now $\dot{\theta}$, the angular velocity of OQ, is ω;

$$\dot{\phi} = \frac{a\cos\theta}{l\cos\phi}\qquad\frac{OM}{MP}\qquad\frac{RQ}{QP} = \frac{\omega}{l}\,.\,RQ.$$

Also, substituting in (i) the value of $\dot{\phi}$ obtained from (ii)

$$\dot{x} = -a\sin\theta\,.\,\omega - l\sin\phi\,.\,\frac{a\cos\theta}{l\cos\phi}\,.$$

$$= -a\omega\left(\sin\theta + \frac{\sin\phi\,.\,\cos\theta}{\cos\phi}\right)$$

$$= -a\omega\sin(\theta+\phi)/\cos\phi$$

$$= -a\omega\sin OQR/\sin ORQ$$

$$= -a\omega\, OR/a$$

$$= -\omega\, OR.$$

Hence $\dot{\phi}$ and $\dot{x}$ are represented by RQ and OR respectively.

To find the acceleration of P, take the equation just obtained,

$$\dot{x} = -a\omega\sin(\theta+\phi)/\cos\phi,$$

and differentiate with respect to t;

$$\ddot{x} = -a\omega\,\frac{\cos\phi\,.\,\cos(\theta+\phi)\,[\dot{\theta}+\dot{\phi}] - \sin(\theta+\phi)\,.\,(-\sin\phi)\,\dot{\phi}}{\cos^2\phi}$$

$$\cos^2\phi\,[\dot{\theta}\cos\phi\cos(\theta+\phi) + \dot{\phi}\,\{\cos\phi\cos(\theta+\phi) + \sin\phi\sin(\theta+\phi)\}]$$

$$-\frac{a\omega}{\cos^2\phi}\left[\omega\cos\phi\cos(\theta+\phi) + \frac{a\cos\theta}{l\cos\phi}\,\omega\,.\,\cos\theta\right]$$

$$\frac{a\omega^2}{\cos\phi}\left[\cos(\theta+\phi) + \frac{a}{l}\,\frac{\cos^2\theta}{\cos^2\phi}\right].$$

This equation gives the acceleration in any position in terms of θ and ϕ, which can both be found when either of them or when x is given.

When Q is at A, $\theta = \phi = 0$, and $\ddot{x} = -a\omega^2(1+a/l)$.

When Q is at A', $\theta = \pi$, $\phi = 0$, and $\ddot{x} = -a\omega^2(-1+a/l) = a\omega^2(1-a/l)$.

If l be large compared with a, so that a/l may be neglected, the accelerations of P when Q is at A and A' become nearly $-a\omega^2$ and $a\omega^2$ respectively, i. e. approximately the same as if the motion of P were simple harmonic. In this case, the angle ϕ is always small, and the general expression for $\ddot{x}$ becomes approximately $-a\omega^2\cos\theta$, i. e. $-\omega^2\,OM$. Since PQ makes a small angle with OX, the distance of P from the centre of its path is very nearly equal to OM; hence the motion of P approximates to simple harmonic motion.

Examples XXV.

1. A point P moves with uniform velocity u along a straight line OX; find the angular velocity of P about a point A at a perpendicular distance a from OX.
2. A point P describes a circle, centre O and radius r, with uniform angular velocity ω, and a point B is taken on a radius OA produced, at distance a from O; find the angular velocity of P about B (i) when P is at A, (ii) when P is at an extremity of the diameter perpendicular to OA.
3. The lengths of a crank and connecting-rod (as in Art. 69) are 6 inches and 2 feet respectively; find the accelerations of P when Q is at A and at A', the crank making 100 revolutions per minute.
4. A crank OQ is made to rotate uniformly about O, and the end P of a connecting-rod PQ moves in a straight line at a perpendicular distance b from O; find the velocity and acceleration of P in any position.
5. A rod AB turns about one end A with uniform angular velocity ω; it is freely jointed at B to another rod BD which is constrained to pass through a fixed ring C; if AE be drawn perpendicular to BC, prove that at any instant the velocity of the point of the rod which is passing through the ring is $AE.\omega$, and that the angular velocity of BD is equal to $\omega BE/BC$.
6. In Art. 69, prove that the velocity of P is equal to the velocity of $Q \times PN/PM$ where PN is the perpendicular from P to OQ.

CHAPTER IX

SIMPLE INTEGRATION WITH APPLICATIONS

70. Introductory.

In the preceding chapters we have shown how to find the differential coefficient or rate of change of a given continuous function of x. In many branches of mathematics, both pure and applied, we are frequently confronted with the inverse problem, viz. given the rate of change of a function, to find the function. This process of finding a function which has a given rate of change is known as **integration.**

An integral may be defined in two quite distinct ways; either as the inverse of a differential coefficient or as the limit of the sum of a certain series. The former of these two definitions is the one which leads to the methods of evaluating integrals; the latter is the one upon which many of the simpler applications depend, but it does not yield convenient methods of evaluation. We shall, therefore, begin by considering the first definition, and later, when we have learned how to calculate the simpler forms of integrals, we shall consider the second definition and show the relation between the two kinds of integrals.

71. Definitions.

The integral of a function $\mathrm{f(x)}$ *with respect to* x *is the function whose differential coefficient with respect to* x *is* $\mathrm{f(x)}$, *and is written* $\int \mathrm{f(x)\,dx}$.

The symbol $\int$ is really an elongated S, the first letter of the word 'sum', and the necessity for the insertion of the factor dx will be seen when we come to consider integrals from the second point of view mentioned above. At present it may be regarded as part of the symbol of integration, indicating the variable with respect to which the required function must be differentiated in order to give $f(x)$, so that $\int \ldots dx$ stands for 'the integral of ... with respect to x'.

Generally,

$$\textit{if } \frac{\mathrm{d}}{\mathrm{dx}}\mathrm{F(x)} \textit{ be denoted by } \mathrm{F'(x)}, \textit{ then } \int \mathrm{F'(x)\,dx} = \mathrm{F(x)}.$$

Briefly, in the differential calculus, the first problem to consider is: Given y, find dy/dx; in the integral calculus, the first problem to consider is the converse of this, viz. given dy/dx, find y.

E.g. the d. c. of x^3 with respect to $x = 3x^2$, hence $\int 3x^2\,dx = x^3$; the d. c. of $\tan x$ with respect to $x = \sec^2 x$, hence $\int \sec^2 x\,dx = \tan x$. Unfortunately, there is no general method of retracing the steps in the process of differentiation. In the calculation of differential coefficients, terms are frequently added and subtracted, and factors multiplied together or cancelled out, and when the final result only is given, there are no means of recovering these terms and factors. A few of the commonest and simplest integrals are collected from knowledge of differential coefficients; these are usually referred to as 'Standard Forms'.*

The first part of the Integral Calculus then consists, in the eyes of the beginner, of a collection of various haphazard methods and devices by means of which other expressions can be reduced to one or other of these standard forms or to some combination of them;† and the degree of difficulty experienced by the student in dealing with these will depend to a great extent upon the thoroughness of his knowledge of the substitutions and formulas of elementary algebra and trigonometry.

For the discussion as to whether a function always possesses an integral or in what cases a function possesses an integral, the student is referred to more advanced works. It can be shown that every continuous function has an integral. The functions which are encountered in elementary applications generally possess integrals which can be expressed in terms of the functions we have already considered, together with those which will be dealt with in the next chapter; but there are many comparatively simple functions whose integrals cannot be expressed in terms of such functions, e. g. $(1-2\sin^2 x)^{-1/2}$, $\sqrt{(1-x^3)}$, $\sqrt{\sin x}$, $(\cos x)/x$, cannot be integrated in terms of such functions.

72. Arbitrary constant. Indefinite integral.

The first point to notice is that the above definition does not give a perfectly definite value for the integral; since the d. c. of a constant is zero, it follows that the integral of $F'(x)$ with respect to x is not necessarily $F(x)$ only, but is $F(x)+C$, where C is an arbitrary constant, i. e. any quantity whatever which does not involve x. On this account these integrals are often referred to as 'indefinite integrals'.

* These Standard Forms *must be committed to memory.* It is not really necessary to remember more than a dozen or so, but these must be thoroughly known, and the student must be able to recognize them at once whenever and in whatever form they occur.

† These methods and devices are not really as disconnected as they appear at first sight to the student. See G. H. Hardy's *Integration of Functions of a Single Variable*, Cambridge Tracts in Mathematics, No. 2.

Taking the cases mentioned above, the d. c. of x^3 is $3x^2$, but so also is the d. c. of x^3+4, x^3-10, x^3+a^3, or x^3+any expression which does not involve x. Therefore

$$\int 3x^2 dx = x^3 + C,$$

where C is arbitrary, except only that it is independent of x. Similarly,

$$\int \sec^2 x\, dx = \tan x + C.$$

This arbitrary constant C is usually omitted, but the student must not become oblivious of its existence. In practical examples, as will be seen later on, it often plays a very important part.

It should be noticed at this stage that if a pair of simultaneous values of the function and its integral be given, then the constant ceases to be arbitrary and can be determined.

E. g. given that $dy/dx = 2x-2$, and that $y=3$ when $x=2$, find y in terms of x.

We have

$$\begin{aligned} y &= \int (2x-2)\, dx \\ &= x^2 - 2x + C, \qquad \text{(i)} \end{aligned}$$

since this is the expression which gives $2x-2$ when differentiated.

It is given that $y=3$ when $x=2$, hence, substituting these values, we have

$$3 = 4 - 4 + C,$$

whence $C = 3$, and therefore, substituting in (i),

$$y = x^2 - 2x + 3.$$

73. Geometrical interpretation.

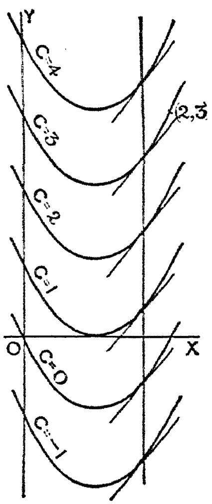


Fig. 74.

The geometrical meaning of this process should be noticed. We have to find y in terms of x, given the value of dy/dx, i.e. of the slope of the curve, in terms of x. Hence we have to find a curve, given the slope at any point in terms of the abscissa.

In the example just considered, equation (i) represents the curves which have the given slope. It obviously represents a system or 'family' of parabolas (Fig. 74). If two different values for C be taken, the ordinates of the two corresponding curves will at each point differ by a constant amount (the difference between the two values of C), i. e. the curves are at a constant distance apart measured along the ordinate. All such curves possess the given slope, i. e. the tangents at the points where they are cut by a straight line parallel to the axis of y are all parallel; hence the arbitrary term C.

The fact that $y=3$ when $x=2$ enables us to select one particular curve of the family, viz. the one which passes through the point (2, 3). There is no arbitrary element now; there is but *one* curve which passes through this particular point and has the given slope.

74. Integral of x^n.

The first and most important standard form to be considered is $\int x^n dx$. What function gives x^n when differentiated? The d. c. of x^{n+1} is $(n+1)x^n$, and therefore the d. c. of $x^{n+1}/(n+1)$ is x^n; hence

$$\int x^n\,dx = x^{n+1}/(n+1).$$

This is true for all rational values of n, $+$ or $-$, integral or fractional (Art. 27), with the single exception of $n=-1$; in this case the integral becomes $\int x^{-1}dx$. We have not yet obtained $1/x$ as the d. c. of any function, but in the next chapter we shall find that it is the d. c. of $\log_e x$.

E.g.
$$\int x^9\,dx = \tfrac{1}{10}x^{10}$$
$$\int 1/x^4 . dx = \int x^{-4}\,dx = x^{-3} \div -3 = -1/(3x^3).$$
$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = x^{3/2} \div \tfrac{3}{2} = \tfrac{2}{3}\sqrt{x^3}.$$
$$\int 1/\sqrt[3]{x} . dx = \int x^{-1/3}\,dx = x^{2/3} \div \tfrac{2}{3} = \tfrac{3}{2}\sqrt[3]{x^2}.$$

Since the d. c. of the sum of a finite number of functions is the sum of their d. c.'s, it follows conversely that the integral of the sum of a finite number of functions is the sum of their integrals separately.

Moreover, since $\dfrac{d}{dx}af(x)$, where a is a constant, is $af'(x)$, it follows that

$$\int af'(x)\,dx = af(x) = a\int f'(x)\,dx;$$

hence a constant factor can be brought outside the integration sign.

These facts enable us to write down at once the integral of any polynomial in x with constant coefficients.

Examples:

$$\int(10x^4-9x^2+5)\,dx = 10 . \tfrac{1}{5}x^5 - 9 . \tfrac{1}{3}x^3 + 5x + C = 2x^5 - 3x^3 + 5x + C.$$
$$\int(ax^2+bx+c)\,dx = a . \tfrac{1}{3}x^3 + b . \tfrac{1}{2}x^2 + cx + d.$$
$$\int(2x-1)^3\,dx = \int[8x^3-12x^2+6x-1]\,dx = 2x^4-4x^3+3x^2-x+C.$$
$$\int\frac{x^2+x+1}{\sqrt{x}}\,dx = \int(x^{3/2}+x^{1/2}+x^{-1/2})\,dx = \frac{x^{5/2}}{\frac{5}{2}} + \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C.$$
$$= \tfrac{2}{5}\sqrt{x^5} + \tfrac{2}{3}\sqrt{x^3} + 2\sqrt{x} + C.$$

In future, the constant C will be omitted.

√Examples XXVI.

Write down the integrals of the expressions in the following examples 1–14.

1. x^6, $21x^6$, $1/x^6$, $10/x^6$, $\sqrt[6]{x}$, $1/\sqrt[6]{x}$. 2. $3x^2-2x+1$.

3. $1/x^2$, $1/x^{10}$, $1/\sqrt{x}$, $1/\sqrt[3]{x}$. 4. $\sqrt[n]{x}$, $1/x^n$, $1/\sqrt[n]{x}$.

5. $x^3+6x^2+10x-5$; $7x^6-10x^4+9x^2-1$.

6. $\dfrac{1}{x^3}+\dfrac{6}{x^2}-5$; $\dfrac{7}{x^6}-\dfrac{10}{x^4}+\dfrac{9}{x^2}-1$.

7. $x^5-4x^3+2x^2-x+3$; $x^6-4x^4+6x^2-8$.

8. $ax^4+bx^3+cx^2+dx+e$; $a/x^4+b/x^3+c/x^2+d$.

9. $\dfrac{6+2x+x^2}{x^2}$ $\dfrac{1+3x+5x^2}{\sqrt{x}}$. 10. $\dfrac{1+x^2+2x^4}{2x^2}$; $\dfrac{1+x^2+2x^4}{\sqrt[3]{x}}$.

11. $(1-3x)^2$; $(1+x^2)^3$. 12. $\dfrac{(ax+b)^2}{x^4}$; $\left(\dfrac{3+4x^2}{x}\right)^2$.

13. $\dfrac{ax^2+bx+c}{n}$ $x^{2n}+x^n a^n+a^2$ $\dfrac{x^m+x^n}{x^p}$. 14. $\dfrac{ax^2+b}{\sqrt{x}}$.

15. Find y in terms of x, given $dy/dx = 8x^3-2x$, and that $y=8$ when $x=2$.

16. Find y in terms of x, given $dy/dx=\sin x$, and that $y=2$ when $x=\frac{1}{3}\pi$.

17. Obtain the equations of the curves in which the slope at any point (x, y) is $3-4x$. Illustrate graphically.

18. In what curves is the slope at a point (x, y) equal to $2-3/x^2$?

19. Find the equation of the curve which passes through the point $(3, 1)$, and has at any point (x, y) the slope x^2-x.

20. Find the equation of the curve whose slope at any point (x, y) is $1/\sqrt{x}$, and which passes through the point $(4, 5)$.

21. The slope at any point (x, y) of a curve is equal to $\cos x$, and the curve passes through the point $(0, 1)$; find its equation.

22. What curve through the origin has its slope given by the equation $dy/dx=(1+x)^2$?

23. From any point P on a curve a perpendicular PN is drawn to the axis of y, and the tangent at P meets the axis of y in T. Find the equation of the curves in which the rectangle $PN.NT$ has a constant value c^2.

24. If in the preceding question the normal at P cuts OY in G, in what curves is NG constant?

25. Find the function whose rate of change per unit increase of x is equal to $6x^2-4x+3$, and which is equal to 10 when x is equal to 1.

26. Find the function whose rate of change with respect to x is inversely proportional to x^2, and which has the values 6 and 10 when x is equal to 1 and 2 respectively.

75. Two important rules.

We next proceed to consider two rules of the utmost importance, which are constantly being used in integration.

If the d. c. of $f(x)$ be denoted by $f'(x)$, we have

$$\int f'(x)\,dx=f(x);$$

the d. c. of $f(x+b)$ is (Art. 34) $f'(x+b)$

$$\therefore \quad \int f'(x+b)\,dx = f(x+b);$$

the d. c. of $f(ax+b)$ is (Art. 34) $f'(ax+b)\times a$

$$\therefore \quad \int f'(ax+b)\,dx = \frac{1}{a}f(ax+b).$$

On examining and comparing these three results, we see that they enable us, when the integral of any function of x is known, to write down the integral of the same function of $ax+b$, where a and b are constants.

They can be put into the following convenient verbal forms:

If the integral of a function of x *is known, then*

(i) *the addition of a constant to* x *makes no difference in the form of the integral;*

(ii) *if* x *is multiplied by a constant, the integral is of the same form, but is divided by the constant.*

Hence, in any function of x, the replacement of x by a linear function of x does not alter the form of the integral of the function.

These two rules, in conjunction with the standard forms of the preceding article, enable us to write down at once the integral of any power or root of $ax+b$.

E.g. $\quad \int x^3\,dx = \frac{1}{4}x^4;$

$\int (x+5)^3\,dx = \frac{1}{4}(x+5)^4;$ the addition of the constant 5 makes no difference to the form of the integral.

$\int (2x+5)^3\,dx = \frac{1}{4\,.\,2}(2x+5)^4;$ x has the constant coefficient 2, therefore the integral is of the same form and is divided by 2.

$\int (1-x)^3\,dx = -\frac{1}{4}(1-x)^4;$

and generally

$$\int (ax+b)^3\,dx = \frac{1}{4a}(ax+b)^4.$$

From the reference to Art. 34 given above, it is clear that this is merely a simple case of the converse of the rule for differentiating a function of a function, and the reason for the insertion of the factor a in the denominator is obvious at once when the result is differentiated so as to give the original function.

E.g. the d. c. of $\quad \frac{1}{4a}(ax+b)^4 = \frac{1}{4a}\times 4(ax+b)^3\times a = (ax+b)^3.$

It must be carefully noticed that the rules only apply when x is replaced by $ax+b$, i.e. by an expression *of the first degree in* x; they give no information about the values of such integrals as

$\int(x^2+1)^3 dx$, where x is replaced by an expression of different degree, or $\int \sin^3 x\, dx$. These integrals are *not* $\frac{1}{4}(x^2+1)^4$ and $\frac{1}{4}\sin^4 x$, as is obvious at once if we differentiate these latter functions.

The integral $\int(x^2+1)^3 dx$ can be obtained by expanding and integrating each term separately, thus

$$\begin{aligned}\int(x^2+1)^3 dx &= \int(x^6+3x^4+3x^2+1)\,dx \\ &= \tfrac{1}{7}x^7+\tfrac{3}{5}x^5+x^3+x+C.\end{aligned}$$

The student should not make mere mechanical applications of these rules. It is important that he should grasp the principle which underlies them, and for this purpose the argument may be presented in a different form as follows:

Suppose the value of $\int(ax+b)^3 dx$ is required.

Then if y denote $\int(ax+b)^3 dx$, we have $dy/dx=(ax+b)^3$.

Let $ax+b=z$; $\therefore$ $a\,dx/dz=1$.

Then $$\frac{dy}{dz}=\frac{dy}{dx}\times\frac{dx}{dz}=(ax+b)^3\times\frac{1}{a}=\frac{1}{a}.z^3;$$

$$\therefore\quad y=\int\frac{1}{a}z^3\,dz=\frac{1}{a}\frac{z^4}{4}=\frac{(ax+b)^4}{4a}.$$

Any particular case can be treated in a similar manner, but such forms occur so frequently that the student should accustom himself to writing the results down at once.

Some further examples of the rules are appended:

$$\int\frac{1}{\sqrt{x}}dx=\int x^{-1/2}\,dx=\frac{x^{1/2}}{\frac{1}{2}}=2\sqrt{x}.$$

$$\int\frac{1}{\sqrt{(x+3)}}dx=2\sqrt{(x+3)}.$$

$$\int\frac{1}{\sqrt{(5x+3)}}dx=2\sqrt{(5x+3)}\div 5=\tfrac{2}{5}\sqrt{(5x+3)}.$$

$$\int\frac{1}{\sqrt{(a-x)}}dx=2\sqrt{(a-x)}\div -1=-2\sqrt{(a-x)}$$

$$\int\frac{1}{\sqrt{(ax+b)}}dx=2\sqrt{(ax+b)}\div a=\frac{2}{a}\sqrt{(ax+b)}.$$

Again, $$\begin{aligned}\int\cos x\,dx&=\sin x;\\ \therefore\quad \int\cos(x+\alpha)\,dx&=\sin(x+\alpha),\\ \int\cos(\alpha-x)\,dx&=-\sin(\alpha-x),\\ \int\cos 3x\,dx&=\tfrac{1}{3}\sin 3x,\\ \int\cos\tfrac{1}{2}x\,dx&=2\sin\tfrac{1}{2}x,\\ \int\cos(px+q)\,dx&=\{\sin(px+q)\}/p.\end{aligned}$$

The student should pay particular attention to these rules, and must be very careful not to omit the dividing factor a. This factor is frequently overlooked when the practice of integration is first begun, and for this reason considerable stress has been laid upon it above.

It is important to notice also that the correctness of an integration can always be tested at once, by differentiating the expression obtained; this should of course give back the function which was to be integrated.

We have so far, from our knowledge of differential coefficients, the following standard forms:

$$\int x^n\,dx = x^{n+1}/(n+1) \text{ (except when } n=-1);$$
$$\int \sin x\,dx = -\cos x;$$
$$\int \cos x\,dx = \sin x;$$
$$\int \sec^2 x\,dx = \tan x.$$

76. An apparent discrepancy.

One other point may be noticed at this stage. In Art. 74, the integral of $(2x-1)^3$ was found by expanding and integrating each term separately, the result being $2x^4-4x^3+3x^2-x$. The integral, as given by the rule of the preceding article, is $\frac{1}{8}(2x-1)^4$. Do these two results agree?

The latter $= \frac{1}{8}(16x^4-32x^3+24x^2-8x+1)$
$= 2x^4-4x^3+3x^2-x+\frac{1}{8}$,

whence we see that the two results differ by $\frac{1}{8}$. But we have already pointed out that in the integral of any expression, an arbitrary constant is to be understood; hence the presence of the term $\frac{1}{8}$ makes no difference to the integral. If, as in Art. 72, we substitute a pair of simultaneous values of x and y in order to obtain y definitely in terms of x, the expressions obtained for y will coincide exactly; the value obtained for the arbitrary constant in the second case will be less by $\frac{1}{8}$ than the value obtained for the arbitrary constant in the first case, and the final results will be identical.

Many expressions can be integrated by two or more different methods, and the results given by these different methods sometimes take different forms, but, on examination, it will be found that the parts involving the variable x are the same in both; the results are either in exact agreement or differ by a constant only.

The formal proof of this statement is as follows:

Let $f(x)$ and $F(x)$ be two functions which have the same d. c.

Then $f'(x) \equiv F'(x)$, i. e. $f'(x)$ and $F'(x)$ are equal *for all values of* x.

$$\therefore\quad f'(x)-F'(x)\equiv 0,$$

i.e. $$\frac{d}{dx}[f(x)-F(x)]\equiv 0,$$

whence we infer that $f(x)-F(x)$ is constant.

Examples XXVII.

Integrate the following expressions:

1. x^2, $(7+x)^2$, $(5-x)^2$, $(3x-4)^2$, $(px+q)^2$.
2. x^n, $(x-a)^n$, $(9x+4)^n$, $(3-2x)^n$, $(ax+b)^n$, $(p-qx)^n$.
3. $\sin x$, $\sin 4x$, $\sin mx$, $\sin \frac{1}{3}x$, $\sin(px+\alpha)$, $\sin(\alpha-2x)$, $\sin(\frac{1}{4}\pi-x)$.
4. $\sqrt{x}$, $\sqrt{(1+x)}$, $\sqrt{(3-4x)}$, $\sqrt{(px+q)}$, $\sqrt{(1+x/a)}$, $\sqrt{(3x)}$, $\sqrt{(mx)}$.
5. $\dfrac{1}{x^2}$, $\dfrac{1}{(2-5x)^2}$, $\dfrac{1}{(7x+2)^2}$, $\dfrac{1}{(a-x)^2}$, $\dfrac{1}{(mx-n)^2}$.
6. $\sec^2 x$, $\sec^2(x+\alpha)$, $\sec^2 mx$, $\sec^2(\alpha+2x)$, $\sec^2(x/m)$, $\sec^2(nx+m)$.
7. $\dfrac{1}{x^n}$, $\dfrac{1}{(4x-5)^n}$, $\dfrac{1}{(1-2x)^n}$, $\dfrac{1}{(c-x)^n}$, $\dfrac{1}{(bx+a)^n}$.
8. $\dfrac{1}{\sqrt[n]{x}}$, $\dfrac{1}{\sqrt[n]{(x+3)}}$, $\dfrac{1}{\sqrt[n]{(2x-5)}}$, $\dfrac{1}{\sqrt[n]{(a-x)}}$, $\dfrac{1}{\sqrt[n]{(nx+c)}}$, $\dfrac{1}{\sqrt[n]{(2x)}}$, $\dfrac{1}{\sqrt[n]{(mx)}}$.
9. $\dfrac{1}{x^4}$, $\dfrac{1}{(x-3)^4}$, $\dfrac{1}{(3-x)^4}$, $\dfrac{1}{(3-7x)^4}$, $\dfrac{1}{(px+q)^4}$.

Evaluate

10. $\int(7y-4)^5\,dy$.
11. $\int(a-bt)^{3/2}\,dt$.
12. $\int\sqrt[n]{(p+qz)}\,dz$.
13. $\displaystyle\int\frac{dz^*}{\sqrt[3]{(5-3z)}}$.
14. $\displaystyle\int\frac{du}{\sqrt{(1-u)^3}}$.
15. $\displaystyle\int\frac{d\theta}{(a-3\theta)^3}$.
16. $\int(a-t)^p\,dt$.
17. $\displaystyle\int\frac{du}{(7-4u)}$.
18. $\displaystyle\frac{dy}{\sqrt[n]{(b-ny)}}$.
19. $\int\cos 3\theta\,d\theta$, $\int\cos\frac{1}{3}\theta\,d\theta$, $\int\cos(\alpha-\theta)\,d\theta$, $\int\cos(n\theta+\alpha)\,d\theta$.
20. $\displaystyle\int\frac{dz}{\sqrt{z^3}}$, $\displaystyle\int\frac{dy}{\sqrt{(1-y)^3}}$, $\displaystyle\int\frac{du}{\sqrt{(3u-5)^3}}$, $\displaystyle\int\frac{dv}{\sqrt{(av+b)^3}}$.
21. Given that $dy/dx=(3x-4)^2$, find y in terms of x in two ways, and compare the results. Find from each expression the value of y in terms of x, given that $y=10$ when $x=2$.

We now discuss a few simple applications which involve only such integrals as have just been considered.

77. Applications to geometry.

The integral calculus can be used to find what curves possess a given property. Two simple examples will be given at this stage. Others will occur later.

Examples:

(1) *In what curves does the slope at any point vary inversely as the square of the abscissa of the point?*

Here we have $\qquad dy/dx=k/x^2$.

$\therefore$ integrating $\qquad y=\int k/x^2\,.\,dx=-k/x+C,$

which may be written $\qquad x(y-C)=-k.$

It is customary to write $\displaystyle\int\frac{1}{f(x)}dx$ in the form $\displaystyle\int\frac{dx}{f(x)}$; and similarly $\int 1\,dx$ as $\int dx$.

Hence the curves which possess the given property are rectangular hyperbolas (p. 21).

If $C = 0$, we get a rectangular hyperbola with the axes as asymptotes (the quadrants in which the curve lies depending upon the sign of k); as the value of C is varied (and any value can be assigned to it at will) one of the asymptotes moves parallel to the axis of y.

(ii) *Find the curves in which the subtangent at any point* P *is proportional to the tangent of the angle which* OP *makes with the axis of* x.

The subtangent is (Art. 48) $y \Big/ \frac{dy}{dx}$.

Therefore $$y \Big/ \frac{dy}{dx} = k\frac{y}{x}; \quad \text{i.e. } dy/dx = x/k,$$

and $$y = \int x/k \,.\, dx = x^2/2k + C.$$

Therefore the curves required are parabolas (p. 18) whose axes lie along the axis of y.

78. Application to mechanics.

It has been shown (Art. 62) that if s be the distance, measured from a fixed point of its path, of a moving point at time t (measured from some fixed instant), then the velocity v is equal to ds/dt, and the acceleration is equal to dv/dt, d^2s/dt^2, or $v\,dv/ds$; and several examples were worked in which the velocity and acceleration at any instant were deduced from a given expression for s in terms of t.

By the use of the integral calculus we can reverse this process; i.e. given the acceleration, we can determine, first, the velocity and thence the distance travelled. These examples illustrate the part played by the constant of integration.

Examples:

(i) *A point moves in a straight line with constant acceleration* a; *if* v *be its velocity and* s *its distance from some fixed point in the line at the end of time* t, *find* v *and* s *in terms of* t.

Taking the first of the three expressions for the acceleration (since we want v in terms of t) we have $dv/dt = a$,

$$\therefore \quad v = \int a\,dt = at + C.$$

If $t = 0$, $v = C$; therefore C is the velocity when $t = 0$, i.e. the initial velocity; denoting this by u, we have

$$v = u + at,$$

i.e. $$ds/dt = u + at;$$

$$\therefore \quad s = \int (u + at)\,dt = ut + \tfrac{1}{2}at^2 + C. \qquad \text{(i)}$$

Here C is the value of s when $t = 0$, and therefore depends upon the position of the point from which s is measured. If the starting-point be taken as the origin, then $s = 0$ when $t = 0$; therefore, substituting in (i), we get $C = 0$, and $s = ut + \frac{1}{2}at^2$.

If we take for the acceleration the expression $v\,dv/ds$, we shall get the relation between v and s. We have $v\,dv/ds = a$. Now the d. c. of v^2 with respect to $s = 2v\,dv/ds$ (Art. 31). Therefore integrating the preceding equation with respect to s, we have

$$\tfrac{1}{2}v^2 = as + C. \qquad \text{(ii)}$$

C is the value of $\frac{1}{2}v^2$ when $s = 0$; therefore, taking the starting-point as origin, so that $v = u$ when $s = 0$, we have $\frac{1}{2}u^2 = C$,

and $$\tfrac{1}{2}v^2 = as + \tfrac{1}{2}u^2, \quad \text{i. e.} \quad v^2 = u^2 + 2as.$$

Thus we obtain the three well-known equations of uniformly accelerated rectilinear motion.

If the point starts with velocity u at distance s_0 from the origin, then in finding s in terms of t, $s = s_0$ when $t = 0$; therefore, substituting these values in (i), $s_0 = C$, so that

$$s = s_0 + ut + \tfrac{1}{2}at^2.$$

In finding v in terms of s, $v = u$ when $s = s_0$; therefore, substituting these values in (ii),

$$\tfrac{1}{2}u^2 = as_0 + C;$$

substituting this value of C in (ii),

$$\tfrac{1}{2}v^2 = as + \tfrac{1}{2}u^2 - as_0, \quad \text{i. e.} \quad v^2 = u^2 + 2a(s - s_0).$$

(ii) *A point moves in a straight line under the influence of an acceleration which varies as the square of the time the point has been in motion; find the velocity at any instant and the distance travelled.*

Here $dv/dt = kt^2$, where k is a constant,

$$\therefore \quad v = \int kt^2\,dt = \tfrac{1}{3}kt^3 + C.$$

If u be the initial velocity, $v = u$ when $t = 0$, $\quad \therefore u = 0 + C$;

i. e. $$v = u + \tfrac{1}{3}kt^3.$$

This gives the velocity at the end of time t.

Next $$ds/dt = v = u + \tfrac{1}{3}kt^3;$$

$$\therefore \quad s = ut + \tfrac{1}{12}kt^4 + C.$$

If s be measured from the starting-point, $s = 0$ when $t = 0$;

$$\therefore \quad C = 0, \text{ and } s = ut + \tfrac{1}{12}kt^4.$$

This gives the distance travelled in t seconds.

(iii) *A particle is projected vertically upwards with velocity* 40 *ft. secs., and, in addition to being acted upon by gravity, is subject to a retardation which varies as the time from the commencement of the motion, and which at the end of the first second is equal to* 16 *ft. secs. per second. To what height will the particle ascend?*

The variable retardation is kt and this is equal to 16 when $t = 1$; therefore $k = 16$. Hence the total retardation $= g + kt = 32 + 16t$.

$$\therefore \quad dv/dt = -32 - 16t,$$

and, integrating, $$v = -32t - 8t^2 + C.$$

$v = 40$ when $t = 0$. Substituting these values in the equation just obtained, $40 = C$,

and therefore $$v = 40 - 32t - 8t^2.$$

Again $$ds/dt = v = 40 - 32t - 8t^2.$$

Therefore, integrating, $s = 40t - 16t^2 - \frac{8}{3}t^3 + C$.

$s = 0$ when $t = 0$, whence $0 = C$, and $s = 40t - 16t^2 - \frac{8}{3}t^3$.

This gives the height after t seconds.

At the highest point, $v = 0$,

$$\therefore\ 40 - 32t - 8t^2 = 0, \quad \text{i.e.}\quad 8(5+t)(1-t) = 0.$$

Taking the positive root of this equation, $t = 1$, it follows that the particle reaches its greatest height at the end of 1 second, and the distance travelled in that second is found by putting $t = 1$ in the expression for s. This gives $s = 40 - 16 - \frac{8}{3} = 21\frac{1}{3}$. Therefore the particle attains the height of $21\frac{1}{3}$ feet.

Examples XXVIII.

1. In what curves is the slope proportional to the abscissa ?
2. Find the curves in which the subnormal is proportional to the abscissa.
3. Find the equation of the curves in which the slope varies as the n^{th} power of the abscissa.
4. In what curves is the subnormal constant ?
5. In what curves is the sum of the abscissa and subnormal constant ? Explain your answer geometrically.
6. In what curves does the rectangle contained by the abscissa and the subtangent vary as the square of the ordinate ?
7. Find the equation of the curves in which the cube of the ordinate varies as the product of the subtangent and the square of the abscissa.
8. Find the equation of the curves in which the rectangle contained by the ordinate and the subnormal varies as the abscissa.
9. In what curves does the slope vary as the cube of the ordinate ?
10. In what curves does the subtangent vary as the square of the ordinate ?
11. The acceleration of a moving point, at the end of t seconds from the commencement of its motion, is $18 - 2t$ ft. secs. per sec.; find the velocity at the end of 3 seconds, and the distance travelled in that time, if the initial velocity be 20 ft. secs.
12. A particle starts with velocity u and moves with an acceleration $f \cos \frac{1}{6}\pi t$; find the velocity and the distance travelled at the end of 3 seconds.
13. A particle starts from rest at a distance a from a fixed point O, and is subject to an acceleration towards O which varies as the distance from O; find the velocity in any position. [Use $v\,dv/ds$ for the acceleration.]
14. A particle starting with velocity 21 ft. secs. has an acceleration $5 - 4t^2$ ft. secs. per second; when does it first come to rest, and how far has it then travelled ?
15. The acceleration of a moving point which starts from rest is $\sqrt{(1+t)}$ after t seconds; find the velocity at the end of 8 seconds, and the distance from the starting-point at the end of 3 seconds.

16. The acceleration of a moving point is $7-2s$, where s is the distance from the starting-point, which it leaves with velocity 40 ft. secs.; how far does it go before first coming to rest ?

17. A particle falls vertically from rest, and, in addition to being acted upon by gravity, is subject to a retardation which varies as the time and which at the end of 2 seconds is 20 ft. secs. per second; find the velocity at the end of 6 seconds and the distance fallen through in that time.

18. If, in the preceding example, the retardation varies as the distance fallen through and is 20 ft. secs. per second after falling 5 feet, find the velocity after falling 15 feet.

19. A particle moves in a straight line towards a fixed point O in the line, starting from rest at a distance of 40 feet from O; it is under the influence of a force which gives it an acceleration towards O of $100/s^2$ ft. secs. per second, where s is its distance from O; find its velocity (i) when it is half-way to O, (ii) when it has moved 30 feet.

20. The velocities of a moving point parallel to the axes of x and y respectively are, after t secs., $8-2t$ and $8\sqrt{t}$; find the coordinates of the point at the end of 4 seconds, taking the origin as the starting-point.

21. In the preceding question, find the velocity along the arc and the distance travelled along the arc. (See Art. 67.)

22. The velocities of a moving point parallel to the axes are, after t secs., t^2-9 and $6t$, and the point starts from the origin; find the equation of its path.

79. Areas of curves.

We have, in Art. 14, defined what is meant by the area bounded by a curved line.

Let AP (Fig. 75) be an arc of a curve measured from some fixed point A to a variable point P whose coordinates are (x, y). Let the

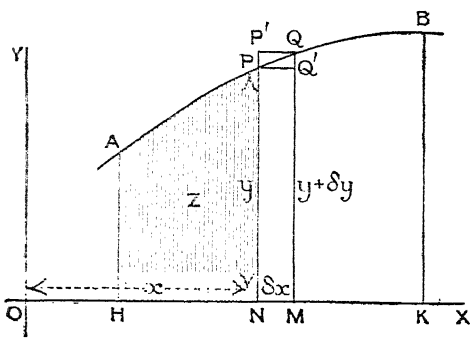


Fig. 75.

area $AHNP$ between the curve, the axis of x and the ordinates AH and PN be denoted by z. This area may be regarded as generated by the motion of the ordinate NP starting from HA and moving to

the right; to each value of x corresponds a value of z, so that z is a function of x.

Let x increase to $x+\delta x$, and let Q be the corresponding point on the curve and MQ its ordinate; then the area $PNMQ$ is δz, the increase in z due to the increase δx in x, and MQ is $y+\delta y$.

Complete the rectangles $PNMQ'$ and $QMNP'$.

Then, if the slope from P to Q be positive,

$$\delta z > \text{the rect. } PNMQ' \text{ and } < \text{the rect. } P'NMQ,$$

i.e. $$> NP.NM \text{ and } < MQ.NM,$$

i.e. $$> y\,\delta x \text{ and } < (y+\delta y)\,\delta x.$$

$\therefore$ $$\delta z/\delta x > y \text{ and } < y+\delta y.$$

[If the slope from P to Q be negative, the inequality signs will be reversed, as is obvious by drawing a figure. We have assumed that the slope has the same sign from P to Q. The range can be taken sufficiently small for this to be the case.]

When $\delta x \longrightarrow 0$, δy also $\longrightarrow 0$; therefore $\delta z/\delta x$, which we have just proved to differ from y by a smaller quantity than δy, tends to the limiting value y.

But dz/dx is, by definition, the limit of $\delta z/\delta x$ when $\delta x \longrightarrow 0$.

$$\therefore \quad dz/dx = y.$$

If the equation of the curve AP is given, we can find y in terms of x. We have therefore $dz/dx =$ a function of x, and z is found by integration.

Examples:

(i) *Find the area between the parabola* $ay = x^2$, *the axis of* x *and the ordinate* $x = h$.

If z be the area from the origin to the ordinate of a point (x, y),

$$dz/dx = y = x^2/a;$$

$$\frac{x^2}{a}\,dx = \frac{x^3}{3a} + C.$$

Clearly the area OPN (Fig. 76) is zero when P coincides with O, i.e. $z = 0$, when $x = 0$. Therefore $C = 0$, and substituting in the preceding equation,

$$z = x^3/3a.$$

This gives the area from the origin to the ordinate PN.

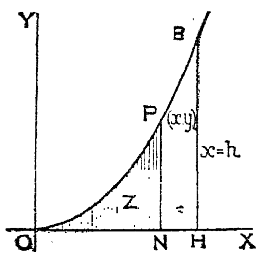


Fig. 76.

The area OBH is the value of z when $x = h$, and therefore is equal to $h^3/3a$.

Since the point B is on the curve,

$$a.HB = h^2;$$

$$\therefore \text{ the area } OHB = \frac{h^3}{3a} = \frac{h}{3}\cdot\frac{h^2}{a} = \frac{1}{3}h.HB = \frac{1}{3} \text{ area of rect. } OH.HB.$$

(ii) *Find the area between the curve* $y = 11x - 24 - x^2$ *and the axis of* x.

We have $y = (8-x)(x-3)$; and the curve cuts the axis of x where $y = 0$, i. e. where $x = 3$ and $x = 8$. Let these two points be A and B (Fig. 77).

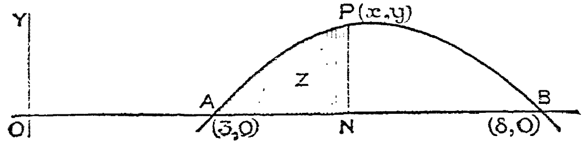


Fig. 77.

If z denote the area from A to PN, we have

$$dz/dx = y = 11x - 24 - x^2;$$

$$\therefore \quad z = \int (11x - 24 - x^2)\, dx,$$

$$= \tfrac{11}{2}x^2 - 24x - \tfrac{1}{3}x^3 + C.$$

In this case, $z = 0$ when P is at A, i. e. when $x = 3$. Therefore, substituting in preceding equation,

$$0 = \tfrac{99}{2} - 72 - \tfrac{27}{3} + C, \text{ whence } C = \tfrac{63}{2},$$

and $$z = \tfrac{11}{2}x^2 - 24x - \tfrac{1}{3}x^3 + \tfrac{63}{2}.$$

The area from A to B is obtained by putting $x = 8$ in this result, giving the required area APB

$$= \tfrac{11}{2} \times 64 - 192 - \tfrac{1}{3} \times 512 + \tfrac{63}{2} = 20\tfrac{5}{6} \text{ units of area.}$$

If the area to be determined is on the negative side of the axis of x, y will be negative, and the value obtained for the area will be negative if it be measured in the direction in which x increases. For instance, if, in the preceding example, the equation of the curve had been $y = x^2 - 11x + 24$, the area would have been on the other side of the axis of x, and the answer would have appeared as $-20\tfrac{5}{6}$.

Further examples of the determination of areas will be given in Chapter XVI.

The geometrical meaning of the existence of the arbitrary constant of integration is now easily seen. It will be noticed that the investigation of this article does not involve the position of the initial ordinate AH from which the area z is measured, and the result will be the same wherever AH may be. The arbitrary position of this ordinate corresponds to the arbitrary value of the constant of integration. When the position of AH is assigned, the constant of integration can be determined, as is shown in the working of the two examples just considered. In Ex. (i), when AH was taken at the origin, C was found to be 0, and in Ex. (ii) when AH was taken at $x = 3$, C was found to be $\tfrac{63}{2}$. If the areas had been measured from some other ordinates, different values for C would have been obtained.

80. Substitution of limits of integration. Definite integrals.

It will be noticed, from the last example, that the final value of z is obtained in the following manner. Taking the expression $\frac{11}{2}x^2-24x-\frac{1}{3}x^3$ obtained on integration, C is the result of substituting $x=3$ in this expression with the sign changed.

Then the final value of z is the result of substituting $x=8$ in the integral $+C$

= result of substituting 8 − result of substituting 3 in the integral

= the difference of the results of substituting in the integral the extreme values of x.

That the area is always obtained by this procedure can be shown as follows:

Let $y=f(x)$ be the equation of a curve, and let $f(x)$ be the derived function of $F(x)$. Suppose that the area between the curve, the axis of x, and the ordinates $x=a$ and $x=b$ is required.

Then $$dz/dx = y = f(x),$$

$$\therefore \quad z = \int f(x)\,dx + C = F(x) + C.$$

Now the area z, being measured from the ordinate $x=a$, is equal to 0 when $x=a$. Therefore $0=F(a)+C$ and $C=-F(a)$.

Hence $$z = F(x) - F(a).$$

The area from $x=a$ to $x=b$ is found by putting $x=b$ in this expression,

i.e. the required area $= F(b)-F(a)$

= the difference of the results of substituting in the integral of $f(x)$ the extreme values of x.

This operation is generally indicated in the following way:

$$z = \int_a^b f(x)\,dx,$$

which is read 'z is the integral, from a to b, of $f(x)$ with respect to x'. This is called a *definite integral*, and a and b are often referred to as the *limits* of integration. This name is not a fortunate one, and the meaning of the word limit used in this sense has no connection whatever with a limit as defined in Art. 12.

81. Volumes of solids of revolution.

If (Fig. 78) the area $AHKB$ be rotated about the axis of x, a solid is generated such that its section by any plane perpendicular to the axis of x is a circle. Such a solid is called a *solid of revolution*. See Art. 14 (4).

The volume V between the sections through AH and PN will be a function of x, the abscissa of P.

If x be increased to $x+\delta x$, then, as in Art. 79 in the case of an area, the increase δV in volume thereby produced is intermediate

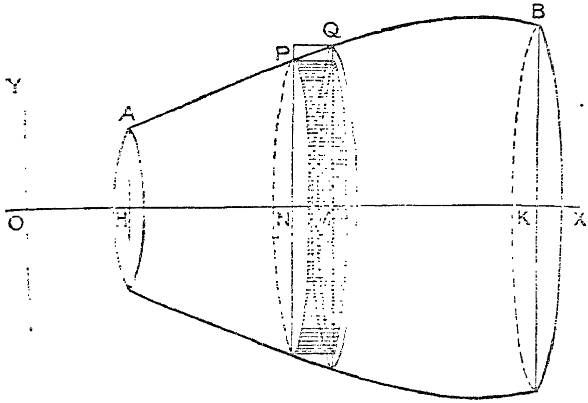


Fig. 78.

between the volumes formed by the rotation of the rectangles PM and QN, and these volumes are cylinders of height δx and radii y and $y+\delta y$ respectively.

Therefore δV is between $\pi y^2\,\delta x$ and $\pi(y+\delta y)^2\,\delta x$,

$$\therefore\quad \delta V/\delta x \text{ is between } \pi y^2 \text{ and } \pi(y+\delta y)^2.$$

As $\delta x \to 0$, $\delta y \to 0$ and $\pi(y+\delta y)^2 \to \pi y^2$,

$\therefore$ $\delta V/\delta x$, which is between these two, also $\to \pi y^2$ as its limit.

i.e. $$dV/dx = \pi y^2.$$

From the equation of the rotating curve, y and therefore πy^2 can be found in terms of x, and V is found by integration.

Exactly as in the case of areas, as explained in Art. 80, the volume V, between the two circular sections through the ordinates $x=a$ and $x=b$, is obtained by subtracting the result of substituting $x=a$ from the result of substituting $x=b$ in the integral of πy^2 with respect to x, and may be written in the form

$$V=\int_a^b \pi y^2\,dx.$$

Examples:

(i) *Find the volume of a right circular cone of height* h *and radius* r.

Let α (Fig. 79) be the semi-vertical angle of the cone. The equation of OA is $y = x\tan\alpha$.

Hence $$dV/dx = \pi y^2 = \pi x^2 \tan^2\alpha;$$

$$\therefore \quad V = \int \pi x^2 \tan^2 \alpha \, dx = \pi \tan^2 \alpha \, . \tfrac{1}{3} x^3 + C.$$

$V = 0$ when $x = 0$; $\quad \therefore \quad C = 0$, and $V = \frac{1}{3}\pi x^3 \tan^2 \alpha$.

This is the volume generated by the revolution of ONP.

At B, $x = h$; $\quad \therefore$ volume of cone $= \frac{1}{3}\pi h^3 \tan^2 \alpha$

$$= \tfrac{1}{3}\pi r^2 h, \text{ since } r = h \tan \alpha.$$

(ii) *Find the volume of a sphere, and of the part of a sphere cut off by two parallel planes.*

A sphere is formed by the revolution of a semicircle about its diameter. Let the equation of the circle be $x^2 + y^2 = r^2$.

Here $\qquad dV/dx = \pi y^2 = \pi (r^2 - x^2)$;

$$\therefore \quad V = \int \pi (r^2 - x^2) \, dx = \pi (r^2 x - \tfrac{1}{3} x^3) + C.$$

For the hemisphere, let V be measured from OB; then $V = 0$ when $x = 0$, $\therefore \quad C = 0$ and $V = \pi (r^2 x - \frac{1}{3} x^3)$.

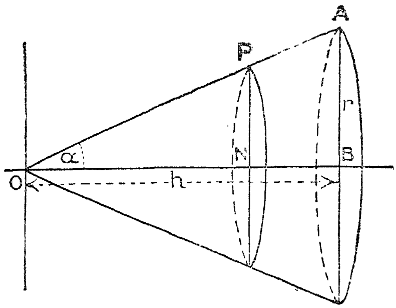


Fig. 79.

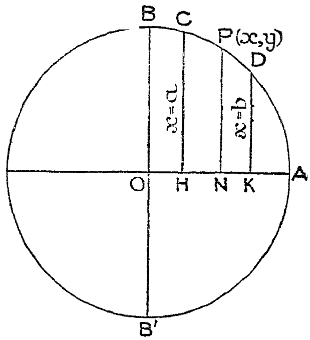


Fig. 80.

This is the volume generated by $OBPN$.

At A, $x = r$. Hence the volume of the hemisphere

$$= \pi (r^3 - \tfrac{1}{3} r^3) = \tfrac{2}{3}\pi r^3,$$

and the volume of the sphere $= \frac{4}{3}\pi r^3$.

If the volume between $x = a$ and $x = b$ be required, then, returning to the equation

$$V = \pi (r^2 x - \tfrac{1}{3} x^3) + C,$$

we have, measuring V from $x = a$, $V = 0$ when $x = a$,

therefore $$C = -\pi (r^2 a - \tfrac{1}{3} a^3),$$

and $$V = \pi (r^2 x - \tfrac{1}{3} x^3) - \pi (r^2 a - \tfrac{1}{3} a^3).$$

This is the volume generated by $HCPN$.

Therefore the volume required, obtained by putting $x = b$,

$$= \pi (r^2 b - \tfrac{1}{3} b^3) - \pi (r^2 a - \tfrac{1}{3} a^3)$$

$$= \pi r^2 (b - a) - \tfrac{1}{3}\pi (b^3 - a^3)$$

$$= \pi (b - a) \{ r^2 - \tfrac{1}{3} (b^2 + ab + a^2) \}.$$

If $b = r$, the figure is referred to as a spherical cap.

Using the notation explained in Art. 80, the working is generally set down as follows:

Volume of hemisphere

$$\int_0^r \pi (r^2 - x^2)\, dx$$

$$= \Big[\pi (r^2 x - \tfrac{1}{3} x^3)\Big]_0^r = \pi (r^3 - \tfrac{1}{3} r^3) - \pi (0) = \tfrac{2}{3}\pi r^3.$$

Volume of slice of sphere

$$= \int_a^b \pi (r^2 - x^2)\, dx$$

$$= \Big[\pi (r^2 x - \tfrac{1}{3} x^3)\Big]_a^b = \pi (r^2 b - \tfrac{1}{3} b^3) - \pi (r^2 a - \tfrac{1}{3} a^3).$$

Examples XXIX.

Find the areas whose boundaries are given in Examples 1–10. Find also the volumes generated when these areas rotate about the axis of x.

1. The axis of x, the curve $y = x^3$, and the ordinate $x = 3$.
2. The axis of x, the curve $y = \frac{1}{4}(x+1)^3$, and the ordinates $x = 2$, $x = 4$.
3. The axis of x, the parabola $y^2 = 12x$, and the ordinate $x = 3$.
4. The parabola $y^2 = 12x$, and the double ordinate $x = 12$.
5. The axis of x, and the curve $y = 9x - x^2 - 14$.
6. The axis of x, and the curve $y = (x-1)^2 - 25$.
7. One semi-undulation of $y = \sin x$, and the axis of x.
   [In finding the volume, use the formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.]
8. The curve $4ay^2 = 3x^3$, and the double ordinate $x = a$.
9. The curve $x^2 y = 36$, the axis of x, and the ordinates $x = 2$, $x = 6$.
10. The curve $9y = x^2(x+3)$, and the axis of x.
11. If P be a point on the curve $y^m = kx^n$, and PM, PN be drawn perpendicular to the axes; prove that the curve divides the rectangle $OMPN$ into two parts whose areas are as $m : n$.
12. Find the area between the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the axes of coordinates.
13. An ellipse whose semi-axes are 8 and 4 inches in length rotates about its major axis; find the volume of the solid formed (which is called a *prolate spheroid*).
14. The parabola $y^2 = 4ax$ rotates about the axis of x; prove that the volume of a segment, measured from the vertex, of the solid formed (called a paraboloid of revolution) is half the volume of the circumscribing cylinder.
15. The rectangular hyperbola $x^2 - y^2 = a^2$ [p. 20] revolves about the axis of x; prove that the volume of a segment of the hyperboloid of height a measured from the vertex is equal to the volume of a sphere of radius a.
16. Find the volume of the solid formed (called an *oblate spheroid*) by the rotation of the ellipse mentioned in Ex. 13 about the minor axis.
17. The curve $ay^2 = x^3$ rotates about the axis of x; prove that the volume of the resulting solid, cut off by a plane perpendicular to the axis, is a quarter of the volume of the circumscribing cylinder.

18. Find the area between the curve $y = x^2$ and the straight line $y = 4x$.
19. Find the area between the curves $y = x^2$ and $x = y^2$.
20. Find the area between the curves $y^2 = x$ and $y^2 = x^3$.
21. Find the volume formed by the rotation of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the axis of x.
22. The radii of the ends of a frustum of a right circular cone are 2 and 5 inches respectively, and its length is 1 foot; find its volume.
23. Prove that the volume of a spherical cap of height h is $\pi h^2 (r - \frac{1}{3} h)$, where r is the radius of the sphere.
24. Find the volume formed by the rotation of the loop of the curve $ay^2 = x(x-a)^2$ about the axis of x.
25. Find the area of the maximum circular section of this solid.

The axis of x intercepts two portions of the curve

$$a^2 y = (x-a)(x-2a)(x-3a);$$

prove that they are equal in area.

27. The segment of the parabola $y^2 = 9x$, cut off by the straight line $x = y$, rotates about the axis of x; find the volume generated.

The coordinates of two points A and B on the curve $a^{n-2} y = x^{n-1}$ are (x_1, y_1) and (x_2, y_2); prove that the area between the curve, the axis of x and the ordinates of A and B is $(x_2 y_2 - x_1 y_1)/n$.

The curve $y^2 = a^2 \cos \frac{1}{2}(x/a)$ rotates about the axis of x; find the volume between $x = -\pi a$ and $x = +\pi a$.

30. The curve $y(3a - 2x)^2 = a^3$ rotates about the axis of x; find the volume between $x = 0$ and $x = a$.
31. Find the area between the curves $y = (3x-5)^3$ and $y^3 = 3x - 5$.
32. Find the area between the graph of $pv^{\gamma} = k$, the axis of v, and the ordinates $v = v_1$, $v = v_2$.

82. Length of arc of a curve.

The length of an arc of a curve has been defined in Art. 14 (3).

Let s be the length of the arc, measured from some fixed point A (Fig. 81) on the curve, to a point P whose coordinates are (x, y), and let $s + \delta s$ be the length of the arc from A to a neighbouring point Q whose coordinates are $(x + \delta x,\ y + \delta y)$.

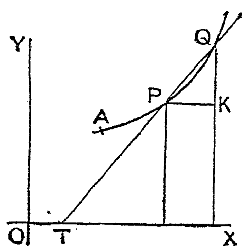


Fig. 81.

Draw PK perpendicular to the ordinate of Q, and let QP meet OX in T.

Then $PK = \delta x$, $KQ = \delta y$, arc $PQ = \delta s$.

Now
$$\sin XTP = \sin KPQ = \frac{KQ}{PQ} = \frac{\delta y}{\delta s} \times \frac{\delta s}{PQ}$$

$$\cos XTP = \cos KPQ = \frac{PK}{PQ} = \frac{\delta x}{ds} \times \frac{\delta s}{PQ}.$$

It was proved in Art. 13 (10) that, in the case of a circle, Lt (arc PQ/chord PQ), when Q approaches indefinitely near to P, is 1. *We shall assume this property to be true for all curves*, and therefore, since the limiting position of PQ is the tangent at P, and $\delta y/\delta s$, $\delta x/\delta s \to$ the limits dy/ds, dx/ds respectively, the preceding relations become

$$\sin\psi = dy/ds, \quad \cos\psi = dx/ds,$$

where ψ is the inclination to OX of the tangent at P.

Since $\sin^2\psi + \cos^2\psi = 1$, we have $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 = 1.$

Also
$$\left(\frac{ds}{dx}\right)^2 = \sec^2\psi = 1 + \tan^2\psi = 1 + \left(\frac{dy}{dx}\right)^2;$$

$$\therefore \quad \frac{ds}{dx} = \pm\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

and
$$\left(\frac{ds}{dy}\right)^2 = \operatorname{cosec}^2\psi = 1 + \cot^2\psi = 1 + \left(\frac{dx}{dy}\right)^2$$

$$\therefore \quad \frac{ds}{dy} = \pm\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}.$$

The + signs must be taken if the variables increase together.

From the equation of the curve, dy/dx can be found in terms of x, or dx/dy in terms of y, and then s will be obtained by integration.

If the coordinates x and y of the point P are expressed in terms of a variable θ, then since $\delta x^2 + \delta y^2 = PQ^2 = \delta s^2 \times (PQ/\delta s)^2$, we have, dividing by $\delta\theta^2$,
$$\left(\frac{\delta x}{\delta\theta}\right)^2 + \left(\frac{\delta y}{\delta\theta}\right)^2 = \left(\frac{\delta s}{\delta\theta}\right)^2\left(\frac{PQ}{\delta s}\right)^2;$$

therefore, when $\delta\theta \to 0$,
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{ds}{d\theta}\right)^2.$$

Examples:

(i) *Find the value of* ds/dx *in the curve* $4y^2 = x^3$, *and deduce the length of the curve from the origin to the point* (4, 4).

We have

$$y = \tfrac{1}{2}x^{3/2}; \quad \therefore \quad dy/dx = \tfrac{3}{4}x^{1/2} \quad \text{and} \quad ds/dx = \pm\sqrt{(1 + \tfrac{9}{16}x)}.$$

Since s, measured from the origin, increases as x increases, the + sign must be taken;

$$\therefore \quad s = \int(1 + \tfrac{9}{16}x)^{1/2}dx = (1 + \tfrac{9}{16}x)^{3/2}/(\tfrac{3}{2} \times \tfrac{9}{16}) + C = \tfrac{32}{27}(1 + \tfrac{9}{16}x)^{3/2} + C.$$

$$s = 0 \text{ when } x = 0, \quad \therefore \quad 0 = \tfrac{32}{27} + C, \text{ i.e. } C = -\tfrac{32}{27}.$$

$$\therefore \quad s = \tfrac{32}{27}[(1 + \tfrac{9}{16}x)^{3/2} - 1].$$

This is the length of the arc from the origin to the point whose abscissa is x; therefore the length of the arc to the point whose abscissa is 4

$$= \tfrac{32}{27}[(\tfrac{13}{4})^{3/2} - 1] = 5\cdot76 \text{ nearly}.$$

(ii) *Find the length of the arc of the curve* $6xy = 3+y^4$, *between the points whose ordinates are* 1 *and* 4.

In this case, we must find ds/dy and integrate with respect to y.

$$x = \frac{1}{2y} + \frac{y^3}{6}; \qquad \therefore \; \frac{dx}{dy} = -\frac{1}{2y^2} + \frac{y^2}{2},$$

and
$$\left(\frac{ds}{dy}\right)^2 = 1 + \frac{1}{4y^4} - \quad + \frac{y^4}{4} = \left(\frac{1}{2y^2} + \frac{y^2}{2}\right)^2$$

$\therefore \; \dfrac{ds}{dy} = \dfrac{1}{2y^2} + \dfrac{y^2}{2}$, taking the $+$ sign, since s is being measured from $y=1$ to $y=4$, and therefore is increasing as y increases.

Hence
$$s = \int\left(\frac{1}{2y^2} + \frac{y^2}{2}\right)dy = -\frac{1}{2y} + \frac{y^3}{6} + C.$$

$s = 0$ when $y=1$; $\quad \therefore \; 0 = -\frac{1}{2}+\frac{1}{6}+C$, and $C = \frac{1}{3}$.

Therefore
$$s = -\frac{1}{2y} + \frac{y^3}{6} + \tfrac{1}{3},$$

and the length of the arc from $y=1$ to $y=4$ is $-\frac{1}{8}+\frac{64}{6}+\frac{1}{3}$, i.e. $10\frac{7}{8}$.

(iii) *Find* $ds/d\theta$ *in the cycloid* (Art. 50), *and deduce the length of one arch of the curve.*

Here
$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= a^2(1-\cos\theta)^2 + a^2\sin^2\theta = a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)$$
$$= a^2(2-2\cos\theta) = 4a^2\sin^2\tfrac{1}{2}\theta;$$
$$\therefore \; ds/d\theta = \pm 2a\sin\tfrac{1}{2}\theta.$$

Measuring s from O (Fig. 53), s increases with θ;

$$\therefore \; ds/d\theta = 2a\sin\tfrac{1}{2}\theta.$$

Hence
$$s = \int 2a\sin\tfrac{1}{2}\theta\,d\theta = -4a\cos\tfrac{1}{2}\theta + C.$$

$s=0$ when $\theta = 0$; $\quad \therefore \; 0 = -4a + C$, and $C = 4a$,

and
$$s = 4a(1-\cos\tfrac{1}{2}\theta).$$

When the tracing-point has completed one arch, $\theta = 2\pi$ and $\cos\frac{1}{2}\theta = -1$;

$$\therefore \; s = 8a,$$

i.e. the length of one arch is four times the diameter of the rolling circle.

83. Area of surface of a solid of revolution.

This has been defined in Art. 14 (5). Referring to the figure of Art. 81, let S be the area traced out by the rotation of the arc AP, and δS the area traced out by the rotation of the arc PQ. The area of the frustum of a cone generated by the rotation of the chord PQ

$$= PQ \times 2\pi(y+\tfrac{1}{2}\delta y) \;[\text{Art. } 14] = PQ/\delta s \times 2\pi(y+\tfrac{1}{2}\delta y)\,\delta s.$$

As $\delta s \to 0$, the ratio of the area traced out by δs to that traced out by $PQ \to 1$, also $PQ/\delta s \to 1$ and $y+\frac{1}{2}\delta y \to y$.

$\therefore$ $\delta S \to 2\pi y\,\delta s$; i.e. $\delta S/\delta s \to 2\pi y$, and in the limit, $dS/ds = 2\pi y$.

If the equation of the curve be given, we may write

$$\frac{dS}{dx} = \frac{dS}{ds} \times \frac{ds}{dx} = 2\pi y \frac{ds}{dx} = 2\pi y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

from the preceding article, and then S is found by integrating with respect to x.

If more convenient, we may take

$$\frac{dS}{dy} = \frac{dS}{ds}\cdot\frac{ds}{dy} = 2\pi y \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}},$$

and then S is found by integrating with respect to y.

Example:

Find the area of the surface formed by the rotation of the parabola $y^2 = 4ax$ *about the axis of* x, *from the origin to the section* $x = 3a$.

We have $y = 2\sqrt{(ax)}$, $dy/dx = \sqrt{(a/x)}$ and $(ds/dx)^2 = 1 + a/x$.

$$\therefore \quad dS/dx = 2\pi y\, ds/dx = 2\pi \,.\, 2\sqrt{(ax)} \,.\, \sqrt{(1 + a/x)} = 4\pi\sqrt{a} \,.\, \sqrt{(a+x)};$$

$$\therefore \quad S = 4\pi\sqrt{a} \,.\int\sqrt{(a+x)}\,dx = 4\pi\sqrt{a} \,.\, \tfrac{2}{3}(a+x)^{3/2} + C.$$

$S = 0$ when $x = 0$, since S is measured from the origin,

$$\therefore \quad 0 = \tfrac{8}{3}\pi\sqrt{a} \,.\, a^{3/2} + C, \quad \text{and} \quad C = -\tfrac{8}{3}\pi a^2.$$

Hence $$S = \tfrac{8}{3}\pi\left[\sqrt{a}\,(a+x)^{3/2} - a^2\right].$$

Therefore the area as far as the section $x = 3a$

$$= \tfrac{8}{3}\pi\left[\sqrt{a} \,.\, (4a)^{3/2} - a^2\right] = \tfrac{56}{3}\pi a^2.$$

On account of the radical sign which occurs in the expressions for ds/dx, ds/dy, dS/dx, and dS/dy, the integration is often complicated, and few examples can be worked out until further methods of integration have been considered.

Examples XXX.

1. In the curve $y = \frac{1}{4}x^3$, find approximately the length of the arc between the points on the curve where $x = 2$ and $x = 2{\cdot}01$ [i. e. given $\delta x = {\cdot}01$, find δs].
2. In the circle $x^2 + y^2 = 100$, find approximately the length of the arc from the point (8, 6) to the point on the circle where $x = 8{\cdot}03$.
3. In the cycloid $x = 10(\theta - \sin\theta)$, $y = 10(1 - \cos\theta)$, find the approximate length of the arc between the points where $\theta = \frac{1}{2}\pi$ and $\theta = \frac{91}{180}\pi$.
4. In the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, find approximately the length of the arc between the points where $\theta = 44°$ and $\theta = 45°$.

Find the lengths of the arcs of the following curves:

5. $27y^2 = x^3$, from the origin to the point where $x = 15$.
6. $16x^2 = y^5$, from the origin to $y = 1$.
7. $x^4 + 3 = 6xy$, from $x = 2$ to $x = 8$.
8. $4y^2 = (x+2)^3$, from $x = 2$ to $x = 7$.

9. The curve whose slope at the point (x, y) is $2\sqrt{(x+x^2)}$, from $x = 1$ to $x = 10$.

10. The curve whose slope at the point (x, y) is $1/\sqrt{\{3y(2+3y)\}}$, from $y = 0$ to $y = 5$.

Find the areas of the following surfaces:

11. The paraboloid formed by rotation of $y^2 = 8x$ about the axis of x, (i) from the origin to $x = 16$, (ii) from $x = 6$ to $x = 16$.

12. The surface formed by rotation of $y = \frac{1}{4}x^2$ about the axis of y, from the origin to $y = 8$.

13. The surface formed by rotation of $x^4 + 3 = 6xy$ about the axis of x, from $x = 1$ to $x = 4$.

14. The surface formed by the rotation of $8x^3y = 2 + x^6$ about the axis of x, from $x = 1$ to $x = 2$.

15. The surface formed by the rotation of $9y^2 = x(x-3)^2$ about the axis of x, from $x = 0$ to $x = 3$.

16. Find also the perimeter of the loop of this curve.

17. Prove that in the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, $ds/d\theta = \frac{3}{2}a\sin 2\theta$, and deduce the total length of the curve.

18. A circle of radius 4 inches rolls along a fixed straight line OX; find the distance travelled by a point P on the circumference in one-quarter of a revolution, starting from O.

CHAPTER X

EXPONENTIAL, HYPERBOLIC, AND INVERSE FUNCTIONS

84. Convergent and Divergent Series.

If each term of a series be finite, the sum of any finite number n of terms is also finite. If $n \to \infty$, the sum of n terms may increase without bounds or may approach a limiting value.

If, as $n \to \infty$, the sum of n terms of a series tends to a definite (and therefore finite) limit S, the series is said to be *convergent* and S is called its sum. If the sum of n terms of a series $\to \infty$ as $n \to \infty$, the series is *divergent.*

An example of a convergent series was fully discussed in Art. 13 (3). That series was a particular case of a Geometrical Progression. The sum of n terms of the series

$$a + ar + ar^2 + \ldots + ar^n$$

is proved in text-books on elementary algebra to be

$$a(1-r^n)/(1-r).$$

If $|r| < 1$, r^n can be made as small as we please by taking n sufficiently large, and therefore the sum of n terms of the series approaches the limit $a/(1-r)$; i.e. an infinite G. P. is convergent if its common ratio r is numerically < 1. If r is equal to or greater than 1, it is obvious that the sum of the series may be made as large as we please by taking n sufficiently large, and the series is divergent.

The question of convergency or divergency only arises of course in connection with infinite series. If a series consists of only a finite number of terms (each finite) the sum of the series is necessarily finite. In dealing with infinite series, it is of essential importance to know whether, or under what circumstances, the series are convergent, because infinite series obey the ordinary laws of elementary algebra and can be added, subtracted, multiplied, rearranged, &c. only when they are *absolutely convergent* [i.e. convergent when all their terms are taken with the same sign].

85. Conditions for convergency.

In the first place, it is obviously necessary, if a series is to be convergent, that the n^{th} term should $\to 0$ as $n \to \infty$, for if the terms remained finite, the sum of n of them would clearly $\to \infty$ as $n \to \infty$; but this alone is not sufficient.

If S_n denote the sum of the first n terms, the definition of a convergent series states that $S_n \to S$, a finite limit, as $n \to \infty$.

$\therefore$ $S_{n+1}, S_{n+2}, \dots S_{n+m}$ (being equal to S_n + additional terms), also $\to S$ as $n \to \infty$ [m being any positive integer].

$\therefore$ they differ from one another by quantities which $\to 0$ as $n \to \infty$;

$$S_{n+m} - S_n \to 0 \text{ as } n \to \infty.$$

Now $S_{n+m} - S_n$ = sum of first $n+m$ terms − sum of first n terms
= the sum of m terms after the n^{th},

and m may have any integral value; therefore not only must

(i) *the* n$^{\text{th}}$ *term* $\to 0$ *as* n $\to \infty$, but also

(ii) *the sum of any number of terms after the* n$^{\text{th}}$ $\to 0$ *as* n $\to \infty$.

For instance, in the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

(which is called the harmonic series), the n^{th} term, $1/n$, $\to 0$ as $n \to \infty$, but the series is not convergent, because the second of the conditions mentioned above is not satisfied.

The sum of n terms following the n^{th} term

$$= \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}.$$

This is greater than $n \times 1/2n$ [the number of terms $\times$ the smallest of them], i.e. $> \frac{1}{2}$, which is not indefinitely small. In fact by taking, after the first and second terms, the next 2 [which are $> 2 \times \frac{1}{4}$], then the next 4 [which are $> 4 \times \frac{1}{8}$], the next 8 [which are $> 8 \times \frac{1}{16}$], the next 16 and so on, we get an infinite number of groups of terms, such that the terms in each group add up to more than $\frac{1}{2}$; and therefore, by taking a sufficient number of groups, we can obtain a sum as large as we please.

86. Tests for convergency.

It is often possible to find whether or not a series is convergent, i.e. whether or not S_n tends to a finite limit S, even if the exact values of S_n and S cannot be found.

The three tests which are the simplest and the most frequently used in elementary cases are the following:

I. The obvious test, that *if each term of a given series is numerically less than the corresponding term of another series which is known to be convergent, then the given series will be convergent.*

This is evidently true, because the sum of n terms of the given series is numerically less than the sum of the corresponding terms of the other series, and since the latter tends to a finite limit, so must the former.

It is obviously immaterial whether the inequality holds at the commencement of the series; it is sufficient if it be true for all after a finite number of terms.

For instance, after the first two terms, each term of the series

$$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+$$

is less than the corresponding term of the series

$$1+\frac{1}{2}+\frac{1}{2.2}+\frac{1}{2.2.2}+\ldots$$

which is convergent [it is a G. P. whose sum to infinity is 2]; hence the given series is convergent.

II. *If the terms of a series diminish continually to the limit zero and are alternately + and −, the series is convergent.*

For the series

$$u_1-u_2+u_3-u_4+u_5-u_6+\ldots$$
$$=(u_1-u_2)+(u_3-u_4)+(u_5-u_6)+\ldots$$

and therefore $>u_1-u_2$, since all the numbers in the brackets are + if the given conditions be satisfied.

Also the series may be written

$$u_1-(u_2-u_3)-(u_4-u_5)-(u_6-u_7)-\ldots$$

which $<u_1$, since again all the numbers in the brackets are +.

Hence the sum of n terms of the series must tend to a limit which is between u_1 and u_1-u_2, and is therefore finite. Therefore the series is convergent.

E.g. the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$ is convergent.

This last series is of the kind known as *semi-convergent* or *conditionally convergent.* This term is applied to series which are convergent, but which lose their convergency and become divergent when their terms are all taken with the same sign.

A series which is convergent, and which remains convergent when its terms are all taken with the same sign, is said to be *absolutely* or *unconditionally convergent.*

Such a series is $\quad 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots.$

III. *A series is convergent if, after a finite number of terms, the ratio of each term to the preceding term is always less than some fixed quantity which is itself less than unity.*

In both cases, 'less' means 'numerically less'.

Suppose that, from the n^{th} term onwards, the ratio of each term to the preceding term $< k$, where $|k| < 1$.

$$\begin{aligned} \text{i.e. } u_{n+1}/u_n &< k, &\quad \therefore\ u_{n+1} &< ku_n; \\ u_{n+2}/u_{n+1} &< k, &\quad \therefore\ u_{n+2} &< ku_{n+1} < k(ku_n) < k^2u_n; \\ u_{n+3}/u_{n+2} &< k, &\quad \therefore\ u_{n+3} &< ku_{n+2} < k(k^2u_n) < k^3u_n; \end{aligned}$$

and so on.

$\therefore$ adding together,

$$u_{n+1}+u_{n+2}+u_{n+3}+\dots < ku_n+k^2u_n+k^3u_n+\dots$$

(a G. P. whose common ratio k is numerically <1)

$$< ku_n/(1-k).$$

Since the sum after the first n terms is finite, and the sum of the first n terms is finite, it follows that the series is convergent.

In applying this test, we see that it is only the value of the ratio when n is large that is of importance; it does not matter about a finite number of terms at the commencement of the series. Hence we write down the ratio of the $(n+1)^{\text{th}}$ term to the n^{th} term, or of the n^{th} term to the $(n-1)^{\text{th}}$ term if more convenient, and examine the value of this ratio when n is very large.

If, as $n \to \infty$, *the ratio* u_{n+1}/u_n *approaches a limit which is numerically less than* 1, *the series is convergent; if* $u_{n+1}/u_n \to 1$ *as* $n \to \infty$, *the test fails; if* $u_{n+1}/u_n \to$ *a limit greater than* 1, *the series is divergent.*

Examples:

(i) $$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots \qquad \text{(The Exponential Series.)}$$

Here $$\frac{(n+1)^{\text{th}}\text{ term}}{n^{\text{th}}\text{ term}} = \frac{x^n/n!}{x^{n-1}/(n-1)!} = \frac{x}{n};$$

which, whatever be the (finite) value of x, $\to 0$ as $n \to \infty$, and therefore obviously satisfies the first condition.

If x be equal to 100, then (putting $n = 100$ in the ratio x/n) the 101^{st} term is equal to the 100^{th} term, and for all subsequent terms, the ratio is <1, and, moreover, continually diminishes.

(ii) $$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \qquad \text{(The Logarithmic Series.)}$$

Here $$\frac{(n+1)^{\text{th}}\text{ term}}{n^{\text{th}}\text{ term}} = -\frac{x^{n+1}/(n+1)}{x^n/n} = -\frac{n}{n+1}x = -\frac{x}{1+1/n},$$

which, as $n \to \infty$, tends to the limit x.

Therefore the convergency or divergency of the series depends upon the numerical value of x.

If $|x| < 1$, the test-ratio approaches a limit less than 1; $\therefore$ the series is convergent.

If $|x| > 1$, the test-ratio approaches a limit greater than 1; $\therefore$ the series is divergent.

If $|x| = 1$, the test fails; the ratio is then numerically $1/(1+1/n)$, which, although less than 1, has 1 as its limit, and can be made as nearly equal to 1 as we please. Therefore we cannot say that the ratio is less than a fixed number which is less than 1. In fact, if we select any fixed number k as little below 1 as we please, we can always get a little nearer to 1 than k is, by taking n large enough, as follows from the definition of a limit.

In this case, if $x = +1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, which, as pointed out above, is semi-convergent; and if $x = -1$, the series is

$$-1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \ldots,$$

which we have shown to be divergent.

For further tests of convergency, and an account of the properties of convergent series, the student is referred to works on Algebra, such as those by Chrystal and C. Smith.

Examples XXXI.

Test the following series for convergency:

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots.$

2. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

3. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots.$

4. $\frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.4.6} + \frac{1}{2.4.6.8} + \ldots.$

5. $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{6} + \ldots.$

6. $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \ldots.$

7. $\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \frac{1}{400} + \ldots.$

8. $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \ldots.$

9. $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots.$ $[x \neq 1.]$

10. $1 + \frac{2}{2} + \frac{2^2}{3} + \frac{2^3}{4} + \ldots.$

11. $1 + \frac{-}{2!} + \frac{-}{3!} + \frac{-}{4!} + \ldots.$

12. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots.$

13. $1 + \frac{2x}{5} + \frac{3x^2}{25} + \frac{4x^3}{125} + \ldots.$

14. $\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \ldots.$ $[x \text{ positive.}]$

We now proceed to discuss a very important limit, a particular case of which has been already considered in Art. 13 (9).

87. Limiting value of $(1+x/m)^m$ as $m \to \infty$.

First, let m be a positive integer.

Expanding by the Binomial Theorem, we have

$$\left(1+\frac{x}{m}\right)^m = 1 + m\cdot\frac{x}{m} + \frac{m(m-1)}{2!}\cdot\frac{x^2}{m^2} + \frac{m(m-1)(m-2)}{3!}\cdot\frac{x^3}{m^3} + \dots$$
$$+ \frac{m(m-1)\dots(m-r+1)}{r!}\cdot\frac{x^r}{m^r} + \dots$$
$$= 1 + x + \left(1-\frac{1}{m}\right)\frac{x^2}{2!} + \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\frac{x^3}{3!} + \dots$$
$$+ \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{r-1}{m}\right)\frac{x^r}{r!} + \dots.$$

As $m \to \infty$, $1/m$, $2/m$, ... $(r-1)/m$ (when r is finite) all $\to 0$, and therefore the sum of the first $r+1$ terms of the series tends to the limit

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!}, \quad \text{provided } r \text{ be finite.}$$

But it must not be taken for granted that the $(r+1)^{\text{th}}$ term tends to the limit $x^r/r!$ when r is indefinitely great; for, in this case, the number of factors

$$\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{r-1}{m}\right)$$

in the coefficient of $x^r/r!$ increases indefinitely, and it cannot be assumed without further investigation that the product of an indefinitely great number of factors, each differing from 1 by an indefinitely small amount (which amount moreover gradually increases as we get farther on in the series of factors) tends to the limit 1.

Hence, when m is indefinitely increased, we write the above expansion in the form

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots + R.$$

It can be proved (see next article) that the quantity R tends to the limit 0 as $m \to \infty$; therefore, assuming this for the moment, we see that, for all values of x,

$$\operatorname*{Lt}_{m\to\infty}\left(1+\frac{x}{m}\right)^m = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots,$$

a series which was shown in the last article to be convergent for all finite values of x.

In particular, if $x = 1$,

$$\underset{m \to \infty}{\text{Lt}} \left(1 + \frac{1}{m}\right)^m = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{r!} + \dots;$$

and, since the terms of this convergent series rapidly diminish, an approximate value of the limit can be obtained by taking the first few terms; e. g., to 5 places of decimals,

$$\begin{aligned}
&\text{the first 3 terms together} && = 2{\cdot}5, \\
&\text{the } 4^{\text{th}} \text{ term } 1/3! = \tfrac{1}{6} && = {\cdot}16667 \\
&\text{the } 5^{\text{th}} \text{ term } 1/4! = \tfrac{1}{4} \text{ of the } 4^{\text{th}} \text{ term} && = {\cdot}04167 \\
&\text{the } 6^{\text{th}} \text{ term } 1/5! = \tfrac{1}{5} \text{ of the } 5^{\text{th}} \text{ term} && = {\cdot}00833 \\
&\text{the } 7^{\text{th}} \text{ term } 1/6! = \tfrac{1}{6} \text{ of the } 6^{\text{th}} \text{ term} && = {\cdot}00139 \\
&\text{the } 8^{\text{th}} \text{ term } 1/7! = \tfrac{1}{7} \text{ of the } 7^{\text{th}} \text{ term} && = {\cdot}00020 \\
&\text{the } 9^{\text{th}} \text{ term } 1/8! = \tfrac{1}{8} \text{ of the } 8^{\text{th}} \text{ term} && = {\cdot}00002.
\end{aligned}$$

Therefore, adding up, we find the value of the limit to be approximately $2{\cdot}7183$, agreeing with the rough value obtained in Art. 13 (9) for $\underset{m \to \infty}{\text{Lt}} (1 + 1/m)^m$.

Hence the limit of $(1+1/\text{m})^{\text{m}}$ *as* $\text{m} \to \infty$, *and the sum of the convergent series* $1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dots$ *are each equal to the number* e.

88. Completion of proof.

We will now complete the proof of the preceding article by showing that the quantity $R \to 0$ as $m \to \infty$.

If $a, b, c \dots$ be positive quantities less than 1, we have

$(1-a)(1-b) = 1-(a+b)+ab$, which is $> 1-(a+b)$,

and therefore $\qquad = 1-\theta_1(a+b)$, where θ_1 is a positive proper fraction.

$(1-a)(1-b)(1-c) > \{1-(a+b)\}(1-c) > 1-(a+b+c)$, by the preceding,

and therefore $\qquad = 1-\theta_2(a+b+c)$, where θ_2 is a positive proper fraction, and so on for any number of factors.

Hence, applying this fact,

$$\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{r-1}{m}\right) = 1-\theta\left(\frac{1}{m}+\frac{2}{m}+\dots+\frac{r-1}{m}\right)$$

[θ a positive proper fraction.]

$$= 1-\theta\frac{(r-1)r}{2m}, \text{ summing the A. P. in the brackets.}$$

$$\therefore \left(1+\frac{x}{m}\right)^m = 1+x+\left(1-\frac{1}{m}\right)\frac{x^2}{2!}+\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\frac{x^3}{3!}+\dots$$
$$+\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{r-1}{m}\right)\frac{x^r}{r!}+\dots$$

$$= 1+x+\left(1-\frac{1}{m}\right)\frac{x^2}{2!}+\left(1-\theta_1\frac{3}{m}\right)\frac{x^3}{3!}+\left(1-\theta_2\frac{6}{m}\right)\frac{x^4}{4!}+\dots$$
$$+\left(1-\theta\frac{r(r-1)}{2m}\right)\frac{x^r}{r!}+\dots$$

$$= 1+x+\frac{x^2}{2!}-\frac{x^2}{2m}+\frac{x^3}{3!}-\theta_1\frac{x^3}{2m}+\frac{x^4}{4!}-\theta_2\frac{x^4}{4m}+\dots$$
$$+\frac{x^r}{r!}-\theta\frac{x^r}{2m.(r-2)!}+\dots$$

$$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots+\frac{x^r}{r!}+\dots$$
$$-\frac{x^2}{2m}\left[1+\theta_1 x+\theta_2\frac{x^2}{2!}+\dots+\theta\frac{x^{r-2}}{(r-2)!}+\dots\right].$$

Hence the quantity R of the preceding article is equal to

$$-\frac{x^2}{2m}\left[1+\theta_1 x+\theta_2\frac{x^2}{2!}+\dots+\theta\frac{x^{r-2}}{(r-2)!}+\dots\right];$$

which, since all the θ's are $+$ and < 1, is numerically

$$< -\frac{x^2}{2m}\left[1+x+\frac{x^2}{2!}+\dots+\frac{x^{r-2}}{(r-2)!}+\dots\right]$$

i.e. $< -\dfrac{x^2}{2m}\times$ [a finite quantity], since it was shown in Art. 86 that the series in the brackets is convergent for all finite values of x.

Hence, as $m \to \infty$, $R \to 0$, since, if x be finite, $R =$ (a finite number)$/m$ and therefore, for all finite values of x,

$$\mathop{\mathrm{Lt}}_{m\to\infty}\left(1+\frac{x}{m}\right)^m = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^r}{r!}+\dots.$$

89. Extension to fractional and negative values of m.

In Art. 87, m was supposed to increase indefinitely through a series of positive integral values. This restriction will now be removed, and we will show that the limit is still the same if m increases continuously until it becomes indefinitely great, whether it be positive or negative.

(i) Let m be between n and $n+1$, where n is a positive integer.

Then, taking x positive, we have

$(1+x/m)^m < (1+x/n)^{n+1}$, since the latter is a larger number (>1) raised to a higher power,

and $> \{1+x/(n+1)\}^n$, since the latter is a smaller number (>1) raised to a lower power;

i.e.

$$\left(1+\frac{x}{m}\right)^m \text{ is between } \left(1+\frac{x}{n}\right)^n\left(1+\frac{x}{n}\right) \text{ and } \left(1+\frac{x}{n+1}\right)^{n+1}\Big/\left(1+\frac{x}{n+1}\right).$$

When $m \to \infty$, $n \to \infty$ also, and the first factor in each of the two latter expressions tends to the limit

$$1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{(Art. 87)},$$

while the second factor in each case tends to the limit 1.

Hence each of these expressions, and therefore also $(1+x/m)^m$ which lies between them, tends to the limit

$$1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots.$$

If x be $-$, the necessary changes in the inequality signs are easily seen.

(ii) Let m be negative and equal to $-(n+x)$, where n is positive and $\to \infty$ as $m \to \infty$.

Then

$$\mathrm{Lt}\left(1+\frac{x}{m}\right)^m = \mathrm{Lt}\left(1-\frac{x}{n+x}\right)^{-n-x} = \mathop{\mathrm{Lt}}_{n\to\infty}\left(\frac{n}{n+x}\right)^{-n-x}$$

$$= \mathop{\mathrm{Lt}}_{n\to\infty}\left(\frac{n+x}{n}\right)^{n+x} = \mathop{\mathrm{Lt}}_{n\to\infty}\left(1+\frac{x}{n}\right)^{n+x} = \mathop{\mathrm{Lt}}_{n\to\infty}\left(1+\frac{x}{n}\right)^n \cdot \left(1+\frac{x}{n}\right)^x;$$

and of these two factors, the first, by the preceding case, since n is $+$, tends to the limit

$$1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots,$$

and the second tends to the limit 1.

Therefore

$$\mathop{\mathrm{Lt}}_{m\to\infty}\left(1+\frac{x}{m}\right)^m = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots$$

for all finite values of x, whether m be $+$ or $-$, integral or fractional.

90. The exponential theorem.

From the foregoing results we can now deduce this extremely important theorem. In the expression $(1+x/m)^m$, put $m = nx$; since x is finite, $n \to \infty$ when $m \to \infty$; therefore we have

$$\mathrm{Lt}\left(1+\frac{x}{m}\right)^m = \mathrm{Lt}\left(1+\frac{x}{nx}\right)^{nx} = \mathrm{Lt}\left(1+\frac{1}{n}\right)^{nx}$$

$$= \mathop{\mathrm{Lt}}_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left[\mathop{\mathrm{Lt}}_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^x = e^x,$$

since it follows from Art. 15 that $\mathrm{Lt}\,(a^x) = [\mathrm{Lt}\,(a)]^x$.

Also it has just been proved that

$$\mathop{\mathrm{Lt}}_{m\to\infty}\left(1+\frac{x}{m}\right)^m = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots;$$

therefore, for all finite values of x,

$$e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \qquad + \frac{x^r}{r\,!} +$$

This is known as the EXPONENTIAL THEOREM, and the series on the right-hand side is called the EXPONENTIAL SERIES.

The function e^x is of very frequent occurrence, and the form of its graph should be noticed.

We have $$y = e^x = 1 + x + \frac{x^2}{2\,!} + \dots + \frac{x^r}{r\,!} +$$

If $x = 0$, $y = 1$; as x increases, each term after the first increases and $\to \infty$ when $x \to \infty$. Therefore y increases from 1 to ∞ as x increases from 0 to ∞.

If x be $-$, then since $e^{-x} = 1/e^x$, it follows that y decreases from 1 to 0 as x goes from 0 to $-\infty$; hence the axis of x is an asymptote. e^x is a one-valued continuous function * of x, which increases from 0 to ∞ as x increases from $-\infty$ to $+\infty$, as shown in Fig. 82.

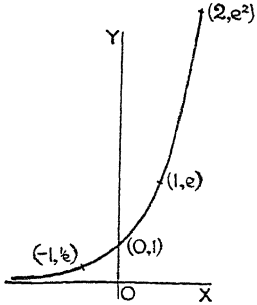


Fig. 82.

91. The logarithmic function $\log_e x$.

This is the inverse of the exponential function e^x just considered. If $x = e^y$, then y is called the logarithm of x to the base e, which fact is written: $y = \log_e x$. e is called the natural base of logarithms, and logarithms to base e are called *Napierian* or *hyperbolic* logarithms. In numerical work, such as is involved in arithmetic and the solution of triangles, 10 is the most convenient base for logarithms, and the common logarithms are calculated to base 10, but the logarithms used in the Calculus are always referred to the base e, and these logarithms occur very frequently, especially in the integral calculus. The symbol $\log x$, with no base indicated, will always be used for such logarithms, and common logarithms will then be denoted by the symbol $\log_{10} x$.

The process of transforming logarithms from base 10 to base e, or vice versa, is quite simple, for

$$\text{if} \quad \log_{10} x = y, \quad \text{we have} \quad x = 10^y.$$

Therefore, taking logarithms to base e, we have

$$\log_e x = y \log_e 10, \quad \text{or} \quad y = \log_e x / \log_e 10\,;$$
$$\text{i.e.} \quad \log_{10} x = \log_e x \times 1/\log_e 10.$$

* A table of values of e^x and e^{-x} is given at the end of the book.

Hence the logarithm of any number is changed from base e to base 10 by multiplying it by the constant factor $1/\log_e 10$, which is equal numerically to $\cdot 43429\ldots$, and is often denoted by the letter μ.

In treatises on Algebra, series are obtained from which logarithms to base e can be calculated, and thence, multiplying by μ, logarithms to base 10 are obtained.* Since the logarithmic function is the inverse of the exponential function, their graphs will be of the same form with the axes interchanged, i.e. they will be symmetrical about the bisector of the angle XOY, cf. Art. 9 (iv). In the case of the exponential function, it was seen that as x increased from $-\infty$ to $+\infty$, y increased from 0 to ∞; therefore in the case of $y=\log x$, as x increases from 0 to ∞, y increases from $-\infty$ to $+\infty$. If x is $-$, y is imaginary (Fig. 83).

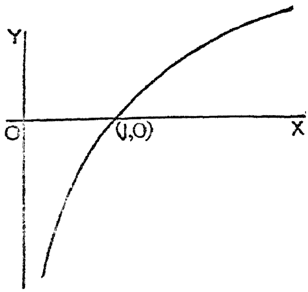


Fig. 83.

92. The hyperbolic functions.

We have $$e^x = 1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\frac{x^4}{4\,!}+\ldots;$$

changing the sign of x, $$e^{-x} = 1-x+\frac{x^2}{2\,!}-\frac{x^3}{3\,!}+\frac{x^4}{4\,!}-\ldots.$$

adding, $$e^x+e^{-x} = 2\left[1+\frac{x^2}{2\,!}+\frac{x^4}{4\,!}+\ldots\right],$$

and subtracting, $$e^x-e^{-x} = 2\left[x+\frac{x^3}{3\,!}+\frac{x^5}{5\,!}+\ldots\right].$$

The function $\frac{1}{2}(e^x+e^{-x})$ is denoted by the symbol $\cosh x$, and $\frac{1}{2}(e^x-e^{-x})$ by the symbol $\sinh x$,

i.e. $\cosh x = 1+\frac{x^2}{2\,!}+\frac{x^4}{4\,!}+\ldots$, and $\sinh x = x+\frac{x^3}{3\,!}+\frac{x^5}{5\,!}+\ldots$,

so that $\cosh x$ is an even function of x, and $\sinh x$ an odd function of x [Art. 5].

These symbols are used because these functions possess properties analogous to those possessed by the circular functions $\cos x$ and $\sin x$.

The quotient $\sinh x/\cosh x$ is written $\tanh x$, and the reciprocals of $\cosh x$, $\sinh x$ and $\tanh x$ are written $\operatorname{sech} x$, $\operatorname{cosech} x$, and $\coth x$

* Tables of logarithms to base 10 and also to base e are given at the end of the book.

respectively. These six functions are called the *hyperbolic functions*,* and are often referred to as the 'hyperbolic sine, cosine', &c. This name is due to the fact that they bear certain relations to the rectangular hyperbola, similar to those that the 'circular functions' $\sin x$, $\cos x$, &c., bear to the circle. E.g. just as the point $(a\cos\theta,\ a\sin\theta)$ is always on the circle $x^2+y^2=a^2$, whatever the value of θ, so the point $(a\cosh u,\ a\sinh u)$ is always on the rectangular hyperbola $x^2-y^2=a^2$, whatever the value of u.

It will be sufficient for our purpose if we prove the fundamental relation

$$\cosh^2 x-\sinh^2 x=1. \qquad [\text{cf. } \cos^2 x+\sin^2 x=1.]$$

This follows at once from the definitions above; for

$$\begin{aligned}\cosh^2 x-\sinh^2 x &= \tfrac{1}{4}(e^x+e^{-x})^2-\tfrac{1}{4}(e^x-e^{-x})^2\\ &= \tfrac{1}{4}(e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2)\\ &= 1.\end{aligned}$$

There are relations between these functions analogous to all the well-known formulae of Trigonometry, most of which can be proved as above. Some of them are given in the examples at the end of the chapter.

93. Graphs of the hyperbolic functions.

The graphs of these functions are best deduced from that of e^x in the following manner:

(i) Draw the graph of e^x; (ii) in the same figure draw the graph of e^{-x}, which is obtained by changing the sign of x, and therefore is the reflexion of the first graph in the axis of y; (iii) for each value of x plot a point P (Fig. 84) whose ordinate is half the sum of the ordinates of the first two graphs; the locus of these points is the graph of $\cosh x$; (iv) plot the points, such as P', whose ordinates are half the differences of the ordinates of the first two curves. The locus of these points is the graph of $\sinh x$.

Cosh x, being an even function of x, has a graph which is symmetrical about the axis of y; as x increases from 0 to ∞, $\cosh x$ increases from 1 to ∞.

The graph of $y=\cosh x$ is a particular case of a curve which is well known in mechanics and is called a *catenary*, because it is the form assumed by a uniform chain suspended between two points and hanging in a vertical plane under the action of its own weight (see Art. 197).

* For full information as to the properties of these functions, and as to their relations to a rectangular hyperbola, the student is referred to such treatises as Chrystal's *Algebra* and Hobson's *Trigonometry*.

Sinh x, being an odd function of x, has a graph which is symmetrical about the origin. As x increases from 0 to ∞, $\sinh x$ increases from 0 to ∞. It is evident from the definitions that $\sinh x$ is always less than $\cosh x$, but becomes very nearly equal to it as x becomes large, since e^{-x} then $\rightarrow 0$.

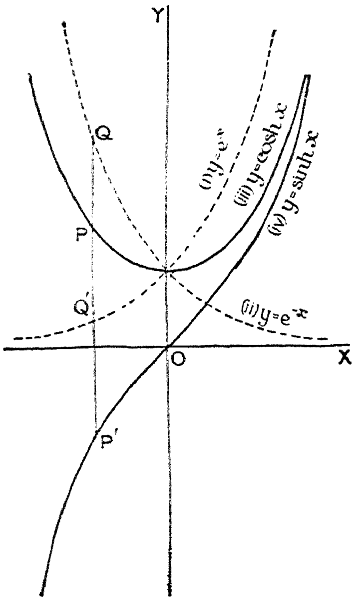


Fig. 84.

The graph of $\tanh x$ can easily be deduced from the facts that $\tanh x$ (i) is an odd function of x, (ii) is equal to 0 when $x = 0$, (iii) increases as x increases, and (iv) approaches the limit 1 as $\cdot\infty$, and is never > 1.

These functions are of comparatively recent introduction, and the calculations in many investigations are expedited by their use. Tables of their numerical values * for different values of the argument x have been compiled, as in the case of the circular functions. A table of values of $\sinh x$ and $\cosh x$ is given at the end of the book.

94. Inverse hyperbolic functions.

These functions bear to $\sinh x$, $\cosh x$, &c., the same relation that $\sin^{-1}x$, $\cos^{-1}x$, &c., bear to $\sin x$, $\cos x$, ...

If $x = \cosh y$, *we may write* $y = \cosh^{-1} x$, *and if* $x = \sinh y$, $y = \sinh^{-1} x$.

Since $\cosh x$ and $\sinh x$ were defined in terms of e^x, it might be expected that the inverse hyperbolic functions can be expressed in terms of $\log x$, the inverse of e^x, and this is the case.

If $y = \sinh^{-1} x$, then $\sinh y = x$, and

$$\therefore \quad \cosh y = \pm\sqrt{(1+\sinh^2 y)} = \sqrt{(1+x^2)}.$$

The + sign is taken since it follows from the definitions in Art. 92 that $\cosh y$ is always +.

From the definitions,
$$e^y = \sinh y + \cosh y$$
$$= x + \sqrt{(x^2+1)};$$
$$\therefore \quad \log\{x+\sqrt{(x^2+1)}\} = y = \sinh^{-1} x.$$

* See J. W. L. Glaisher, 'Tables, Mathematical,' in *Encyclopaedia Britannica* (11th ed.).

Similarly, if $y = \cosh^{-1} x$, then $x = \cosh y$, and

$$\sinh y = \pm \sqrt{(\cosh^2 y - 1)} = \pm \sqrt{(x^2 - 1)}.$$

$$\therefore \quad e^y = \sinh y + \cosh y = x \pm \sqrt{(x^2 - 1)},$$

and
$$\log \{x \pm \sqrt{(x^2 - 1)}\} = y = \cosh^{-1} x.$$

In this case, either sign may be taken; $\cosh^{-1} x$ is not a single-valued function of x [Art. 3].

The two values of $\cosh^{-1} x$ given by this equation, viz.: $\log\{x + \sqrt{(x^2 - 1)}\}$ and $\log\{x - \sqrt{(x^2 - 1)}\}$, differ in sign only, since their sum

$$\begin{aligned} &= \log\{x + \sqrt{(x^2 - 1)}\}\{x - \sqrt{(x^2 - 1)}\} \\ &= \log\{x^2 - (x^2 - 1)\} \\ &= \log 1 \\ &= 0, \end{aligned}$$

so that, for any value of x, there are two real values of $\cosh^{-1} x$ equal in magnitude and opposite in sign, as is obvious from the graph in Fig. 84. This is the graph of $y = \cosh x$, i.e. $x = \cosh^{-1} y$, and from the figure, it is evident that, taking any point on the axis of y, there are two points on the graph corresponding to it, which have equal and opposite abscissæ, i.e. to any value of y correspond two values of $\cosh^{-1} y$, equal in magnitude and opposite in sign.

In the case of $\sinh^{-1} x$, to each value of x corresponds one and only one value of $\sinh^{-1} x$.

Again, if $y = \tanh^{-1} x$, $x = \tanh y = (e^{2y} - 1)/(e^{2y} + 1)$,

whence
$$e^{2y} = \frac{1+x}{1-x}, \text{ and } y = \tfrac{1}{2} \log \frac{1+x}{1-x}.$$

This gives $\tanh^{-1} x$ in terms of logarithms.

Examples XXXII.

1. Find Lt $(1+m)^{1/m}$ as $m \to 0$. 2. Find Lt $(e^x - 1)/x$ as $x \to 0$.

3. Evaluate $\underset{x \to \infty}{\text{Lt}}\, xe^{-x}$, $\underset{x \to 0}{\text{Lt}}\, x \log x$, $\underset{x \to 0}{\text{Lt}}\, x^{-1} \log(1+x)$.

4. Calculate, from the series of Art. 90, the values, to 4 places of decimals, of $1/e^4$, $\sqrt[3]{e}$ and $1/\sqrt{e}$.

5. Prove that $\dfrac{1}{e} = \dfrac{2}{3!} + \dfrac{4}{5!} + \dfrac{6}{7!} + \dots$.

6. Expand $(e^{3x} + e^x)/e^{2x}$ in a series of ascending powers of x.

7. Prove that the series

$$2 + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \dots$$

is convergent, and find its sum.

8. Sum the series

$$1+\frac{3}{2}+\frac{9}{2.4}+\frac{27}{2.4.6}+\dots \text{ to infinity.}$$

9. Draw the graphs of e^{x^2} and e^{-x^2}.

10. Draw the graph of $\tanh x$.

11. Draw the graphs of $\pm e^{-x}$, $e^{-x}\sin x$ and $\pm e^{-x/2}$, $e^{-x/2}\cos x$.

12. Prove that

$$\sinh 2x = 2\sinh x\cosh x$$
$$\cosh 2x = \cosh^2 x+\sinh^2 x = 2\cosh^2 x-1 = 1+2\sinh^2 x.$$

13. Find, from the definitions, the values to 4 decimal places of
    (i) $\cosh 1$, $\sinh 1$, $\tanh 1$; (ii) $\cosh \frac{1}{4}$, $\sinh \frac{1}{4}$, $\tanh \frac{1}{4}$.

14. Given $\log_{10} 2 = {\cdot}3010$, find, by the aid of Art. 91, the values of $\log_e 20$, $\log_e 16$, $\log_2 e$.

15. Prove that $\quad (1+\tanh x)/(1-\tanh x) = e^{2x}$.

16. Show that

$$\cosh(x+y) = \cosh x\cosh y+\sinh x\sinh y.$$
$$\sinh(x+y) = \sinh x\cosh y+\cosh x\sinh y.$$

17. Draw graphs of $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$.

18. Draw the graph of $y = \log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$.

19. Prove that the functions $\tanh(1/x)$ and $e^{1/x}$ are discontinuous when $x = 0$. Draw their graphs.

20. Show that, if $|x| < a$,

$$\tanh^{-1}\frac{x}{a} = \frac{1}{2}\log\frac{a+x}{a-x},$$

and if $|x| > a$, $\qquad \coth^{-1}\frac{x}{a} = \frac{1}{2}\log\frac{x+a}{x-a}.$

21. Find $\quad \mathrm{Lt} \qquad \underset{x\to 0}{\mathrm{Lt}}\dfrac{\tanh x}{x}$, and $\quad \mathrm{L} \quad \dfrac{1-\cosh x}{}$

22. Find from Art. 94 and Table IX, the values of $\tanh^{-1}\frac{1}{3}$, $\sinh^{-1} 1$, $\cosh^{-1} 2$.

23. If $u = \log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)$, prove that

$$\sinh u = \tan\theta,\quad \cosh u = \sec\theta,\quad \tanh u = \sin\theta,\quad \tanh\tfrac{1}{2}u = \tan\tfrac{1}{2}\theta.$$

24. Prove that the coordinates of any point on the hyperbola $x^2/a^2-y^2/b^2 = 1$ can be expressed in the form $x = a\cosh u$, $y = b\sinh u$.

25. Calculate, by the aid of Art. 91 and a table of ordinary logarithms, the values of $\log_e 2$, $\log_e 10$, $\log_e 15$.

26. Calculate also the values of $e^{1/2}$, $e^{-2/3}$, e^{-2}; and compare with the results obtained by expanding by the exponential theorem, and retaining only terms of value greater than ${\cdot}001$.

27. Obtain, by aid of Table X, the values of $\sinh \frac{2}{3}$, $\cosh 2$, $\tanh 1{\cdot}5$, $\sinh^{-1} 1{\cdot}4$, $\cosh^{-1} 3$, $\coth \frac{1}{2}$.

28. Prove that $\sinh^{-1}\dfrac{x}{a} \quad \log\dfrac{x+\sqrt{(x^2+a^2)}}{a}$, $\cosh^{-1}\dfrac{x}{a} = \log\dfrac{x\pm\sqrt{(x^2-a^2)}}{a}$.

CHAPTER XI

DIFFERENTIATION OF EXPONENTIAL AND INVERSE FUNCTIONS

95. Introductory.

We will now show how to find the differential coefficients of the functions considered in the last chapter.

The differential coefficients of e^x and $\log x$ may be obtained in two ways. (i) We may find the d. c. of $\log x$ by the aid of the limit of Art. 87, and then deduce from the result the d. c. of e^x. [One advantage of this method is that it does not require the use of the exponential theorem. This may then be taken later on as a particular case of Taylor's Theorem (Chapter XXII).] (ii) We may find the d. c. of e^x first by the aid of the exponential theorem, and deduce from it the d. c. of $\log x$.

96. Differentiation of $\log x$ and e^x. First Method.

Taking the first of the two methods mentioned above, we have,

if $y = \log x$, $$\frac{dy}{dx} = \operatorname*{Lt}_{h \to 0} \frac{\log(x+h) - \log x}{h}$$

$$= \operatorname*{Lt}_{h \to 0} \frac{1}{h} \log \frac{x+h}{x} = \operatorname*{Lt}_{h \to 0} \frac{1}{h} \log\left(1 + \frac{h}{x}\right).$$

Let $h/x = 1/m$; then, as $h \longrightarrow 0$, $m \longrightarrow \infty$, and therefore, since $1/h = m/x$,

$$\frac{dy}{dx} = \operatorname*{Lt}_{m \to \infty} \frac{m}{x} \log\left(1 + \frac{1}{m}\right) = \frac{1}{x} \operatorname*{Lt}_{m \to \infty} m \log\left(1 + \frac{1}{m}\right)$$

$$= \frac{1}{x} \operatorname*{Lt}_{m \to \infty} \log\left(1 + \frac{1}{m}\right)^m = \frac{1}{x} \log e \text{ [Art. 87]}.$$

If e be the base of the logarithms, $\log e = 1$, and $dy/dx = 1/x$.

Hence *the d. c. of* $\log_e x = 1/x = x^{-1}$.

If the base be any other number a, then $dy/dx = x^{-1} \log_a e$.

The d. c. of e^x can now be at once deduced from this by the theorem of Art. 35.

For if $y = e^x$, $x = \log_e y$,

$$\therefore \quad dx/dy = 1/y, \text{ and } dy/dx = y = e^x;$$

i.e. *the d. c. of* $e^x = e^x$.

97. Differentiation of e^x. Second method.

Taking now the other method mentioned in Art. 95, we have, using the general method of Art. 26,

if $y = e^x$, $\dfrac{dy}{dx} = \underset{h\to 0}{\mathrm{Lt}}\dfrac{e^{x+h}-e^x}{h} = \underset{h\to 0}{\mathrm{Lt}}\dfrac{e^x(e^h-1)}{h}$

$$= \underset{h\to 0}{\mathrm{Lt}}\frac{e^x}{h}\left[h+\frac{h^2}{2\,!}+\frac{h^3}{3\,!}+\dots\right] \quad \text{(Art. 90)}$$

$$= \underset{h\to 0}{\mathrm{Lt}}\, e^x\left[1+\frac{h}{2\,!}+\frac{h^2}{3\,!}+\frac{h^3}{4\,!}+\dots\right]$$

$$= \underset{h\to 0}{\mathrm{Lt}}\, e^x\left[1+h\left(\frac{1}{2\,!}+\frac{h}{3\,!}+\frac{h^2}{4\,!}+\dots\right)\right].$$

The series within the inner brackets is convergent (Test 3, Art. 86), and therefore has a finite sum S.

$$\therefore\quad \frac{dy}{dx} = \underset{h\to 0}{\mathrm{Lt}}\, e^x[1+hS] = e^x.$$

The same result may be otherwise obtained as follows. The series

$$1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\dots$$

satisfies the conditions referred to in Art. 29 (ii). *Assuming this*, we have, on differentiating each term,

$$\text{d. c. of } e^x = 0+1+\frac{2x}{2\,!}+\frac{3x^2}{3\,!}+\frac{4x^3}{4\,!}+\dots$$

$$= 1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\dots = e^x.$$

Hence e^x is a function whose rate of change is, for any particular value of x, always equal to its own value, e.g. when e^x is equal to 4, it is increasing 4 times as fast as x; when $e^x = 100$, it is increasing 100 times as fast as x; and so on.

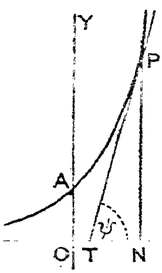


Fig. 85.

Geometrically, this means that, if P (Fig. 85) be any point on the graph of e^x, and if the tangent and ordinate of P meet the axis of x in T and N respectively, then

$$\tan PTN = dy/dx = e^x = y = NP.$$

Hence, since $\tan PTN = NP/TN$, it follows that TN is of unit length, wherever the point P be on the graph.

From the theorem of Art. 34, it follows that

$$\text{the d. c. of } e^{3x} = 3e^{3x},$$

$$\text{the d. c. of } e^{ax+b} = ae^{ax+b},$$

$$\text{the d. c. of } e^{\sin x} = e^{\sin x} \times \cos x,$$

and generally, the d. c. of e^u with respect to $x = e^u \times du/dx$, where u is any function of x.

Hence the rate of increase of the function e^{ax+b} is ae^{ax+b}, which is always proportional to the value of the function, and it will be seen later (Art. 99) that e^{ax+b} is the only function for which this is true. This is the reason that this function occurs so frequently in the investigation of natural phenomena. See Art. 181.

98. Differentiation of $\log x$. Second method.

The d. c. of $\log x$ can at once be deduced from that of e^x by Art. 35.

If $y = \log x$, then $x = e^y$,

$$\therefore \quad dx/dy = e^y = x, \text{ and } dy/dx = 1/x.$$

Hence *the d. c. of* $\log_e x = 1/x = x^{-1}$.

To find the d. c. of $\log_{10} x$, the result of Art. 91 may be used.

$$\log_{10} x = \log_e x/\log_e 10 = \mu \log_e x;$$

$$\therefore \quad \text{the d. c. of } \log_{10} x = \mu \times 1/x = \cdot 434\ldots/x,$$

and generally, the d. c. of $\log_a x = \dfrac{1}{\log_e a} \cdot \dfrac{1}{x}$.

From Art. 34, it follows that

$$\text{the d. c. of } \log(x+5) = 1/(x+5),$$

$$\text{the d. c. of } \log(3x-2) = 3/(3x-2),$$

$$\text{the d. c. of } \log(x^2+1) = 2x/(x^2+1),$$

$$\text{the d. c. of } \log \sin x = \cos x/\sin x = \cot x,$$

and generally, the d. c. of $\log u = u^{-1}\, du/dx$, where u is any function of x.

A rather more complicated case is

$$\text{the d. c. of } \log\{x + \sqrt{(x^2 \pm a^2)}\} = \frac{1}{x+\sqrt{(x^2 \pm a^2)}} \left(1 + \frac{2x}{2\sqrt{(x^2 \pm a^2)}}\right)$$

$$= \frac{1}{x+\sqrt{(x^2 \pm a^2)}} \cdot \frac{\sqrt{(x^2 \pm a^2)} + x}{\sqrt{(x^2 \pm a^2)}}$$

$$= \frac{1}{\sqrt{(x^2 \pm a^2)}}.$$

This is an important result, to which we shall have occasion to refer later. [See Art. 128.]

In differentiating expressions which involve logarithms, it is advisable to begin by making use of the properties of logarithms as

shown in the following examples. In many cases, the work of differentiation is thereby rendered much less complicated.

Examples:

(i) The d. c. of $\log\dfrac{x(x-3)}{4-x}$ = the d. c. of $[\log x+\log(x-3)-\log(4-x)]$

$$=\frac{1}{x}+\frac{1}{x-3}+\frac{1}{4-x}.$$

(ii) The d.c. of $\log\dfrac{x}{\sqrt{(x^2-1)}}$ = the d. c. of $[\log x-\frac{1}{2}\log(x^2-1)]$

$$=\frac{1}{x}-\frac{1}{2}\cdot\frac{2x}{x^2-1}=\frac{-1}{x(x^2-1)}.$$

99. Integrals of e^x and $1/x$ or x^{-1}.

Corresponding to the two differential coefficients of the preceding articles, we have the two very important integrals,

$$\int e^x\,dx=e^x,$$

$$\int\frac{1}{x}\,dx=\log x.$$

The latter supplies the one case which was missing in the result of Art. 74. It was there shown that $\int x^n\,dx=x^{n+1}/(n+1)$ *except when* $n=-1$. When $n=-1$, the integral becomes $\int x^{-1}\,dx$, which we now see to be $\log x$.

Using the theorems of Art. 75, we have

$$\int e^{x+a}\,dx=e^{x+a};\qquad \int e^{2x}\,dx=\tfrac{1}{2}e^{2x};$$

$$\int e^{x/a}\,dx=ae^{x/a};\qquad \int e^{mx}\,dx=e^{mx}/m,$$

and generally, $\int e^{ax+b}\,dx=e^{ax+b}/a.$

Similarly, $\displaystyle\int\frac{1}{x+3}\,dx=\log(x+3)$, $\displaystyle\int\frac{1}{5x-2}\,dx=\tfrac{1}{5}\log(5x-2)$,

$\displaystyle\int\frac{1}{a-x}\,dx=-\log(a-x)$, and generally, $\displaystyle\int\frac{1}{ax+b}\,dx=\frac{1}{a}\log(ax+b)$.

We have $\int dx/(x-a)=\log(x-a)$; but, if $x<a$, $x-a$ is $-$, and $\log(x-a)$ imaginary. In this case, we may write

$$\int\frac{dx}{x-a}=-\int\frac{dx}{a-x}=\text{(Art. 75)}\log(a-x).$$

In particular, if x is $-$, $\displaystyle\int\frac{1}{x}\,dx$ is not $\log x$ (which is imaginary), but $\log(-x)$.

We can now prove the statement made in Art. 97 that a function whose rate of change is proportional to its own value is of the form e^{ax+b} [which may be written $e^{ax}\times e^b$, i. e. ke^{ax}, on writing k for the constant factor e^b]. For if y be such a function, we have, since dy/dx is the rate of change of y with respect to x,

$$dy/dx=ay,\text{ which may be written }dx/dy=1/ay.$$

Hence $x = \int \frac{1}{ay}\,dy = \frac{1}{a}\log y + C$, where C is an arbitrary constant.

Therefore $\log y = ax - aC$, i.e. $y = e^{ax-aC} = ke^{ax}$,
writing k for the arbitrary constant e^{-aC}.

Examples XXXIII.

Differentiate:

1. e^{4x}, e^{3-x}, e^{a+bx}, e^{x^2}, $e^{x/a}$, e^{p-qx}.
2. $e^{\sin x}$, $e^{\cos x}$, $e^{\sin ax}$, $e^{a\cos x}$, $e^{\tan x}$.
3. $x^3 e^{3x}$, $x^n e^{ax}$, $e^{ax}\sin ax$, $e^{-3x}\cos 3x$, $e^{ax}\cos bx$, $e^{ax}\sin^2 x$.
4. e^{2x}/x^2, $e^{ax}/\sqrt{x}$, $e^x/\tan x$, $(ax^2+bx+c)/e^x$.
5. $\log(2x-1)$, $\log(2-x)$, $\log(x^2-1)$, $\log(a+bx^2)$.
6. $\log(5+7x)$, $\log(p-qx)$, $\log(x^2-3x-1)$, $\log(1-x^3)$.
7. $\log\cos x$, $\log\tan x$, $\log(a+b\sin x)$, $\log(3-4\cos x)$. $\log(1+\cos^2 x)$.
8. $x^n\log x$, $x^2\log(2-x)$, $x\log(1-x^2)$.
9. $\frac{\log x}{x}$, $\frac{\log x}{x^n}$, $\frac{\log(ax+b)}{x^2}$, $\frac{\log x}{\sqrt{x}}$.
10. $\log\{x^n(x+2)\}$, $\log\sqrt[n]{(1+x^2)}$, $\log\frac{x}{\sqrt{(x^2+a^2)}}$, $\log\frac{x(1-x)}{(3-x)^2}$.
11. $\log\sqrt{\sin x}$, $\log\sqrt{[x(1-x)]}$, $\log\frac{(x-1)^2}{2x-3}$, $\log\frac{\sin x}{2-\cos x}$.
12. $\log[x+\sqrt{(x^2-1)}]$, $\log[\sqrt{(x-1)}+\sqrt{(x+1)}]$, $\log[\sqrt{(bx-a)}+\sqrt{(a+bx)}]$.

Find the 2nd, 3rd, and nth differential coefficients of:

13. e^{ax}.
14. $\log x$.
15. e^{-x}.
16. e^{a-bx}
17. $\log(1-x)$.
18. $\log(a+bx)$.
19. Prove that the equation $d^2y/dx^2 = a^2y$ is satisfied by $y = Ae^{ax}+Be^{-}$ where A and B denote any constants.

Write down the integrals of:

20. e^{2x}, e^{5-x}, $e^{x/a}$, e^{px+q}, $\sqrt{e^x}$, $\sqrt[n]{e^x}$, e^{-x}, $e^{-x/a}$.
21. $\frac{1}{5x+3}$, $\frac{1}{7-2x}$, $\frac{1}{x-a}$, $\frac{1}{p-qx}$, $\frac{1}{bx+c}$, $\frac{1}{8+3x}$.
22. $\frac{1}{8-5x}$, $\frac{1}{1-x}$, $\frac{4}{4x-5}$, $\frac{b}{a-bx}$, $\frac{a}{bx+c}$, $\frac{2}{3(5-2x)}$.

100. Differential coefficient of $\sin^{-1}x$.

The differential coefficients of the inverse circular functions are easily obtained by the rule of Art. 35.

If $y = \sin^{-1}x$, $x = \sin y$,

$$\therefore\quad dx/dy = \cos y = \pm\sqrt{(1-\sin^2 y)} = \pm\sqrt{(1-x^2)},$$

$$\therefore\quad \frac{dy}{dx} = \pm\frac{1}{\sqrt{(1-x^2)}}.$$

The double sign $\pm$ needs some consideration.

The function $\sin^{-1} x$ is a many-valued function of x; it is undefined for values of x which are greater than unity (notice that for such values the d. c. is imaginary), but if x has any value between -1 and $+1$, both inclusive, there is an infinite number of values of $\sin^{-1} x$ corresponding thereto [e.g. if $x = \frac{1}{2}$, $\sin^{-1} x$ may be $\frac{1}{6}\pi$ or $\frac{5}{6}\pi$, or either of these $\pm$ any multiple of 2π], but among all these values there will be one and only one between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

The angle between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, whose sine is equal to x, is called the *principal value* of $\sin^{-1} x$.

If we take therefore the principal value of $\sin^{-1} x$, then, as x increases from -1 to $+1$, $\sin^{-1} x$ increases from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, and hence its d. c. will be $+$ (Art. 25). In this case

$$dy/dx = +1/\sqrt{(1-x^2)}.$$

There will be one angle between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, whose sine is equal to x; if we were to take this value of $\sin^{-1} x$, then $\sin^{-1} x$ increases from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$ as x decreases from $+1$ to -1; hence in this case its d. c. would be $-$,

i.e. $\quad -1/\sqrt{(1-x^2)}$.

The working of course gives both signs, because the selection of one angle as principal value is a mere arbitrary convention of which the analysis takes no account. In the general case, the sign of dy/dx is the same as the sign of $\cos y$.

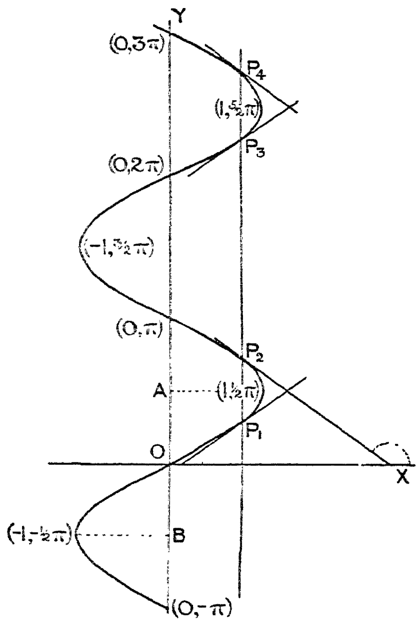


Fig. 86.

The meaning of the double sign is perhaps best seen geometrically. The graph of $y = \sin^{-1} x$ is shown in Fig. 86. It bears the same relation to the axes of y and x as the graph of $y = \sin x$ bears to the axes of x and y. An ordinate corresponding to a value of $|x| > 1$ does not meet the graph at all. For such values of x, the function is undefined. An ordinate corresponding to a value of $|x| < 1$ cuts the graph at an infinite number of points $P_1, P_2, P_3 \ldots$.

The value of the d. c. at any point is the slope of the curve at that point, and it is evident from the figure that at the points P_1, P_3, ... the tangents to the curve make acute angles with the axis of x, and the slope is $+$; whereas at the points P_2, P_4, ... the angles are obtuse, and the slope $-$. The principal value is represented by the ordinate between OB and OA, and between these two the slope is everywhere $+$.

101. Differential coefficient of $\cos^{-1} x$.

If $y = \cos^{-1} x, \quad x = \cos y,$

$$\therefore \quad dx/dy = -\sin y = \pm\sqrt{(1-\cos^2 y)} = \pm\sqrt{(1-x^2)},$$

$$\therefore \quad dy/dx = \pm 1/\sqrt{(1-x^2)}.$$

The double sign is accounted for in the same way as in the preceding case. There is one and only one angle between 0 and π, whose cosine is equal to x (if $|x| < 1$). This is taken as the principal value of $\cos^{-1} x$. [The range $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ would not serve in this case, since throughout this range the cosine is always $+$.]

Taking the principal value, $\cos^{-1} x$ increases from 0 to π, as x decreases from $+1$ to -1, therefore its d. c. is $-$,

$$\text{i.e. the d. c. of } \cos^{-1} x = -1/\sqrt{(1-x^2)}.$$

In the general case, the sign of dy/dx is opposite to the sign of $\sin y$. This can be illustrated geometrically as in the case of $\sin^{-1} x$.

This result can also be deduced from the preceding result, for, taking the principal values, $\cos^{-1} x = \frac{1}{2}\pi - \sin^{-1} x$.

$$\therefore \quad \text{d. c. of } \cos^{-1} x = -\text{d. c. of } \sin^{-1} x,$$

since the d. c. of the constant $\frac{1}{2}\pi$ is zero.

102. Differential coefficient of $\tan^{-1} x$.

If $y = \tan^{-1} x, \qquad x = \tan y;$

$$\therefore \quad dx/dy = \sec^2 y = 1 + \tan^2 y = 1 + x^2,$$

$$\therefore \quad dy/dx = 1/(1+x^2).$$

There is no ambiguity of sign in this case, since, as was pointed out in Art. 52, y and x always increase together, and therefore dy/dx is always $+$. $\operatorname{Tan}^{-1} x$ is a many-valued function of x, which is defined for all real values of x; there is one and only one angle between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, which has a tangent equal to x, and this is taken as the principal value of $\tan^{-1} x$. [Either of the ranges $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ and 0 to π would serve in the case of $\tan^{-1} x$; $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ is the one adopted.]

Geometrically, any ordinate cuts the graph of $\tan^{-1} x$ in an infinite number of points $\dots P_{-1}, P_1, P_2, \dots$, but at all these points the tangents make acute angles with the axis of x, and the slope is $+$. (Fig. 87.)

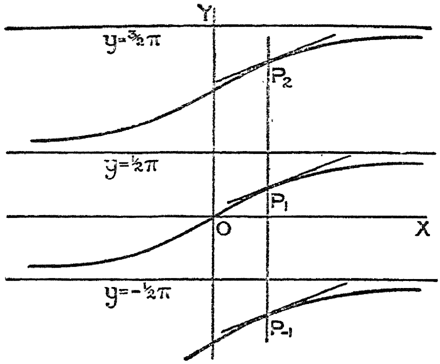


Fig. 87.

The differential coefficients of $\cot^{-1} x$, $\sec^{-1} x$, and $\operatorname{cosec}^{-1} x$, which occur much less frequently, can be obtained in a similar manner. The differential coefficients of all the inverse circular functions can also be obtained geometrically from the figure of Art. 39.

By Art. 34, the d. c. of $\sin^{-1}\dfrac{x}{a} = \dfrac{1}{\sqrt{(1-x^2/a^2)}} \times \dfrac{1}{a} = \dfrac{1}{\sqrt{(a^2-x^2)}}$;

the d. c. of $\tan^{-1}\dfrac{x}{a} = \dfrac{1}{1+x^2/a^2} \times \dfrac{1}{a} = \dfrac{a}{a^2+x^2}$;

the d. c. of $\sin^{-1}(2\sin x) = \dfrac{1}{\sqrt{(1-4\sin^2 x)}} \times 2\cos x$.

Expressions involving inverse circular functions can sometimes be simplified before differentiation.

The d. c. of $\tan^{-1}\dfrac{1}{x} = \dfrac{1}{1+1/x^2} \times -\dfrac{1}{x^2} = -\dfrac{1}{1+x^2}$,

or, as is obvious geometrically, $\tan^{-1}(1/x) = \frac{1}{2}\pi - \tan^{-1} x$;

$\therefore$ the d. c. $= -1/(1+x^2)$.

Again, the d. c. of $\cos^{-1}\sqrt{(1-x^2)} =$ d. c. of $\sin^{-1} x = 1/\sqrt{(1-x^2)}$.

The relations between the functions can often be seen geometrically, by drawing a right-angled triangle.

Reversing the first two of these examples, we get the important integrals:—

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a}; \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},$$

which we shall have occasion to use frequently later.

103. Differential coefficients and integrals of hyperbolic functions.

The d. c. of $\sinh x$, i.e. of $\frac{1}{2}(e^x - e^{-x})$, $= \frac{1}{2}(e^x + e^{-x}) = \cosh x$.

The d. c. of $\cosh x$, i.e. of $\frac{1}{2}(e^x + e^{-x})$, $= \frac{1}{2}(e^x - e^{-x}) = \sinh x$.

The differential coefficients of the other hyperbolic functions may, if required, be deduced from these two in exactly the same way as in the case of the circular functions (Art. 42).

By Art. 34, the d. c. of $\sinh 2x = 2\cosh 2x$,

the d. c. of $\sinh mx = m\cosh mx$,

$$\text{the d. c. of } \cosh\frac{x}{a} = \frac{1}{a}\sinh\frac{x}{a}.$$

Conversely

$$\int \cosh x\, dx = \sinh x, \qquad \int \sinh x\, dx = \cosh x.$$

$$\int \cosh mx\, dx = (\sinh mx)/m, \qquad \int \sinh mx\, dx = (\cosh mx)/m.$$

104. Differential coefficients of the inverse hyperbolic functions and corresponding Integrals.

(i) If $y = \sinh^{-1} x$, $\quad x = \sinh y$,

$\therefore\ dx/dy = \cosh y = +\sqrt{(1 + \sinh^2 y)}$ (Art. 92) $= +\sqrt{(1+x^2)}$.

The + sign is taken since $\cosh y$ cannot be negative,

$$\therefore\ \frac{dy}{dx} = +\frac{1}{\sqrt{(1+x^2)}}.$$

(ii) If $y = \cosh^{-1} x$, $\quad x = \cosh y$,

$$\therefore\ dx/dy = \sinh y = \pm\sqrt{(\cosh^2 y - 1)} = \pm\sqrt{(x^2-1)},$$

$$\therefore\ \frac{dy}{dx} = \pm\frac{1}{\sqrt{(x^2-1)}}.$$

Either sign may be taken here, because y is a two-valued function of x [which is defined for values of x such that $|x| > 1$; if $|x| < 1$, the d. c. is imaginary]. To each value of x (> 1) correspond two values of $\cosh^{-1} x$, equal in magnitude and opposite in sign (Art. 94). Taking the positive value, $\cosh^{-1} x$ increases from 0 to ∞ as x increases from 1 to ∞; therefore dy/dx is +. Taking the negative value, $\cosh^{-1} x$ decreases from 0 to $-\infty$ as x increases from 1 to ∞; therefore dy/dx is $-$. Hence the d. c. of $\cosh^{-1} x = \pm 1/\sqrt{(x^2-1)}$, according as the positive or negative value of $\cosh^{-1} x$ is taken.

By Art. 34, the d. c. of $\sinh^{-1}\dfrac{x}{a} = \dfrac{1}{\sqrt{(1+x^2/a^2)}} \times \dfrac{1}{a} = \dfrac{1}{\sqrt{(a^2+x^2)}}$;

$$\text{the d. c. of } \cosh^{-1}\frac{x}{a} = \pm\frac{1}{\sqrt{(x^2/a^2-1)}} \times \frac{1}{a} = \pm\frac{1}{\sqrt{(x^2-a^2)}}.$$

Conversely, we get the two important integrals:

$$\int \frac{1}{\sqrt{(a^2+x^2)}}\,dx = \sinh^{-1}\frac{x}{a}; \quad \int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \cosh^{-1}\frac{x}{a}.$$

These two integrals can also be expressed as logarithms. See Ex. XXXII. 28.

Examples XXXIV.

Differentiate

1. $\sin^{-1}\frac{1}{2}x$, $\sin^{-1}(a/x)$, $\sin^{-1}\sqrt{x}$, $\sin^{-1}(1-1/x)$.
2. $\cos^{-1}(x^2)$, $\cos^{-1}mx$, $\cos^{-1}(1/\sqrt{x})$, $\cos^{-1}(\sin x)$.
3. $\tan^{-1}(a-x)$, $\tan^{-1}(\cot x)$, $\tan^{-1}(x^2/a^2)$, $\tan^{-1}\sqrt{x}$.
4. $\cot^{-1}x$, $\cot^{-1}(x/a)$, $\cot^{-1}(1/x)$.
5. $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, $\sec^{-1}(a/x)$, $\operatorname{cosec}^{-1}(1-x^2)^{-1/2}$.
6. $\sinh\frac{3}{4}x$, $\sinh(x^2)$, $\sinh(1/x)$, $\sinh^2 x$.
7. $\cosh(ax+b)$, $\cosh^3 x$, $\sinh x+\frac{1}{3}\sinh^3 x$.
8. $\tanh x$, $\coth x$, $\tanh(x/a)$, $\coth(a/x)$.
9. $\sinh^{-1}\frac{1}{3}x$, $\cosh^{-1}(x^2/a^2)$.
10. $\tanh^{-1}x$, $\coth^{-1}x$.
11. $(1+x^2)\tan^{-1}x$, $x\cos^{-1}x$.
12. $\sqrt{(1-x^2)}\sin^{-1}x$.
13. $\sin^{-1}\sqrt{(1-x^2)}$.
14. $\operatorname{cosec}^{-1}(1/x)$.
15. $\cos^{-1}\dfrac{1-x^2}{1+x^2}$.
16. $\tan^{-1}\dfrac{2x}{1-x^2}$

Integrate

17. $\dfrac{1}{\sqrt{(9-x^2)}}$, $\dfrac{1}{\sqrt{(5-x^2)}}$, $\dfrac{1}{\sqrt{[1-(x+1)^2]}}$, $\dfrac{1}{\sqrt{(9-4x^2)}}$, $\dfrac{1}{\sqrt{(a^2-b^2x^2)}}$.
18. $\dfrac{1}{1+x^2}$, $\dfrac{1}{100+x^2}$, $\dfrac{1}{x^2+7}$, $\dfrac{1}{9x^2+4}$, $\dfrac{1}{2x^2+5}$, $\dfrac{1}{a^2x^2+b^2}$, $\dfrac{1}{bx^2+c}$
19. $\dfrac{1}{\sqrt{(x^2+16)}}$, $\dfrac{1}{\sqrt{(x^2-9)}}$, $\dfrac{1}{\sqrt{(5+x^2)}}$, $\dfrac{1}{\sqrt{(4x^2-1)}}$, $\dfrac{1}{\sqrt{(25x^2+9)}}$, $\dfrac{1}{\sqrt{(b^2x^2-a^2)}}$.
20. $\sinh 3x$, $\cosh 2x$, $\sinh(x/a)$, $\cosh(x/a)$.

105. Applications.

We will now work out a few more examples in illustration of the principles of Chapters V–VIII, introducing some of the differential coefficients just obtained.

Examples:

(i) *Prove that in the catenary* $y = a\cosh(x/a)$ (Art. 93), *the length of the perpendicular* NK (Fig. 88) *from the foot of the ordinate* PN *to the tangent at* P *is constant. Show also that the length of the arc, measured from the vertex* A *of the curve to the point* P, *is equal to* PK.

When $x=0$, $\cosh(x/a)=1$, and $y=a$.

$$NK = y\cos\psi, \text{ and } \tan\psi = \frac{dy}{dx} = a\sinh\frac{x}{a}\times\frac{1}{a} = \sinh\frac{x}{a};$$

$$\therefore \quad \cos\psi = \frac{1}{\sqrt{[1+(dy/dx)^2]}} = \frac{1}{\sqrt{\{1+\sinh^2(x/a)\}}} = \frac{1}{\cosh(x/a)} \text{ (Art. 92)};$$

$$\therefore \quad NK = y\cos\psi = a\cosh(x/a) \times \frac{1}{\cosh(x/a)} = a, \text{ which is constant.}$$

It follows from this that $OA = a$, for if P be taken at A, the tangent at A is parallel to OX, and AO is the ordinate; hence NK in this case becomes OA, which is therefore equal to a.

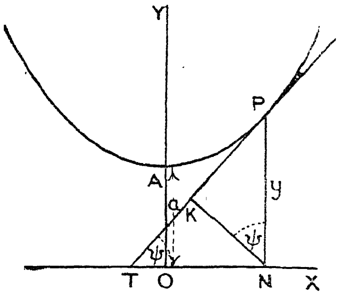


Fig. 88.

To prove the second part of the question, we have (Art. 82)

$$ds/dx = \sec\psi = \cosh(x/a);$$

$$\therefore \quad s = \int \cosh(x/a)\,dx = a\sinh(x/a) + C.$$

Since s is measured from A, $s = 0$ when $x = 0$. $\quad\therefore\quad 0 = C$,

and
$$s = a\sinh(x/a) = a\tan\psi$$
$$= KN\tan\psi = PK.$$

(ii) *Find the maxima, minima and points of inflexion of the curve* $y = x^2e^{-x^2}$, *and draw it roughly.*

The d.c. of $e^{-x^2} = e^{-x^2} \times$ d.c. of $-x^2 = e^{-x^2} \times -2x$.

$$\therefore \quad dy/dx = x^2 \, . \, e^{-x^2}(-2x) + e^{-x^2} \, . \, 2x$$
$$= 2xe^{-x^2}(1-x^2). \tag{1}$$

Writing this as $2e^{-x^2}(x-x^3)$, for convenience in differentiating again, we have

$$d^2y/dx^2 = 2e^{-x^2}(1-3x^2) + (x-x^3)\,2e^{-x^2}(-2x)$$
$$= 2e^{-x^2}(1-3x^2-2x^2+2x^4)$$
$$= 2e^{-x^2}(1-5x^2+2x^4). \tag{2}$$

From (1), $dy/dx = 0$, when $x = 0$ or ± 1.

If $x = 0$, $d^2y/dx^2 = +2$; $\therefore$ $x = 0$ makes y a minimum, and equal to 0.

If $x = \pm 1$, $d^2y/dx^2 = 2e^{-1}(-2) = -4/e$; $\therefore$ $x = \pm 1$ makes y a maximum, and equal to $1 \times e^{-1}$, i.e. $\cdot 37$ nearly.

Hence $(0, 0)$ is a minimum, and $(1, \cdot 37)$, $(-1, \cdot 37)$ are maxima.

From (2), $d^2y/dx^2 = 0$, when $2x^4 - 5x^2 + 1 = 0$,

i.e. when
$$x^2 = \tfrac{1}{4}(5 \pm \sqrt{17}) = 2\cdot 28 \text{ or } \cdot 22 \text{ nearly},$$
and
$$x = \pm 1\cdot 51 \text{ or } \pm\cdot 47 \text{ nearly}.$$

d^2y/dx^2 changes sign in passing through each of these values, since none of them are repeated; hence there are 4 points of inflexion.

When $x^2 = 2{\cdot}28$, $y = 2{\cdot}28\,e^{-2\cdot28} = {\cdot}23$; and $dy/dx = \pm 3{\cdot}02\,e^{-2\cdot28} \times -1{\cdot}28 = \mp {\cdot}39$ nearly;

when $x^2 = {\cdot}22$, $y = {\cdot}22e^{-\cdot22} = {\cdot}18$; and $dy/dx = \pm {\cdot}94e^{-\cdot22} \times {\cdot}78 = \pm {\cdot}59$ nearly.

These give the coordinates and the slopes of the tangents at the four points of inflexion.

The graph is symmetrical about the axis of y, y can never be $-$, and $y \to 0$ as $x \to \pm\infty$; hence the graph is roughly as shown in Fig. 89.

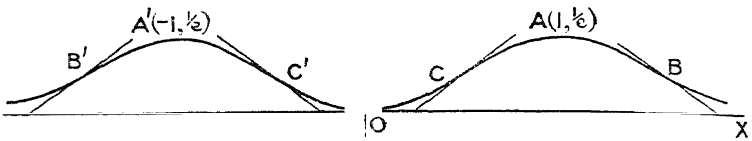


Fig. 89.

The minimum is at O, the maxima at A and A', and B, B', C, C' are the points of inflexion.

(iii) *Find the difference for* 1 *minute in the neighbourhood of* 60°, *in a table of logarithmic tangents.*

The function whose change we have to find is $\log_{10} \tan x$.

The d.c. of $\log_{10} \tan x$ = the d.c. of $\mu \log_e \tan x$ (Art. 91)

$$= \mu \cdot \frac{1}{\tan x} \sec^2 x = \frac{\mu}{\sin x \cos x};$$

i.e. if x increases by a very small amount (in radian measure), the logarithmic tangent will increase by approximately $\mu/\sin x \cos x$ times as much.

Now the given increase in x is the circular measure of $1'$, i.e. $\pi/10800$, and the value of x is $\frac{1}{3}\pi$.

Hence the increase in $\log_{10} \tan x = \dfrac{\mu}{\sin\frac{1}{3}\pi \cos\frac{1}{3}\pi} \times \dfrac{\pi}{10800}$

$$= \frac{{\cdot}434\ldots}{{\cdot}866\ldots \times {\cdot}5} \times \frac{3{\cdot}1416}{10800}$$

$$= {\cdot}00029 \text{ approximately};$$

therefore $\log_{10} \tan 60°\,1'$ exceeds $\log_{10} \tan 60°$ by ·00029, as can be verified by reference to a book of mathematical tables.

(iv) *A point moves in a straight line, so that its distance* s *from a fixed point* O *in the line at the end of time* t *is given by the equation* $s = ae^{-kt} \sin bt$. *Determine the nature of the motion.*

The velocity is given by

$$v = ds/dt = a\left[e^{-kt}\, b \cos bt + \sin bt \,.\, (-ke^{-kt})\right]$$
$$= ae^{-kt}[b \cos bt - k \sin bt].$$

$ds/dt = 0$, i.e. the velocity is zero, and s is a maximum or minimum, when $b \cos bt = k \sin bt$ (since e^{-kt} cannot be 0), i.e. when $\tan bt = b/k$, or $bt = \alpha + n\pi$, where α is any one angle whose tangent is equal to b/k.

Hence maximum and minimum values of s occur, and the particle is (for an instant) at rest, when bt increases by a multiple of π (from the value α), i.e. they occur at intervals of time π/b.

If $bt = \alpha$, $\qquad s = ae^{-k\alpha/b} \sin \alpha.$

The next maximum or minimum is given by $bt = \alpha + \pi$,

and then
$$\begin{aligned} s &= ae^{-k(\alpha+\pi)/b} \sin (\alpha + \pi) \\ &= ae^{-k\alpha/b - k\pi/b} \times -\sin \alpha \\ &= -ae^{-k\alpha/b} \sin \alpha \times e^{-k\pi/b}. \end{aligned}$$

The $-$ sign indicates that this is on the opposite side of the origin, and this distance is equal to the preceding one multiplied by $e^{-k\pi/b}$; also this is

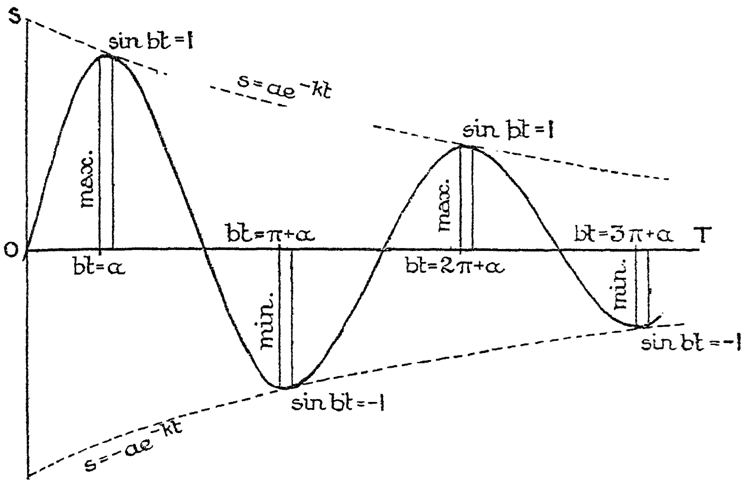


Fig. 90.

true for any two consecutive stationary points, since α is any value which makes ds/dt vanish. Hence the point oscillates to and fro through the origin, over distances which decrease in geometrical progression with a common ratio $e^{-k\pi/b}$, and the turning-points occur at equal intervals of time, the time from any one to the next one being π/b; s decreases as t increases, on account of the factor e^{-kt}, and the oscillations gradually die away.

This is the case of a particle performing 'damped oscillations'; it should be compared with ordinary simple harmonic motion, given by the equation $s = a \sin bt$, without the exponential factor e^{-kt}. In the series of maximum and minimum values of s, the ratio of any term to the next term is $e^{k\pi/b}$; the logarithm of this ratio, $k\pi/b$, is called the *logarithmic decrement*.

The student should pay attention to the graphical representation of the motion. It has been noticed that the maximum and minimum values of $ae^{-kt}\sin bt$ occur at constant intervals π/b, so that there is the same interval between consecutive maxima and minima as in the case of $a\sin bt$; but the actual values of t which give maxima and minima do not coincide for the two functions unless $\alpha = \frac{1}{2}\pi$, in which case $k=0$, and the two functions coincide.

Again, the graphs of $s = ae^{-kt}\sin bt$ and $s = ae^{-kt}$ meet where $\sin bt = 1$, and therefore $\cos bt = 0$; at these points on the first graph, ds/dt becomes $ae^{-kt} \times -k$, which is also the d.c. of ae^{-kt}. Hence the graphs of the two functions have the same slope where they meet, and therefore they touch each other at their common points. Similarly the graphs of the given function and of $s = -ae^{-kt}$ meet and touch one another when $\sin bt = -1$. Fig. 90 shows the form of the graph; its actual dimensions depend upon the numerical values of a, b, k.

Examples XXXV.

1. Prove that, in the curve $y = ce^{x/a}$, the subtangent is constant, and the subnormal varies as the square on the ordinate.
2. Find the lengths of the subtangent and subnormal in the catenary $y = c\cosh(x/c)$.
3. Show that, at the point of intersection of $y = ce^{x/c}$ and $y = c\cosh(x/c)$, the subnormal in the former curve is equal to the normal in the latter.
4. Prove that the curves $y = ae^{-kx}$ and $y = ae^{-kx}\cos bx$ touch at the points where $x = 2n\pi/b$.
5. In the curve $y = b\log(x/a)$, the tangent at any point P meets the axis of y in T, and PM is drawn perpendicular to the axis of y; prove that MT is of constant length.
6. Find the equation of the tangent and normal to the curve $y = \log x$, at the point where it cuts the axis of x.
7. Prove that, in the catenary $y = a\cosh(x/a)$, $ds/dx = y/a$. Hence find the length of the arc s from the vertex to any point (x, y) on the curve, and prove that $y^2 = s^2 + a^2$.
8. Find the angle between the tangents at two consecutive points of intersection of the ordinate $x = \frac{1}{2}$ with the curve $y = \sin^{-1}x$.
9. When is the ratio of the logarithm of a number to the number itself a maximum?
10. Examine $a\cosh x + b\sinh x$ for maxima and minima.
11. Find the minimum value of $ae^{kx} + be^{-kx}$.
12. Find the minimum value of $x/\log x$, and the points of inflexion on its graph. Sketch the graph.
13. Find the maximum value of xe^{-x} and the points of inflexion of $y = xe^{-x}$. Sketch the graph.
14. If x be the ratio of the radius of the core of a submarine cable to the thickness of the covering, the speed of signalling varies as $x^2\log(1/x)$. For what value of x will the speed be greatest?

15. The graphs of $y = \sinh x$ and $y = 8\tanh x$ are drawn with the same axes; find where the distance between them, measured parallel to the axis of y, is greatest.

16. Find the maximum and minimum values of $e^{-ax}\sin ax$.

17. Find the maxima and minima of $e^{-7x}\cos(10x - 35°)$ [$\tan 35° = \cdot 7$].

18. Find the points of inflexion of $y = e^{-x^2}$, and draw the graph.

19. Find the maxima, minima and points of inflexion of $y = xe^{-x^2}$. Trace the curve.

20. Show that the origin is a point of inflexion on the graphs of $\sinh x$ and $\tanh x$, and that the graphs of e^x, $\cosh x$ and $\log x$ have no points of inflexion.

21. Prove that, in the curve $y = a\log\sec(x/a)$, $ds/dx = \sec(x/a)$.

22. Find the difference for 1 minute in a table of logarithmic cosines in the neighbourhood of 45°.

23. Find the difference for 1 minute in a table of logarithmic sines in the neighbourhood of 120°.

24. Find the area between the axes of coordinates, the graph of e^x, and the ordinate $x = 3$.

25. If this area rotates about the axis of x, find the volume of the solid generated.

26. Find the area between the rectangular hyperbola $xy = 20$, the axis of x, and the ordinates $x = 2$, $x = 5$.

27. Find the area between the catenary $y = a\cosh(x/a)$, the axes and the ordinate $x = b$.

28. Find the area between the axis of x, $y = \sinh x$, and $x = 4$.

29. Find the area between $y = \cosh x$, $y = \sinh x$, and the ordinates $x = 1$, $x = 5$.

30. Find the area between the axis of y, $y = \cosh x$, $y = \sinh x$, and $x = a$. If the ordinate $x = a$ recedes to a very great distance, to what limit does this area tend?

31. Find the area between the axis of y, the curve $(a^2 + x^2)y^2 = a^4$ and (i) $x = a$, (ii) $x = b$.

32. The two areas in the preceding question rotate about the axis of x; find the volumes generated. To what limit does the latter volume tend as $b \to \infty$?

33. Find the area between the axis of y, the curve $y^2 = a^4/(a^2 - x^2)$, and the ordinate $x = \frac{1}{2}a$.

34. If the distance travelled by a moving point be given by the equation $s = ae^{kt} + be^{-kt}$, prove that the acceleration is proportional to the distance travelled.

35. The acceleration of a point moving in a straight line varies inversely as its distance from a point in the line 2 feet behind the starting-point; if it starts from rest with initial acceleration 1 ft. sec. per sec., find its velocity after travelling 20 feet.

36. A particle starts from rest and moves under the influence of an acceleration, which at the end of t seconds is $12/(t+1)^2$; find the distance travelled in 9 seconds.

37. If $s = e^{-t/4}\cos\frac{1}{2}\pi t$, make a table giving the position, velocity, and acceleration of the particle, initially and after 1, 2, 4, 10 seconds.

38. Draw the graphs of $s = \pm e^{-t/10}$, $s = e^{-t/10}\sin 2t$. Where do they touch one another? Where and at what angle does the latter graph cut the axis of t?

39. Draw the graphs of $s = e^{-t/2}\cos(3t - \frac{1}{4}\pi)$ and $s = e^{-t/4}\sin(2t + \frac{1}{3}\pi)$.

40. The distance of a moving point from the origin at the end of time t is given by the equation $s = e^{-t/2}\cos\frac{1}{4}\pi t$; find the velocity and acceleration at the end of 4 seconds.

41. A point moves in a straight line so that its acceleration towards a fixed point O in the line varies as its distance from O; if it starts from rest at distance a from O, find its velocity in any position, and its position at any time.

42. Given that $s = Ae^{-kt}\sin(pt + \alpha)$, prove that

$$\frac{d^2s}{dt^2} + 2k\frac{ds}{dt} + (k^2 + p^2)s = 0.$$

CHAPTER XII

HARDER DIFFERENTIATION

106. Extension of theorem of Art. 34.

It is proposed in this chapter to consider the differentiation of expressions of a more complicated nature than those we have hitherto considered.

We have seen (Art. 34) how to differentiate a function of a function, e.g. $(x^2+1)^n$, $\log \sin x$. This method can be extended.

For example, let $y = \log(1+\sin^2 x)$.

Here $y = \log u$, where $u = 1+v^2$, where $v = \sin x$.

Exactly as in Art. 34 we shall have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} \\ &= (1/u) \times 2v \times \cos x \\ &= \frac{2 \sin x \cos x}{1+\sin^2 x}.\end{aligned}$$

It is hardly necessary in practice to introduce the u, v ..., explicitly. The results may be written down thus:

(i) the d. c. of

$$\begin{aligned}\sqrt{(1+\sin^n x)} &= \frac{1}{2\sqrt{(1+\sin^n x)}} \times \text{d. c. of } (1+\sin^n x) \\ &= \frac{1}{2\sqrt{(1+\sin^n x)}} \times n \sin^{n-1} x \times \text{d. c. of } \sin x \\ &= \frac{1}{2\sqrt{(1+\sin^n x)}} \times n \sin^{n-1} x \times \cos x.\end{aligned}$$

(ii) the d. c. of

$$\begin{aligned}(\log \tan \tfrac{1}{2}x)^4 &= 4(\log \tan \tfrac{1}{2}x)^3 \times \text{d. c. of } \log \tan \tfrac{1}{2}x \\ &= 4(\log \tan \tfrac{1}{2}x)^3 \times \frac{1}{\tan \frac{1}{2}x} \times \text{d. c. of } \tan \tfrac{1}{2}x \\ &= 4(\log \tan \tfrac{1}{2}x)^3 \times \frac{1}{\tan \frac{1}{2}x} \times \sec^2 \tfrac{1}{2}x \times \text{d. c. of } \tfrac{1}{2}x \\ &= 4(\log \tan \tfrac{1}{2}x)^3 \times \frac{1}{\tan \frac{1}{2}x} \times \sec^2 \tfrac{1}{2}x \times \tfrac{1}{2}.\end{aligned}$$

107. Taking logarithms before differentiation.

(1) In the case of some expressions of a complicated type, e.g. if the expression consists of a root or power of a product or quotient of several factors, it is advisable to take logarithms before differentiating, and use the result of Art. 98.

Examples:

(i) If $\quad y = \sqrt[n]{[x(x-a)(x-b)(x-c)]}$.

$$\log y = [\log x + \log(x-a) + \log(x-b) + \log(x-c)]/n;$$

$\therefore$ differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{n}\left[\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\right],$$

and
$$\frac{dy}{dx} = \frac{y}{n}\left[\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\right]$$
$$= \frac{\sqrt[n]{[x(x-a)(x-b)(x-c)]}}{n}\left[\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\right].$$

(ii) If $\quad y = e^{ax}\sin^3 x\cos^2 x$,

$$\log y = ax + 3\log\sin x + 2\log\cos x;$$

$$\therefore\quad \frac{1}{y}\cdot\frac{dy}{dx} = a + 3\frac{1}{\sin x}\cdot\cos x + \frac{2}{\cos x}\,-\sin x);$$

$$\therefore\quad \frac{dy}{dx} = e^{ax}\sin^3 x\cos^2 x\,[a + 3\cot x - 2\tan x].$$

(2) If the expression to be differentiated contains an index involving x, it is advisable to begin by taking logarithms, except in the case of e^u where u is a function of x; the d. c. of this was seen in Art. 97 to be $e^u du/dx$.

Examples:

(i) If $\quad y = 2^{ax+b}$, then $\quad \log y = (ax+b)\log 2$,

$\therefore$ differentiating, $\quad \dfrac{1}{y}\cdot\dfrac{dy}{dx} = a\,.\log 2,$

and $\quad \dfrac{dy}{dx} = ay\log 2 = a\,.\,2^{ax+b}\log 2.$

(ii) If $\quad y = a^x\tan x$, $\quad \log y = x\log a + \log\tan x;$

$$\frac{1}{y}\frac{dy}{dx} = \log a + \frac{1}{\tan x}\cdot\sec^2 x$$
$$= \log a + \sec x\operatorname{cosec} x;$$
$$\therefore\quad \frac{dy}{dx} = a^x\tan x\,[\log a + \sec x\operatorname{cosec} x].$$

108. Inverse circular functions.

Some simple examples of these have been given in Art. 102. Here are two of a more complicated nature.

(i) the d. c. of $\quad \sin^{-1}[2x\sqrt{(1-x^2)}]$

$$= \frac{1}{\sqrt{[1-4x^2(1-x^2)]}} \times \text{d. c. of } 2x\sqrt{(1-x^2)}$$
$$= \frac{1}{\sqrt{(1-4x^2+4x^4)}} \times \left[2x\cdot\frac{-2x}{2\sqrt{(1-x^2)}} + 2\sqrt{(1-x^2)}\right]$$
$$= \frac{1}{1-2x^2} \times \left[\frac{-2x^2+2(1-x^2)}{\sqrt{(1-x^2)}}\right]$$
$$\frac{\quad}{\sqrt{(1-x^2)}}.$$

The result can also be obtained by Trigonometry, for if $x = \sin\theta$ [and this is a legitimate substitution, since $|x|$ must be < 1 if the given expression be real], $\sqrt{(1-x^2)}$ will be $\cos\theta$, and

$$2x\sqrt{(1-x^2)} = 2\sin\theta\cos\theta = \sin 2\theta\,;$$

$$\therefore\quad \sin^{-1} 2x\sqrt{(1-x^2)} = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1} x,$$

whence its d. c. $= \dfrac{2}{\sqrt{(1-x^2)}}$, as before.

(ii) the d. c. of

$$\tan^{-1}\sqrt{\left(\frac{x}{1-x}\right)} = \frac{1}{1+x/(1-x)} \times \text{d. c. of } \sqrt{\left(\frac{x}{1-x}\right)}$$

$$= \frac{1-x}{1-x+x} \times \frac{1}{2\sqrt{[x/(1-x)]}} \times \text{d. c. of } \frac{x}{1-x}$$

$$= \frac{1-x}{1} \times \frac{1}{2}\sqrt{\left(\frac{1-x}{x}\right)} \times \frac{1-x-x(\ \ 1)}{(1-x)}$$

$$= \frac{1}{2}\cdot\sqrt{\left(\frac{1-x}{x}\right)} \times \frac{1}{1-x}$$

$$= \frac{1}{2\sqrt{(x-x^2)}}\,.$$

The result may also be obtained as follows. It is easily seen geometrically * that $\tan^{-1}\sqrt{[x/(1-x)]} = \sin^{-1}\sqrt{x}$.

Hence its d. c. $= \dfrac{1}{\sqrt{(1-x)}} \times \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{(x-x^2)}}\,.$

Examples XXXVI.

Differentiate the following functions:

1. $\log\sin(\alpha - 3x)$.
2. $\log(1-\cos^3 x)$.
3. $\log\tan(\frac{1}{4}\pi + \frac{1}{2}x)$.
4. $\log[\sqrt{(x+1)} + \sqrt{(x-1)}]$.
5. $\sqrt{(2-\sin^2 2x)}$.
6. $(1+\cos^3 ax)^n$.
7. $\dfrac{1}{1-\tan^2 3x}$.
8. $[\log(1+\sqrt{x})]^2$.
9. $(1+\sec\frac{1}{2}x)^{1/n}$.
10. $\log(1+\cos^2 ax)$.
11. $e^{1+\sin^2 x}$.
12. $\sin^{-1}(\sqrt{\sin x})$.
13. $\log(1+\sqrt{e^x})$.
14. $\cos^{-1}\sqrt{(3x-2)}$.
15. $\sin\left(\tan^{-1}\dfrac{x^2}{a^2}\right)$.
16. $\tan^{-1}\left(\dfrac{\sin^2 x}{1-\cos x}\right)$.
17. $\tan^{-1}\dfrac{a}{\sqrt{(x^2-a^2)}}$.
18. $\log\cos(1+\sqrt{x})$.
19. 3^x.
20. 10^{2x-1}.
21. a^{bx-c}.
22. $2^{1/x}$.
23. a^{1/x^n}.
24. $5^{1+2\sin x}$.
25. $\sqrt{\left[\dfrac{x^3(x-1)}{(x-2)^5}\right]}$.
26. $\dfrac{\sqrt[n]{(a-3x)}}{(a-x)^3}$.
27. $\dfrac{1}{\sqrt[3]{[x(1-x)(2-x)]}}$.
28. x^4

* Draw a right-angled triangle with sides $\sqrt{x}$ and $\sqrt{(1-x)}$, and therefore hypotenuse 1.

29. $\dfrac{x^3\sqrt{(3-2x)}}{(1+x)(2-x)}$.

30. $\dfrac{(a-x)^2(b-x)^3}{(c-2x)^4}$.

31. $e^{-x}\sin^m x\cos^n x$.

32. $(a+x)^3\sin x\cos^3 2x$.

33. $\sqrt{[a^x\sin(x+\alpha)\cos(x-\beta)]}$.

34. $\log(\log x)$.

35. x^x.

36. $(\log x)^x$

37. $x^{\sin x}$.

38. $\cos^{-1}\sqrt{(1-x^2)}$.

39. $\tan^{-1}\dfrac{2x}{1-x^2}$.

40. $\operatorname{cosec}^{-1}\dfrac{1+x^2}{2x}$

41. $(1-x^2)^{3/2}\sin^{-1}x$.

42. $\tan^{-1}\dfrac{x\cos\alpha}{1+x\sin\alpha}$.

43. $\sin^{-1}\sqrt{\left(\dfrac{x-1}{x}\right)}$.

44. $\dfrac{\sqrt{(1-x)}+\sqrt{(1+x)}}{\sqrt{(1-x)}-\sqrt{(1+x)}}$.

45. $\dfrac{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}$.

46. $\log\cosh(x/a)$.

47. $\log\sinh(a/x)$.

48. $\log(1+\cosh^2 ax)$.

49. $\sqrt[3]{\left[\dfrac{(a^2+x^2)^2}{a-x}\right]}$.

50. $\dfrac{x^3}{(a^2+x^2)^{3/2}}$.

51. $\left(1+\dfrac{\ }{\log x}\right)^n$

52. $(1+x^4)^n\tan^{-1}(x^2)$.

53. $(a^2+\sin^2 x\cos^2 x)^n$.

54. $\tan^{-1}(\cot\frac{2}{3}x)$.

55. $\log\sin^n(bx+c)$.

56. $\log\dfrac{\sqrt{(x+a)}-\sqrt{(x-a)}}{\sqrt{(x+a)}+\sqrt{(x-a)}}$.

57. $\tan^{-1}\left(\sqrt{\left(\dfrac{a-b}{a+b}\right)}\tan\frac{1}{2}x\right)$.

58. $\log\sqrt{\left[\dfrac{1+\sin x}{1-\sin x}\right]}$.

59. $\sin^{-1}\dfrac{a+b\cos x}{b+a\cos x}$.

60. $x\sqrt{(x^2+a^2)}+a^2\log[x+\sqrt{(x^2+a^2)}]$.

61. $x\sqrt{(a^2-x^2)}+a^2\sin^{-1}(x/a)$.

62. $e^{ax}(a\sin bx-b\cos bx)$.

63. $\tan^{-1}[\sqrt{(x^2+1)}-x]$.

64. $\sec^{-1}[1/\sqrt{(1-x^2)}]$.

65. $\tanh^{-1}(\tan\frac{1}{2}x)$.

66. $\tan^{-1}\sqrt{\left(\dfrac{1-\cos x}{1+\cos x}\right)}$.

67. $\cot^{-1}\dfrac{a}{x}+\log\sqrt{\left(\dfrac{x-a}{x+a}\right)}$.

68. $\log\sec\tan^{-1}x$.

69. $\log_x a$.

70. $\tan^{-1}(x/a)+\tanh^{-1}(x/a)$.

71. $\log\tanh\frac{1}{2}x$.

72. $\log\dfrac{1+\tanh x}{1-\tanh x}$.

73. $\log\sqrt{\left(\dfrac{\tan x-1}{\tan x+1}\right)}-x$.

74. Find $dp/d\theta$ (i) if $p=ab^{\theta/(\gamma+\theta)}$, (ii) if $\log p=a+b\alpha^\theta-c\beta^\theta$.

75. Prove that the d. c. of $\cos^{-1}\dfrac{b+a\cos x}{a+b\cos x}=\dfrac{\sqrt{(a^2-b^2)}}{a+b\cos x}$. If $a^2<b^2$, this is imaginary. Explain this.

109. Successive differential coefficients of implicit functions.

The method of finding dy/dx, when y is given as an implicit function of x, is contained in Art. 36. If differential coefficients of higher orders are required, the method of procedure is indicated in the following example.

Given $x^2+xy+y^2=a^2$, find d^2y/dx^2.

Differentiating with respect to x, $\quad 2x+x\dfrac{dy}{dx}+y+2y\dfrac{dy}{dx}=0, \qquad$ (i)

whence $\quad \dfrac{dy}{dx}=-\dfrac{2x+y}{2y+x}$;

differentiating again with respect to x,

$$\frac{d^2y}{dx^2} = -\frac{(2y+x)\left(2+\frac{dy}{dx}\right)-(2x+y)\left(2\frac{dy}{dx}+1\right)}{(2y+x)^2} = -\frac{3y-3x\frac{dy}{dx}}{(2y+x)^2}$$

$$= -\frac{3y+3x\cdot\frac{2x+y}{2y+x}}{(2y+x)^2}, \text{ on substituting the value of } \frac{dy}{dx} \text{ from (i)}$$

$$= -\frac{6x^2+6xy+6y^2}{(2y+x)^3}$$

$$= -\frac{6a^2}{(2y+x)^3}, \text{ from the given equation between } x \text{ and } y.$$

Or, the same result may be obtained by differentiating equation (i) again as it stands; this gives

$$2+\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)+\frac{dy}{dx}+2\left(y\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}\right)=0,$$

i.e. $$(x+2y)\frac{d^2y}{dx^2} = -2-2\frac{dy}{dx}-2\left(\frac{dy}{dx}\right)^2 = -2\left[1-\frac{2x+y}{2y+x}+\left(\frac{2x+y}{2y+x}\right)^2\right]$$

$$= -2\,.\frac{3x^2+3xy+3y^2}{(2y+x)^2} = -\frac{6a^2}{(2y+x)^2}, \text{ as before.}$$

110. Successive differential coefficients of $e^{-at}\sin(bt+c)$.

The graph of $y=e^{-at}\sin(bt+c)$ is one of great importance in certain physical and engineering problems. It shares the characteristics of the graphs of $y=e^{-at}$ and $y=\sin(bt+c)$, and consists of a number of undulations, whose alternate maxima and minima occur at equal intervals and decrease in geometrical progression. Cf. Art. 105, Ex. (iii).

Differentiating with respect to t,

$$dy/dt = e^{-at}\times b\cos(bt+c)+\sin(bt+c)\times -ae^{-at}$$
$$= -e^{-at}[a\sin(bt+c)-b\cos(bt+c)].$$

This can be put into a more convenient form by the artifice (which is a common one) of putting $a=r\cos\theta$, $b=r\sin\theta$ [whence $\tan\theta=b/a$, and $r^2=a^2+b^2$, so that θ and r can always be found].

$$\therefore\ dy/dt = -e^{-at}[r\cos\theta\sin(bt+c)-r\sin\theta\cos(bt+c)].$$
$$= -e^{-at}\sqrt{(a^2+b^2)}\,.\sin(bt+c-\theta).$$

The higher differential coefficients can now at once be found.

The d.c. of $e^{-at}\sin(bt+c)$ is found by multiplying by $-\sqrt{(a^2+b^2)}$ and subtracting θ (i.e. $\tan^{-1}b/a$) from $bt+c$; this d.c. is an expression of the same form as the original one, and therefore its d.c. is found in the same way, by multiplying it by $-\sqrt{(a^2+b^2)}$ and subtracting θ from $bt+c-\theta$,

i.e. $$d^2y/dt^2 = -\sqrt{(a^2+b^2)}\,e^{-at}\sin(bt+c-\theta-\theta)\times -\sqrt{(a^2+b^2)}$$
$$= +(a^2+b^2)\,e^{-at}\sin(bt+c-2\theta),$$

and so on for any number of differentiations.

After n differentiations,

$$d^n y/dt^n = (-1)^n (a^2+b^2)^{n/2} e^{-at} \sin(bt+c-n\theta).$$

111. Leibnitz's theorem.

This is a theorem which expresses the successive differential coefficients of a product in terms of the differential coefficients of its factors. The theorem is due to Leibnitz, who shares with Newton the distinction of having discovered the principles of the infinitesimal calculus.

It was seen (Art. 30) that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Using the notation Du, D^2u, D^3u ... for $\dfrac{du}{dx}, \dfrac{d^2u}{dx^2}, \dfrac{d^3u}{dx^3} \dots,$

$$D(uv) = u\,Dv + v\,Du.$$

Differentiating again,

$$D^2(uv) = (u\,D^2v + Dv\,.\,Du) + (v\,D^2u + Du\,.\,Dv)$$
$$= u\,D^2v + 2\,Du\,Dv + v\,D^2u;$$

differentiating again,

$$D^3(uv) = (u\,D^3v + Du\,.\,D^2v) + 2(Du\,D^2v + D^2u\,Dv) + (v\,D^3u + Dv\,.\,D^2u)$$
$$= u\,D^3v + 3\,Du\,.\,D^2v + 3\,D^2u\,.\,Dv + v\,D^3u.$$

It will be noticed that the coefficients in these results are the same as the coefficients in the expansions of $(x+y)^2$ and $(x+y)^3$; and if the method of formation of these successive differential coefficients is compared with the method of expanding by multiplication the successive powers of the binomial $x+y$, it is evident that this must always be the case. The coefficients in the expansion of $D^n(uv)$ are the same as in the expansion of $(x+y)^n$ by the Binomial theorem.

Hence
$$D^n(uv) = u\,D^nv + n\,Du\,D^{n-1}v + \frac{n(n-1)}{2!}D^2u\,D^{n-2}v + \frac{n(n-1)(n-2)}{3!}D^3u\,.\,D^{n-3}v + \dots + v\,D^nu.$$

A complete formal proof by induction may be given as follows: Suppose the theorem to be true for some one value of n, i.e. suppose

$$D^n(uv) = uD^nv + {}^nC_1\,DuD^{n-1}v + {}^nC_2\,D^2uD^{n-2}v + \dots + {}^nC_{r-1}\,D^{r-1}uD^{n-r+1}v + {}^nC_r D^r uD^{n-r}v + \dots + vD^nu.$$

Differentiating again, we get

$$D^{n+1}(uv) = (uD^{n+1}v + DuD^nv) + {}^nC_1(DuD^nv + D^2uD^{n-1}v) + \dots$$
$$+ {}^nC_{r-1}(D^{r-1}uD^{n-r+2}v + D^r uD^{n-r+1}v) + {}^nC_r(D^r uD^{n-r+1}v + D^{r+1}uD^{n-r}v) + \dots + DvD^nu + vD^{n+1}u$$
$$= uD^{n+1}v + (1 + {}^nC_1)\,DuD^nv + \dots + ({}^nC_{r-1} + {}^nC_r)\,D^ruD^{n+1-r}v + \dots + vD^{n+1}u.$$

Now $1+{}^nC_1 = 1+n = {}^{n+1}C_1$, and it is shown in text-books on Algebra that

$${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r.$$

$$\therefore D^{n+1}(uv) = uD^{n+1}v + {}^{n+1}C_1 DuD^n v + \ldots + {}^{n+1}C_r D^r u D^{n+1-r} v + \ldots + vD^{n+1}u.$$

Hence, if the theorem be true for any value of n, it must be true for the next value $n+1$. It has been seen that it is true when $n=3$, and therefore it is true when $n=4$, and therefore again when $n=5$, and so on for all values of n.

This theorem is particularly useful when one of the two factors is a small integral power of x; if this be taken as u in the preceding formula, its differential coefficients soon vanish, and the series consists of a few terms only.

E.g. (i) Find the n^{th} d. c. of $(x^2+1)e^{2x}$.

The successive d. c.'s of e^{2x} are $2e^{2x}$, $2^2 e^{2x}$, ... $2^n e^{2x}$, hence, taking x^2+1 as u and e^{2x} as v, $Du = 2x$, $D^2u = 2$, and higher d. c.'s of u are 0.

$$\begin{aligned}\therefore\quad D^n\left[(x^2+1)e^{2x}\right] &= (x^2+1)2^n e^{2x} + n \,.\, 2x \,.\, 2^{n-1}e^{2x} \\ &\qquad + \frac{n(n-1)}{2\,!} \,.\, 2 \,.\, 2^{n-2}e^{2x} \\ &= 2^{n-2}e^{2x}\left[4(x^2+1) + 4nx + n(n-1)\right].\end{aligned}$$

(ii) Find the n^{th} d. c. of $x\log x$.

If $u = x$, $Du = 1$, and higher d. c.'s are 0.

If $v = \log x$, $Dv = 1/x$, $D^2v = -1/x^2, \ldots$

$$D^n v = D^{n-1}\left(\frac{1}{x}\right) = \frac{(-1)^{n-1}(n-1)\,!}{x^n} \text{ [Art. 57];}$$

$$\begin{aligned}\therefore\quad D^n(x\log x) &= x \,.\, \frac{(-1)^{n-1}(n-1)\,!}{x^n} + n \,.\, 1 \,.\, \frac{(-1)^{n-2}(n-2)\,!}{x^{n-1}} \\ &= \frac{(-1)^{n-1}(n-2)\,!}{x^{n-1}}\left[n-1-n\right] \\ &= (-1)^n(n-2)\,!/x^{n-1}.\end{aligned}$$

112. Formation of differential equations.

The following example illustrates how in many cases a relation between successive differential coefficients of a function can be found.

If $y = e^{a\sin^{-1}x}$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = a^2y$.

We have $\dfrac{ay}{dx} = e^{a\sin^{-1}x}$ which may be written

$$\sqrt{(1-x^2)} \,.\, \frac{dy}{dx} = e^{a\sin^{-1}x} = ay;$$

differentiating again,

$$\sqrt{(1-x^2)} \,.\, \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{-x}{\sqrt{(1-x^2)}} = a\frac{dy}{dx} = \frac{a^2y}{\sqrt{(1-x^2)}}, \text{ from (i);}$$

multiplying by $\sqrt{(1-x^2)}$, $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = a^2y$

A relation such as this, between x, y and d. c.'s of y with respect to x, is called a *differential equation.*

If this be differentiated n times by Leibnitz's Theorem, we get a relation between any 3 consecutive differential coefficients of y, viz.:

$$\left[(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}+n\cdot(-2x)\frac{d^{n+1}y}{dx^{n+1}}+\frac{n(n-1)}{2}\cdot(-2)\frac{d^ny}{dx^n}\right]$$
$$-\left[x\frac{d^{n+1}y}{dx^{n+1}}+n\,.\,1\,.\frac{d^ny}{dx^n}\right]=a^2\frac{d^ny}{dx^n};$$

which becomes, on collecting like terms,

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-(2n+1)\,x\frac{d^{n+1}y}{dx^{n+1}}-(n^2+a^2)\frac{d^ny}{dx^n}=0.$$

Examples XXXVII.

1. Given $x^2-axy+y^2=a^2$, find d^2y/dx^2.
2. If $x^3+3\,axy+y^3=a^3$, find d^2y/dx^2.
3. Find d^2y/dx^2 if $x^n+y^n=a^n$.
4. If $x^3+y^3=a^3$, find d^3y/dx^3.
5. Given $a(x+y)=x^2+y^2$, find d^2y/dx^2.
6. Find the 4th d. c. of $e^{-2x}\sin(2x+\alpha)$.
7. Find the nth d. c. with respect to t of $e^{-at}\cos at$.
8. Prove that the 2nd d. c. of $e^{-x}\sin 2x=5e^{-x}\sin(2x-126°\,52')$.
9. Find the 10th d. c. of x^2e^x.
10. Find the nth d. c. of $(x^3+a^3)\,e^{ax}$.
11. Find the 6th d. c. of $x^2\log x$.
12. Find the nth d. c. of $x^2\log x$.
13. Obtain the 5th d. c. of $x^2\sin 2x$.
14. Obtain the nth d. c. of x^3e^{-x}.
15. If $y=(\sin^{-1}x)^2$, prove that $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2$.
16. Differentiate the result of the preceding example n times by Leibnitz's Theorem.
17. If $y=\log[x+\sqrt{(x^2-a^2)}]$, prove that $(x^2-a^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=0$.
18. Find the relation between any consecutive 3 differential coefficients of y in the preceding example.
19. Determine the nth d. c. of $x^3(1+x)^n$.
20. If $y=A\cos(\log x)+B\sin(\log x)$, prove that $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$.
21. Given $y=\sin(m\sin^{-1}x)$, prove that $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+m^2y=0$.
22. Find by Leibnitz's Theorem a relation between any consecutive 3 differential coefficients of $\sin^{-1}x$.

Find the nth d. c. of x^2y with respect to x.

If x and y are given as functions of a third variable t by equations $x=f(t)$, $y=F(t)$; find d^2y/dx^2 in terms of differential coefficients of x and y with respect to t.

25. If $u=x^2v$ and $v=\log x$, prove that $\dfrac{d^nu}{dx^n}=2\dfrac{d^{n-2}v}{dx^{n-2}}$.

CHAPTER XIII

APPLICATION TO THEORY OF EQUATIONS.

MEAN-VALUE THEOREM

113. The differential coefficient of a function vanishes in the interval between two equal values of the function, provided both the function and its differential coefficient are continuous throughout the interval.

Let $y = b$ when $x = a$, and let a' ($> a$) be the next value of x for which $y = b$. After passing through the value b when $x = a$, y must either remain constant or increase or decrease. If it remains constant, its d. c. is zero; if it increases, then before reaching the value b again (when $x = a'$), it must decrease and therefore, if continuous, must pass through a maximum; similarly, if it decreases, then before reaching the value b again, it must increase and therefore, if continuous, must pass through a minimum, and in either case, at the maximum or minimum, its d. c. is zero. [Art. 53.]

Geometrically, it is obvious that, between two consecutive points

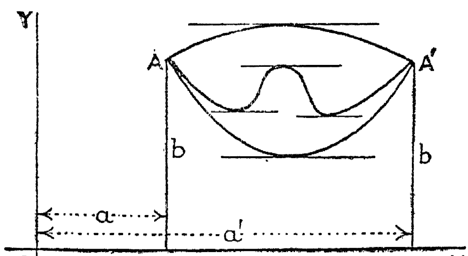


Fig. 91.

with equal ordinates, there must, on a continuous curve of continuous slope (Fig. 91), be a point where the tangent is parallel to the axis of x, i.e. a point where $dy/dx = 0$. There is not necessarily one point only; there may be any odd number of such points. The fact proved in the theorem is that there is *at least one* such point. The

result may also be stated in the form that between two equal values the function must have at least one maximum or minimum. The theorem is clearly not true if either y or dy/dx be discontinuous between $x = a$ and $x = a'$. Cf. (i) the graph of $\tan x$, (ii) Fig. 56 [Art. 54].

In particular, between two values of x for which $y = 0$, there is at least one value for which $dy/dx = 0$. The graphical solution of the quadratic $ax^2 + bx + c = 0$ furnishes an illustration of this. The graph of $y = ax^2 + bx + c$ is a parabola whose axis is vertical, and the roots of the equation are the abscissae of the points where the parabola cuts the axis of x; at the vertex of the parabola, which is between these two points, the tangent is parallel to the axis of x, i.e. $dy/dx = 0$.

114. Application to equations. Rolle's Theorem.

If y be a rational integral function of x (Art. 7), denoted by $f(x)$, y and dy/dx or $f'(x)$ are both continuous so long as x is finite. The above theorem therefore states that *between two real roots of* $\mathrm{f}(\mathrm{x}) = 0$ *there must be at least one real root of* $\mathrm{f}'(\mathrm{x}) = 0$. This is known as Rolle's Theorem. It evidently follows that not more than one real root of $f(x) = 0$ can lie between two consecutive roots α and β of $f'(x) = 0$, for if there were two, then between these two roots of $f(x) = 0$ would lie a root of $f'(x) = 0$, and therefore α and β would not be consecutive roots of $f'(x) = 0$. There is or is not a root of $f(x) = 0$ between α and β, according as $f(\alpha)$ and $f(\beta)$ have opposite signs or the same sign [Art. 17 (4)]. Geometrically, between two consecutive points A and B on a continuous curve, where the tangent is parallel to the axis of x, the curve cannot cut the axis of x more than once. It will or will not cut it according as A and B are on opposite sides of the axis of x or on the same side.

As an example, take the function considered in Art. 55, Ex. (i).

$$f(x) = x^3 - 9x^2 + 15x.$$

$$f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5).$$

The roots of the equation $f'(x) = 0$ are 1 and 5, hence the roots of the equation $f(x) = 0$, if real, will lie between $-\infty$ and 1, 1 and 5, 5 and $+\infty$.

If $x = -\infty$, y is $-$, ⎫ $\therefore$ $y = 0$ at some point between $-\infty$ and 1. [Art. 17 (4).]

If $x = 1$, y is $+$, ⎬ $\therefore$ $y = 0$ at some point between 1 and 5.

If $x = 5$, y is $-$, ⎬

If $x = +\infty$, y is $+$, ⎭ $\therefore$ $y = 0$ at some point between 5 and $+\infty$.

$\therefore$ the equation $f(x) = 0$ has three real roots, as shown in the figure, viz.: 0, 2·2, and 6·8 approximately.

115. Equal roots.

If two of the roots of $f(x)=0$ approach one another and ultimately coincide, the root of $f'(x)=0$ which is intermediate between them must also coincide with them. In the figure of Art. 55, Ex. (i), imagine that the graph gradually ascends vertically, the axes remaining fixed. The points of intersection of the graph with the axis of x gradually approach one another until ultimately, when the curve touches the axis of x, they coincide, and clearly the minimum point coincides with them. [The graph then represents the function there given with each ordinate increased by 25, i.e. it is the graph of $y=x^3-9x^2+15x+25$, and the abscissae of the points of intersection with the axis of x are the roots of the equation $x^3-9x^2+15x+25=0$. Since the vertical ascent of the graph does not alter the abscissa of any point on the curve, it follows that this latter equation has two roots each equal to 5. It is equivalent to $(x-5)^2(x+1)=0$, so that the third root is -1.] Hence, *if a root of* $f(x)=0$ *is repeated, it is also a root of* $f'(x)=0$.

This can also be seen analytically as follows:—If α be a root of the equation $f(x)=0$, where $f(x)$ is a rational integral function of x, the function contains $x-\alpha$ as a factor; if α be a double root, the function contains $(x-\alpha)^2$ as a factor;

$\therefore\ f(x)=(x-\alpha)^2\phi(x)$, where $\phi(x)=0$ gives the remaining roots.

$$\therefore\ f'(x)=(x-\alpha)^2\phi'(x)+\phi(x)\,.\,2(x-\alpha)$$
$$=(x-\alpha)\left[(x-\alpha)\phi'(x)+2\phi(x)\right].$$

$\therefore\ f'(x)=0$ when $x=\alpha$, so that α is a root of $f'(x)=0$.

It follows in a similar manner that, if α be a root of $f(x)=0$ repeated r times, it is a root of $f'(x)=0$ repeated $r-1$ times; then by the same argument, it is a root of $f''(x)=0$ repeated $r-2$ times, and so on. For instance, a triple root of $f(x)=0$ is a double root of $f'(x)=0$, a single root of $f''(x)=0$, and is not a root of $f'''(x)=0$.

Hence, to multiple roots of $f(x)=0$ correspond common factors of $f(x)$ and $f'(x)$; and therefore such multiple roots can be obtained by finding the H. C. F. of $f(x)$ and $f'(x)$ (by the ordinary algebraical method).

Examples:

(i) A simple illustration is furnished by the quadratic $ax^2+bx+c=0$. If it has equal roots, then the root is also a root of $2ax+b=0$, i.e. the root is $-b/2a$, and the condition for equal roots is obtained by substituting this in the given equation,

i.e.
$$a\,.\,\frac{b^2}{4a^2}-\frac{b^2}{2a}+c=0,$$

which reduces to $b^2=4ac$, the well-known condition.

(ii) Solve the equation $x^4-6x^2+8x+24=0$ [cf. Art. 55, Ex. (ii)], having given that it has a multiple root.

$$f'(x)=4x^3-12x+8=4(x^3-3x+2).$$

The H.C.F. of x^3-3x+2 and $x^4-6x^2+8x+24$ is $x+2$; therefore -2 is a double root of the given equation.

Hence $f(x)$ contains the factor $(x+2)^2$; dividing out by this, the other factor is found to be x^2-4x+6, and solving the equation $x^2-4x+6=0$, the other two roots (which are imaginary) are obtained.

Examples XXXVIII.

Between what values do the real roots of the equations 1–6 lie?

1. $x^3-6x^2+2=0$.
2. $x^4-18x^2+12=0$.
3. $x^3-12x^2+36x-10=0$.
4. $x^4+4x^3-20x^2+10=0$.
5. $2x^3-3x^2-36x-5=0$.
6. $x^4-8x^3+22x^2-24x+12=0$.

Solve the equations 7–12, given that each has a multiple root:

7. x^3+2x^2-7x+4.
8. $4x^3-16x^2-19x-5$.
9. $x^4-4x^3+16x-16$.
10. $x^5-7x^4-2x^3+14x^2+x-7$.
11. $12x^3+28x^2+3x-18$.
12. $x^4-6x^3+10x^2-6x+9$.
13. Find the condition that the conic $ax^2+by^2+2gx+2fy+c=0$ may touch (i) the axis of x; (ii) the axis of y.
14. Prove that the curve $x^3+y^3-3x+4y+2=0$ touches the axis of x.
15. Show that the curve $y=2x^3+3x^2-1$ touches the axis of x.

Verify the theorem of Art. 113 in the following cases 16–20, and find the coordinates of the point where the d.c. is zero.

16. $y=3x^2-7x+4$.
17. $y=(x-1)^2(x-3)$.
18. $y=\frac{1}{2}x+x^{-1}+2$.
19. $y=\log[(x^2+8)/6x]$.
20. $y=\sin x-\cos x$.

Discuss the application of Rolle's Theorem to the functions

21. $x(x-4)/(x-1)$.
22. $\tan x$.
23. $4-(8-x)^{2/3}$.
24. Find the condition that the equation $x^3+px+q=0$ may have two equal roots.

116. Mean-value theorem.

If $f(x)$ *and* $f'(x)$ *be continuous throughout the range* $x=a$ *to* $x=b$, *then*

$$\frac{f(b)-f(a)}{b-a}=f'(x)$$

for some value of x *between* a *and* b.

The expression $\dfrac{f(b)-f(a)}{b-a}$ is the ratio of the total increase in the function to the total increase in the variable x, and therefore is the average rate of increase over the range $x=a$ to $x=b$. Hence

the theorem states that the average rate of increase throughout the interval is equal to the actual rate of increase at some point in the interval (e.g. the average velocity of a train between two stations is equal to the actual velocity at some intermediate point). This follows from general reasoning, for the average rate of increase in the interval is evidently intermediate in value between the greatest and least rates of increase; and in passing between its greatest and least values, the rate of increase, being continuous, must pass through every intermediate value, and therefore at some point must equal the average rate of increase.

Geometrically, this can also be seen at once, for let A and B (Fig. 92) be the points on the graph of $f(x)$ for which $x=a$ and $x=b$ respectively; then the ordinates AM and BN are $f(a)$ and $f(b)$ respectively. Let AK, parallel to MN, meet NB in K.

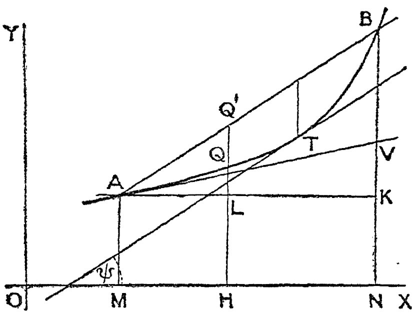


Fig. 92.

Then $$\frac{f(b)-f(a)}{b-a}=\frac{NB-MA}{MN}=\frac{KB}{AK}=\tan BAK.$$

If y and dy/dx are both continuous, there is obviously some point on the curve between A and B at which the tangent is parallel to the chord AB, i.e. there is some point, T say, at which $\tan\psi$, i.e. dy/dx or $f'(x)$, $=\tan BAK$,

i.e. there is a value of x for which $f'(x)=\dfrac{f(b)-f(a)}{b-a}$.

It is easily seen by drawing figures that there is not necessarily such a point, if either y or dy/dx be discontinuous anywhere between A and B.

117. Analytical proof.

The theorem can also be deduced analytically from Art. 113 as follows, and this method is important because it can be used to

extend the given theorem, and ultimately obtain one of the most important theorems in Mathematics. (Chap. XXII.)

Denote the expression $[f(b)-f(a)]/(b-a)$ by R, and consider the function

$$f(x)-f(a)-(x-a)R. \tag{i}$$

If $x=a$, this function $=f(a)-f(a)-0\times R=0$.

If $x=b$, the function $=f(b)-f(a)-(b-a)R$, which is seen to be 0 on substituting the value of R.

Therefore, since the function vanishes when $x=a$ and also when $x=b$, its d. c. must vanish for some intermediate value of x [the function and its d. c. both being continuous between $x=a$ and $x=b$] (Art. 113). The d. c. is $f'(x)-R$, and therefore

$$f'(x)-R=0 \text{ for some value of } x \text{ between } a \text{ and } b.$$

$$f'(x_1)=R=\frac{f(b)-f(a)}{b-a} \quad \text{where } a<x_1<b.$$

Geometrically, it should be noticed that if, in Fig. 92, Q be the point whose abscissa is x and ordinate $f(x)$, and if the ordinate of Q meet the axis of x in H, AK in L, and the chord AB in Q', the expression (i) considered above is equal to

$$HQ-MA-AL\tan BAK=QL-Q'L=-QQ'.$$

Now QQ' is obviously zero at A and at B, and it is a maximum, and its d. c. vanishes, at some intermediate point, viz. at the point T where the tangent is parallel to the chord.

The preceding result may be written

$$f(b)-f(a)=(b-a)f'(x_1), \text{ where } x_1 \text{ is between } a \text{ and } b.$$

Let $b=a+h$, then x_1, being $>a$ and $<b$, i.e. $<a+h$, may be written as $a+\theta h$, where θ is a positive proper fraction; and the theorem takes the form

$$f(a+h)-f(a)=hf'(a+\theta h),$$

$$\text{i.e.} \quad f(a+h)=f(a)+hf'(a+\theta h).$$

It should be noticed that this involves the definition of a d. c., for the last result may be written

$$\frac{f(a+h)-f(a)}{h}=f'(a+\theta h),$$

and when $h\rightarrow 0$, $a+\theta h\rightarrow a$, since $\theta<1$;

$$\underset{h\to 0}{\mathrm{Lt}}\frac{f(a+h)-f(a)}{h}=f'(a), \quad \text{as in Art. 26.}$$

It also indicates the amount of error involved in the use of differentials (Art. 24). It was there pointed out that

$$\delta y=\frac{dy}{dx}\delta x=f'(x)\,\delta x \quad \text{approximately.}$$

From the preceding result we have, if $h = \delta x$,

$$\delta y = f(x+h) - f(x) = hf'(x+\theta h) = \delta x . f'(x+\theta\delta x), \quad \text{where } 0 < \theta < 1.$$

If G and L be the greatest and least values of $f'(x)$ in the interval from x to $x+\delta x$, then the greatest possible value of δy is $G\delta x$, and the least possible value of δy is $L\delta x$; hence the error involved in the statement $\delta y = \dfrac{dy}{dx} . \delta x$ is not greater than $|G-L|\,\delta x$.

All that is known about θ in the general case is the fact that it is intermediate in value between 0 and 1. Usually its value depends upon the values of a and h. In some particular cases, its value can be found, e.g. if $f(x) = ax^2 + bx + c$, then

$$f(x+h) = a(x+h)^2 + b(x+h) + c, \quad f'(x) = 2ax + b$$

and

$$f'(x+\theta h) = 2a(x+\theta h) + b;$$

hence the theorem gives

$$a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + h[2a(x+\theta h) + b];$$

whence, after multiplying out and cancelling,

$$ah^2 = 2a\theta h^2, \quad \text{and} \quad \theta = \tfrac{1}{2}.$$

This is obvious geometrically, for the graph is a parabola with its axis parallel to the axis of y (p. 18), and if any chord of the parabola be drawn, the tangent at the end of the diameter which bisects the chord (and which is parallel to the axis of y) is parallel to the chord; hence the abscissa of the point of contact of the tangent is half the sum of the abscissae of the ends of the chord.

In the preceding case, θ is constant, but if we take $f(x) = x^3$, and therefore $f'(x) = 3x^2$, we get from the mean-value theorem

$$(x+h)^3 = x^3 + h . 3(x+\theta h)^2.$$

Whence, on multiplying out and dividing by h^2, we have

$$3h\theta^2 + 6x\theta = 3x + h,$$

from which θ can be found in terms of x and h.

If h be very small, the terms in this equation which contain h may be neglected in comparison with the others, and it follows then that θ is approximately equal to $\frac{1}{2}$.

118. Indeterminate forms.

The following is a useful application of the mean-value theorem.

Let $f(x)$, $F(x)$ be two functions of x which both become zero when $x = a$; then, if $f'(x)/F'(x)$ approaches a limiting value as $x \longrightarrow a$, $f(x)/F(x)$ will tend to the same limit. This is often called the 'true value' of $f(x)/F(x)$.

For
$$\frac{f(a+h)}{F(a+h)} = \frac{f(a) + hf'(a+\theta h)}{F(a) + hF'(a+\theta' h)} = \frac{f'(a+\theta h)}{F'(a+\theta' h)},$$

since $f(a)$ and $F(a)$ are both zero;

$$\therefore \quad \operatorname*{Lt}_{h\to 0} \frac{f(a+h)}{F(a+h)} = \operatorname*{Lt}_{h\to 0} \frac{f'(a+\theta h)}{F'(a+\theta h)} = \frac{f'(a)}{F'(a)}.$$

If $f'(x)$, $F'(x)$ also both become zero when $x = a$, the same argument shows that the 'true value' is $f''(a)/F''(a)$, and so on.

The 'true value' is found in the same way if $f(x)$ and $F(x)$ both become infinite when $x = a$.

For then $\dfrac{f(a)}{F(a)} = \dfrac{1}{F(a)} \Big/ \dfrac{1}{f(a)}$, and $\dfrac{1}{F(a)}$, $\dfrac{1}{f(a)}$ are both zero; therefore, by the preceding case, the true value

$$A = -\left[\frac{1}{F(a)}\right]^2 F'(a) \Big/ -\left[\frac{1}{f(a)}\right]^2 f'(a) = \left[\frac{f(a)}{F(a)}\right]^2 \cdot \frac{F'(a)}{f'(a)} = A^2 \frac{F'(a)}{f'(a)};$$

whence $A = f'(a)/F'(a)$ as before, provided A be neither zero nor infinity, and it can be shown that the rule holds for these cases also.

Examples:

The true value

(i) of $\dfrac{1-x}{\log x}$, when $x = 1$, is the value of $-\dfrac{1}{1/x}$, i.e. -1;

(ii) of $\dfrac{x \cos x + \pi}{\sin x}$ $(x = \pi)$ = the value of $\dfrac{-x \sin x + \cos x}{\cos x} = 1$;

(iii) of $\dfrac{x^2}{1 - \cos x}$ $(x = 0)$ = the value of $\dfrac{2x}{\sin x}$ $\left(\text{which is still of the form } \dfrac{0}{0}\right)$

the value of $\dfrac{2}{\cos x} = 2$;

(iv) of $\dfrac{5x-2}{3x+4}$ $(x = \infty)$ = the value of $\dfrac{5}{3} = \dfrac{5}{3}$.

119. Extended mean-value theorem.

We have proved (Art. 117) that, provided $f(x)$ and $f'(x)$ are continuous in the interval from $x = a$ to $x = b$,

$$f(b) = f(a) + (b-a) f'(x_1), \text{ where } x_1 \text{ is between } a \text{ and } b.$$

This result can be extended to show that, provided $f''(x)$ is also continuous in the given interval,

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(x_2), \text{ where } x_2 \text{ is between } a \text{ and } b.$$

Using a method of proof similar to that of the preceding case, denote

$$f(b) - f(a) - (b-a) f'(a) \quad \text{by} \quad \tfrac{1}{2}(b-a)^2 R, \tag{i}$$

and consider the function of x

$$f(b) - f(x) - (b-x) f'(x) - \tfrac{1}{2}(b-x)^2 R,$$

which, with its d. c., is continuous within the given range, since $f(x)$, $f'(x)$ and $f''(x)$ are continuous.

This function obviously vanishes when $x = b$. If $x = a$, it becomes

$$f(b)-f(a)-(b-a)f'(a)-\tfrac{1}{2}(b-a)^2 R, \text{ which } = 0 \text{ from (i).}$$

Since the function vanishes when $x = a$ and when $x = b$, its d. c. must vanish for some intermediate value.

Its d. c. $= -f'(x)-(b-x)f''(x)+f'(x)+(b-x)R.$

This therefore must vanish for some value of x between a and b, i.e. after dividing out by the factor $b-x$,

$$-f''(x_2)+R = 0, \text{ or } R = f''(x_2), \text{ where } x_2 \text{ is between } a \text{ and } b.$$

Substituting this value of R in (i) and re-arranging, we get

$$f(b) = f(a)+(b-a)f'(a)+\tfrac{1}{2}(b-a)^2 f''(x_2), \text{ where } a < x_2 < b.$$

If $b = a+h$, then x_2, being between a and $a+h$, may be denoted by $a+\theta' h$, where $0 < \theta' < 1$, and the theorem takes the form

$$f(a+h) = f(a)+hf'(a)+\tfrac{1}{2}h^2 f''(a+\theta' h).$$

The geometrical interpretation of this result should be noticed.

If in Fig. 92 the tangent at A meets BN in V, then

$$\begin{aligned} f(b)-f(a)-(b-a)f'(a) &= NB-MA-MN\tan VAK = KB-KV \\ &= VB. \end{aligned}$$

Hence the preceding theorem gives the result

$$VB = \tfrac{1}{2}MN^2 . f''(a+\theta' h),$$

i.e. when $b-a$ is very small, and therefore $f''(a+\theta' h) \to f''(a)$,

$$\mathrm{Lt}\,\frac{VB}{MN^2} = \tfrac{1}{2}f''(a).$$

Hence, since $f''(a)$ is finite, if B is indefinitely near to A, VB is very small compared with MN, i. e. the distance between the curve and the tangent (measured along the ordinate) is very small, or is of (at least) the second order of small quantities, compared with the difference in the abscissae (Art. 24).

This also includes the results of Art. 59, for VB is + or − according as $f''(a)$ is + or −, and the curve in the neighbourhood of A is above or below the tangent at A according as VB is + or −, i.e. according as $f''(a)$, the value of the second d. c. at A, is + or −.

As in the case of the first mean-value theorem, the value of θ' depends in general upon the values of a and h; a fixed numerical value can be found for it in the case when $f(a) = a^3$, for then $f'(a) = 3a^2$, $f''(a) = 6a$, and the theorem gives

$$(a+h)^3 = a^3+h . 3a^2+\tfrac{1}{2}h^2\, 6(a+\theta' h),$$

whence $\theta' = \frac{1}{3}$.

120. Principle of proportional parts.

The extended mean value theorem can be used to prove the principle of proportional parts, a principle which the student has probably used in elementary work in connection with tables of logarithms, trigonometrical ratios, &c. This principle states that 'if the increase in the variable be small, then the increase in the function is proportional to the increase in the variable'.

We have $\quad f(a+h)-f(a) = hf'(a)+\frac{1}{2}h^2 f''(a+\theta h)$

and $\quad f(a+k)-f(a) = kf'(a)+\frac{1}{2}k^2 f''(a+\theta' k)$;

$$\therefore \quad \frac{f(a+h)-f(a)}{f(a+k)-f(a)} = \frac{h}{k} \cdot \frac{f'(a)+\frac{1}{2}h\,f''(a+\theta h)}{f'(a)+\frac{1}{2}k\,f''(a+\theta' k)}.$$

When h and k are small, the last term in both numerator and denominator *generally* becomes very small compared with the first term, and both numerator and denominator approach the value $f'(a)$. The right-hand side of the equation then becomes approximately h/k, and we have

$$\frac{f(a+h)-f(a)}{f(a+k)-f(a)} = \frac{h}{k},$$

i.e. the increase in the function is proportional to the increase in the variable.

The last terms in the numerator and denominator mentioned above do not become small compared with the first term if $f''(a)$ is large compared with $f'(a)$; hence the principle will usually fail when the second differential coefficient of the function is large compared with the first. E.g. in the case of common logarithms, the 2nd d. c. of $\log_{10} x = -\mu/x^2$, which is large compared with the first d. c. μ/x, when x is small; therefore the principle is not true for the logarithms of small numbers. Again the 2nd d. c. of $\tan x = 2\sec^2 x \tan x$, which is large compared with the first d. c. $\sec^2 x$, when x is nearly $\frac{1}{2}\pi$; therefore the principle does not hold for natural tangents in the neighbourhood of 90° [or of any odd multiple of 90°].

For a more complete discussion, and investigation as to the amount of error involved in using the principle, the student is referred to more advanced works.

Examples XXXIX.

Find the value of θ in the application of the mean-value theorem to the functions 1–4:

1. $1/x$. 2. e^x. 3. $\sin x$. 4. $\log x$.

5. Prove that, with the usual conditions, $f(x) = f(0) + xf'(\theta x)$.

6. Discuss the application of the mean-value theorem to the function $x(x-4)/(x-1)$, in the neighbourhood of $x=1$.

7. Also to the function $\tan x$, in the neighbourhood of $x=\frac{1}{2}\pi$.

8. Also to the function $4-(8-x)^{2/3}$, in the neighbourhood of $x=8$.

Find the value of θ in the application of the extended mean-value theorem to the functions 9–12.

9. ax^3+bx^2+cx+d. 10. x^4. 11. $1/x$. 12. e^x.

13. Prove that, with the usual conditions, $f(x)=f(0)+xf'(0)+\frac{1}{2}x^2f''(\theta x)$.

14. Deduce from the theorem of question 13 that $\cos x > 1-\frac{1}{2}x^2$.

15. Also that $\log(1+x) > x-\frac{1}{2}x^2$.

16. Also that $\log(1+\cos x) < \log 2-\frac{1}{4}x^2$.

17. Prove, by the method of Arts. 117 and 119, that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a+\theta h),$$

where $0<\theta<1$, provided $f(x)$ and its first 3 differential coefficients are continuous in the interval a to $a+h$.

18. Deduce from the preceding result that

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(\theta x).$$

19. Deduce from the result of Question 18 that $\sin x > x-\frac{1}{6}x^3$.

20. Also that $\tan x > x+\frac{1}{3}x^3$.

21. Deduce from the mean-value theorem that, if two functions have the same derivative, their difference is constant.

22. Discuss the application of the extended mean-value theorem to the function $\log(x-4)$, in the neighbourhood of $x=4$.

23. Also to the function $1-(1-x)^{4/3}$, in the neighbourhood of $x=1$.

24. Also to the function $\log\cos x$, in the neighbourhood of $x=\frac{1}{2}\pi$.

25. Find the true values of (i) $\dfrac{x\sin x-\frac{1}{2}\pi}{\cos x}$, when $x=\frac{1}{2}\pi$.

(ii) $\dfrac{\sin x-\sin\alpha}{x-\alpha}$, when $x=\alpha$.

(iii) $\sin ax \operatorname{cosec} bx$, when $x=0$.

26. Find the true values, when $x=0$, of

(i) $\dfrac{\sin x-x}{\tan x-x}$; (ii) $x-\sin x$ (iii) e^x-1

27. Find the true values, when $x=\infty$, of

(i) $\dfrac{\log x}{x}$; (ii) $\dfrac{3x^2-2x+1}{5x^2+3x-2}$; (iii) $\dfrac{x^2}{e^x}$.

CHAPTER XIV

METHODS OF INTEGRATION

121. Introductory.

The integration of very simple functions and some easy applications thereof have already been considered in Chapter IX, and several other integrals have been given in Chapter XI. We now proceed to discuss various methods by means of which more complicated functions may be reduced to some combination of these simpler functions.

The first process that will be considered is the integration of rational algebraical fractions, i.e. fractions whose numerator and denominator contain only positive integral powers of x with constant coefficients.

122. Integration of rational algebraical fractions.

In the first place, *if the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of lower degree than the denominator.* This gives one or more terms whose integrals can be written down at once together with a fraction whose numerator is of lower degree than its denominator, and it remains to consider the integration of such fractions.

1. *Let the denominator be of the first degree.*

After division, the remainder will be independent of x; therefore the process just described gives the integral as the sum of a number of powers of x, together with a logarithm.

E.g.
$$\frac{x^3}{x-2} = x^2+2x+4+\frac{8}{x-2},$$

$$\therefore \quad \int \frac{x^3}{x-2}\,dx = \tfrac{1}{3}x^3+x^2+4x+8\log(x-2).$$

Again, $\dfrac{3x-5}{3-2x}$, after arranging in powers of x and using ordinary division, becomes
$$-\frac{3}{2}-\frac{\frac{1}{2}}{3-2x},$$

$$\therefore \quad \int \frac{3x-5}{3-2x}\,dx = -\frac{3}{2}x - \frac{1}{2}\cdot\int \frac{1}{3-2x}\,dx = -\frac{3}{2}x + \frac{1}{4}\log(3-2x).$$

It must not be forgotten that in every case an arbitrary constant is understood.

Examples XL.

Integrate the following expressions:

$\frac{x}{x+3}$.	2. $\frac{x^2}{x+3}$.	3. $\frac{x^3}{x+3}$.	4. $\frac{}{2x-3}$.
$\frac{x^2}{2x-3}$.	6. $\frac{x}{1-2x}$.	7. $\frac{x^2}{1-2x}$.	8. $\frac{1-x}{1-2x}$.
9. $\frac{2x+3}{x-4}$.	10. $\frac{3-2x}{2x-1}$.	11. $\frac{x^2}{a-x}$.	12. $\frac{x^3}{2x-1}$.
13. $\frac{x}{ax+b}$.	14. $\frac{x^2}{px-q}$.	15. $\frac{x^3}{c-2x}$.	16. $\frac{ax+b}{cx+d}$.

123. 2. Denominator of the second degree.

(i) *If the denominator breaks up into rational factors*, use the method of partial fractions, as illustrated in the following examples:—

Examples:

(i) $\int \frac{5x-4}{x^2-8x+12}\,dx.$

The denominator is equal to $(x-6)(x-2)$.

we assume $\quad \frac{5x-4}{x^2-8x+12} \equiv \frac{A}{x-6} + \frac{B}{x-2}$

We have to find A and B. Clearing of fractions,

$$5x-4 \equiv A(x-2) + B(x-6). \qquad \text{(i)}$$

This, being an identity, is true for all values of x;

$\therefore$ putting $x=6$ (which makes the denominator of A, i.e. the coefficient of B in (i), vanish), $26 = 4A$, and $A = \frac{13}{2}$.

putting $x = 2$ (which makes the denominator of B, i.e. the coefficient of A in (i), vanish), $6 = -4B$, and $B = -\frac{3}{2}$.

Hence $\quad \frac{5x-4}{x^2-8x+12} \equiv \frac{\frac{13}{2}}{x-6} + \frac{-\frac{3}{2}}{x-2}$;

and $\quad \int \frac{5x-4}{x^2-8x+12}\,dx = \frac{13}{2}\log(x-6) - \frac{3}{2}\log(x-2).$

(ii) $\int \frac{x^2}{2x^2+x-3}\,dx.$

Here the numerator is of the same degree as the denominator, and therefore must be divided by it.

$$\frac{x^2}{2x^2+x-3} = \frac{1}{2} - \frac{\frac{1}{2}x-\frac{3}{2}}{2x^2+x-3} = \frac{1}{2} - \frac{1}{2}\cdot\frac{x-3}{(x-1)(2x+3)}.$$

To resolve the latter into partial fractions, assume

$$\frac{x-3}{(x-1)(2x+3)} \equiv \frac{A}{x-1} + \frac{}{2x+3};$$

$$\therefore \quad x-3 \equiv A(2x+3) + B(x-1).$$

To find A, put $x = 1$, $\quad -2 = 5A$, $\quad$ and $\quad A = -\frac{2}{5}$.

To find B, put $x = -\frac{3}{2}$, $\quad -\frac{9}{2} = -\frac{5}{2}B$, $\quad$ and $\quad B = \frac{9}{5}$.

$$\therefore \quad \frac{x^2}{2x^2+x-3} = \frac{1}{2} - \frac{1}{2}\left[\frac{-\frac{2}{5}}{x-1} + \frac{\frac{9}{5}}{2x+3}\right] = \frac{1}{2} + \frac{1}{5(x-1)} - \frac{9}{10(2x+3)};$$

$$\therefore \quad \int \frac{x^2}{2x^2+x-3}\,dx = \tfrac{1}{2}x + \tfrac{1}{5}\log(x-1) - \tfrac{9}{20}\log(2x+3).$$

(iii) The case in which the two factors of the denominator are coincident should be noticed. E.g. find $\int \frac{4x+3}{(x-3)^2}\,dx$.

In this case, let $\frac{4x+3}{(x-3)^2} \equiv \frac{A}{x-3} + \frac{B}{(x-3)^2}$.

[These are the only two fractions whose denominators could have the L. C. M. $(x-3)^2$.]

Clearing of fractions, $\quad 4x+3 \equiv A(x-3) + B$.

To find B, put $x = 3$; $\quad \therefore \quad B = 15$.

To find A, compare the coefficients of x in the preceding identity; these are, on the left-hand side 4, and on the right-hand side A; $\quad \therefore \quad A = 4$.

$$\frac{4x+3}{(x-3)^2} = \frac{}{x-3} + \frac{15}{(x-3)^2},$$

and $\quad \int \frac{4x+3}{(x-3)^2}\,dx = 4\int \frac{dx}{x-3} + 15 \int \frac{dx}{(x-3)^2} \quad 4\log(x-3) - \frac{15}{x-3}.$

The values of A and B can be found in all the preceding examples by comparing coefficients, although the method given above is shorter, e.g. in Ex. (ii),

comparing coefficients of x on both sides, $\quad 1 = 2A + B$,
and comparing constant terms, $\quad -3 = 3A - B$.

These are two simultaneous equations of the first degree for A and B which, when solved, give $A = -\frac{2}{5}$, $B = \frac{9}{5}$ as before.

Two integrals which can be obtained in this way are of special importance, and will be included among the standard forms. These are

$$\int \frac{dx}{x^2-a^2} \quad \text{and} \quad \int \frac{dx}{a^2-x^2}.$$

Taking the former, $\quad \frac{1}{x^2-a^2} \equiv \frac{A}{x-a} + \frac{B}{x+a}$;

$$1 \equiv A(x+a) + B(x-a).$$

Put $x=a$, $\quad 1=A\,.\,2a$, $\quad$ and $A=1/2a$.

Put $x=-a$, $\quad 1=B(-2a)$, and $B=-1/2a$.

$$\therefore \quad \frac{1}{x^2-a^2}=\frac{1}{2a}\cdot\frac{1}{x-a}-\frac{1}{2a}\cdot\frac{1}{x+a},$$

and $$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log(x-a)-\frac{1}{2a}\log(x+a)=\frac{1}{2a}\log\frac{x-a}{x+a}.$$

Similarly, it will be found that $\displaystyle\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\frac{a+x}{a-x}$.

It must be remembered that the logarithm of a negative quantity is imaginary. In the former of these two expressions, x^2 is supposed $> a^2$, and in the latter, $x^2 < a^2$; the logarithms which occur in the results are then real.

Examples XLI.

Integrate the following expressions:

1. $\dfrac{x}{x^2-1}$.
2. $\dfrac{2x-5}{x^2-5x+6}$.
3. $\dfrac{2x+3}{x^2+x-30}$.
4. $\dfrac{x^2}{x^2-4}$.
5. $\dfrac{x^2}{x^2-5x+4}$.
6. $\dfrac{x+1}{3x^2-x-2}$.
7. $\dfrac{x+1}{(x-1)^2}$.
8. $\dfrac{5x+2}{x^2-4x+4}$.
9. $\dfrac{5+x^2}{9-x^2}$.
10. $\dfrac{x^2+1}{x^2-x-2}$.
11. $\dfrac{4x+3}{3x^2-10x+3}$.
12.* $\left(\dfrac{x-2}{x+1}\right)^2$
13. $\dfrac{x^3}{x^2-1}$.
14. $\dfrac{x}{(2x-1)^2}$
15. $\dfrac{x^3}{x^2+x-20}$
16. $\dfrac{1}{x^2-(a+b)x+ab}$.
17. $\dfrac{x^4}{x^2-5}$.
18. $\dfrac{x^2+1}{5x-2x^2}$.

124. 2 (ii). Denominator which does not resolve into rational factors.

It has been shown, in Art. 102, that

$$\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}, \tag{i}$$

and, in the preceding article, that

if $x^2 > a^2$, $$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}, \tag{ii}$$

if $x^2 < a^2$, $$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\frac{a+x}{a-x} \tag{iii}$$

* Divide out before squaring.

The two latter integrals may also (from Art. 94 and Ex. XXXII. 20) be expressed in the alternative forms

$$-\frac{1}{a}\coth^{-1}\frac{x}{a} \text{ and } \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

respectively, which are analogous to the form (i).

Taking the case 2 (ii) mentioned above,

(*a*) *Let the numerator be* 1.

Then, dividing the denominator by the coefficient of x^2, and completing the square of the terms which contain x, ~~the~~ integral reduces to one of the three forms just mentioned.

Examples:

$$\int\frac{dx}{x^2+8x+25} = \int\frac{dx}{(x+4)^2+9} = \frac{1}{3}\tan^{-1}\frac{x+4}{3}, \text{ by (i).}$$

$$\int\frac{dx}{3x^2-4x+7} = \frac{1}{3}\int\frac{dx}{x^2-\frac{4}{3}x+\frac{7}{3}} = \frac{1}{3}\int\frac{dx}{(x-\frac{2}{3})^2+\frac{17}{9}}$$

$$= \frac{1}{3}\cdot\frac{1}{\frac{1}{3}\sqrt{17}}\tan^{-1}\frac{x-\frac{2}{3}}{\frac{1}{3}\sqrt{17}} = \frac{1}{\sqrt{17}}\tan^{-1}\frac{3x-2}{\sqrt{17}}, \text{ by (i).}$$

$$\int\frac{dx}{x^2+6x-4} = \int\frac{dx}{(x+3)^2-13} = \frac{1}{2\sqrt{13}}\log\frac{x+3-\sqrt{13}}{x+3+\sqrt{13}}, \text{ by (ii).}$$

$$\int\frac{dx}{6x-7-x^2} = \int\frac{dx}{2-(x-3)^2} = \frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+x-3}{\sqrt{2}-x+3}, \text{ by (iii).}$$

$$\int\frac{dx}{11-4x-2x^2} = \frac{1}{2}\int\frac{dx}{\frac{11}{2}-2x-x^2} = \frac{1}{2}\int\frac{dx}{\frac{13}{2}-(x+1)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2\sqrt{\frac{13}{2}}}\log\frac{\sqrt{\frac{13}{2}}+x+1}{\sqrt{\frac{13}{2}}-x-1}, \text{ by (iii).}$$

$$= \frac{1}{2\sqrt{26}}\log\frac{\sqrt{26}+2x+2}{\sqrt{26}-2x-2}.$$

Examples XLII.

Integrate

1. $\dfrac{1}{x^2+2x+10}$.
2. $\dfrac{1}{2x^2+2x+5}$.
3. $\dfrac{1}{x^2-4x+12}$.
4. $\dfrac{1}{x^2-2x-1}$.
5. $\dfrac{1}{3x^2+8x-4}$.
6. $\dfrac{1}{4-2x-x^2}$.
7. $\dfrac{1}{10-4x-3x^2}$.
8. $\dfrac{1}{4x^2-4x-7}$.
9. $\dfrac{1}{5x^2-7}$.
10. $\dfrac{x^2+2x}{x^2+2x+2}$.
11. $\dfrac{x^2+4x}{x^2+4x-1}$.
12. $\dfrac{x^4}{x^2+7}$.
13. $\dfrac{x^2+x+2}{x^2+x-1}$.
14. $\dfrac{1}{8+3x-2x^2}$.
15. $\dfrac{1}{10a^2+4ax-x^2}$.
16. $\dfrac{1}{ax^2+bx+c}$ (i) when $b^2>4ac$, (ii) when $b^2<4ac$.

125. A useful rule.

Before proceeding to discuss the next case in the integration of rational fractions, it is necessary to call attention to the following fact:—

The integral of a fraction whose numerator is the d. c. of its denominator is the logarithm of the denominator, i.e. the integral of $f'(x)/f(x)$ with respect to x is $\log f(x)$.

This is obvious from the method of differentiating the logarithm of any function of x (Art. 98).

The d. c. of $\log u$ with respect to $x = \frac{1}{u} \times \frac{du}{dx} = \frac{du}{dx} \Big/ u$, a fraction whose numerator is the d. c. of its denominator. Therefore, conversely, the integral of the fraction $\frac{du}{dx} \Big/ u$ with respect to x is $\log u$.

This is a rule which is often useful in dealing with all kinds of functions, algebraical, trigonometrical, exponential, &c. It is really a particular case of the method of integration by *change of variable* considered later, but the student should try to accustom himself to recognizing at a glance fractions of the above type. A good deal of labour is often thereby saved, e.g. in the first two of the examples immediately following, the integral can be written down at once, whereas if the method of partial fractions be used, the working is long. In some cases, the insertion of a numerical factor is required to make the numerator equal to the d. c. of the denominator.

Examples:

$$\int \frac{3x^2+1}{x^3+x-2}\,dx = \log(x^3+x-2).$$

$$\int \frac{x^3}{x^4-a^4}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4-a^4}\,dx = \frac{1}{4}\log(x^4-a^4).$$

$$\int \frac{x^{n-1}}{x^n+a^n}\,dx = \frac{1}{n}\int \frac{nx^{n-1}}{x^n+a^n}\,dx = \frac{1}{n}\log(x^n+a^n).$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \log \sin x.$$

$$\int \frac{e^x}{e^x+1}\,dx = \log(e^x+1).$$

$$\int \frac{\sin x \cos x}{a+b\cos^2 x}\,dx = -\frac{1}{2b}\int \frac{-2b\cos x \sin x}{a+b\cos^2 x}\,dx = -\frac{1}{2b}\log(a+b\cos^2 x).$$

Examples XLIII.

Integrate:

1. $\dfrac{2x+3}{x^2+3x-4}$. 2. $\dfrac{x^2}{x^3-1}$. 3. $\dfrac{x^3}{a^4-x^4}$.

4. $\dfrac{x+1}{x^2+2x+7}$. 5. $\tan x$. 6. $\cot ax$.

7. $\dfrac{\sin x \cos x}{1+3\sin^2 x}$. 8. $\dfrac{x}{x^2+a^2}$. 9. $\dfrac{\sin x}{a+b\cos x}$.

10. $\dfrac{e^{2x}}{1-e^{2x}}$. 11. $\dfrac{ax+b}{ax^2+2bx+c}$. 12. $\dfrac{\sec^2 x}{3+4\tan x}$.

13. $\tanh x$. 14. $\dfrac{1}{x\log x}$. 15. $\dfrac{x^{n-1}}{1-x^n}$.

16. $\dfrac{\sin x-\cos x}{\sin x+\cos x}$. 17. $\dfrac{e^x(1+x)}{1+xe^x}$. 18. $\dfrac{\sin 2x}{a+b\sin^2 x}$.

126. 2 (ii) *b*. Numerator of the first degree.

Returning to the integration of rational fractions, the case in which the denominator is of the second degree and the numerator of the first degree has next to be considered.

The following method will effect the integration:—Put the numerator equal to $k \times$ (the d. c. of the denominator) $+ l$, where k and l are constants which can be determined by inspection (or by comparing coefficients); the integral can then be divided into two parts, of which the first is a fraction whose numerator is the d. c. of its denominator, and whose integral is therefore the logarithm of the denominator (Art. 125), and the second is a fraction of the kind considered in the previous case (Art. 124). The process is illustrated in the following examples.

If the numerator is of the second or higher degree, it can be divided by the denominator until the remainder is of the first degree, and therefore it has now been shown how to integrate any rational algebraical fraction whose denominator is of the first or second degree.

Examples:

$$\int \frac{4x+5}{x^2+2x+2}\,dx = \int \frac{2(2x+2)+1}{x^2+2x+2}\,dx = 2\int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$$= 2\log(x^2+2x+2) + \int \frac{dx}{(x+1)^2+1}$$

$$= 2\log(x^2+2x+2) + \tan^{-1}(x+1).$$

$$\int\frac{3x-5}{2x^2+6x+1}dx = \int\frac{\frac{3}{4}(4x+6)-\frac{19}{2}}{2x^2+6x+1}dx$$

$$= \frac{3}{4}\int\frac{4x+6}{2x^2+6x+1}dx - \frac{19}{2}\int\frac{dx}{2x^2+6x+1}$$

$$= \frac{3}{4}\log(2x^2+6x+1) - \frac{19}{4}\int\frac{dx}{x^2+3x+\frac{1}{2}}$$

$$= \frac{3}{4}\log(2x^2+6x+1) - \frac{19}{4}\int\frac{dx}{(x+\frac{3}{2})^2-\frac{7}{4}}$$

$$= \frac{3}{4}\log(2x^2+6x+1) - \frac{19}{4}\cdot\frac{1}{2\cdot\frac{1}{2}\sqrt{7}}\log\frac{x+\frac{3}{2}-\frac{1}{2}\sqrt{7}}{x+\frac{3}{2}+\frac{1}{2}\sqrt{7}}$$

$$= \frac{3}{4}\log(2x^2+6x+1) - \frac{19}{4\sqrt{7}}\log\frac{2x+3-\sqrt{7}}{2x+3+\sqrt{7}}.$$

$$\int\frac{x^3+1}{x^2+4}dx = \int\left[x-\frac{4x-1}{x^2+4}\right]dx = \int x\,dx - \int\frac{4x-1}{x^2+4}dx$$

$$= \frac{1}{2}x^2 - \int\frac{2(2x)-1}{x^2+4}dx = \frac{1}{2}x^2 - 2\int\frac{2x}{x^2+4}dx + \int\frac{1}{x^2+4}dx$$

$$= \frac{1}{2}x^2 - 2\log(x^2+4) + \frac{1}{2}\tan^{-1}\frac{1}{2}x.$$

Examples XLIV.

Integrate the following:

1. $\frac{x+1}{x^2+9}$
2. $\frac{4x-3}{x^2-5}$.
3. $\frac{}{x^2+a^2}$.
4. $\frac{1-x}{7-x^2}$.
5. $\frac{6x+3}{x^2+4x+13}$.
6. $\frac{4x-5}{x^2-2x-1}$.
7. $\frac{8x-3}{2x^2+2x+1}$
8. $\frac{3-2x}{3x^2+6x-1}$.
9. $\frac{5x-1}{x^2-3x+5}$.
10. $\frac{x^2-1}{x^2-2x+5}$.
11. $\frac{x^3}{x^2-6x+10}$.
12. $\frac{x}{4x^2-10x-3}$.
13. $\frac{x^2-x+1}{x^2+x+1}$.
14. $\frac{x^2-3x+2}{x^2-2x+3}$.
15. $\frac{x^2-1}{x^2+5x+6}$.
16. $\frac{}{x^2+ax+a^2}$.
17. $\frac{x^3+1}{x^2+1}$.
18. $\frac{x-a}{x^2+2ax-a^2}$

127. 3. Denominator of higher degree than the second.

If the denominator breaks up into rational factors of the first and second degree, use the method of partial fractions.

To illustrate the various cases which may arise, three examples will be worked out, in the first of which the denominator resolves into three factors of the first degree; in the second, into one factor of the first degree and one of the second degree; and in the third, into a factor of the first degree repeated and one of the second degree. In each of the constituent fractions, we take a numerator of lower

degree than the denominator, since the given fraction is of this type, i.e. for the numerator of a partial fraction with denominator of the first degree, we take a numerical quantity A (as in Art. 123); for the numerator of a fraction whose denominator is of the second degree, we assume an expression $Bx+C$ of the first degree; if the given fraction contains a repeated factor $(x-a)^2$ in its denominator, a fraction with denominator $(x-a)^2$ and numerator $Bx+C$ might be assumed, but this would resolve into two simpler fractions $\dfrac{E}{x-a}$ and $\dfrac{F}{(x-a)^2}$, and these are therefore taken as the partial fractions corresponding to a repeated factor $(x-a)^2$. Similarly, to a repeated factor $(x-a)^3$, in the given denominator, would correspond three fractions $\dfrac{A}{x-a}$, $\dfrac{B}{(x-a)^2}$, $\dfrac{C}{(x-a)^3}$, and so on.

For a complete account of the general theory of partial fractions and the integration of rational fractions in general, the student is referred to treatises on Algebra and more advanced works on the Integral Calculus. The examples here considered are sufficient to enable the student to deal with most of the cases he is likely to meet with in elementary work.

Examples:

(i) $\dfrac{x^2+1}{x(x^2-4)}$.

Let $$\frac{x^2+1}{x(x^2-4)} \equiv \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}.$$

Clearing of fractions, $x^2+1 \equiv A(x^2-4)+Bx(x+2)+Cx(x-2)$.

To find A, put $x=0$; $\therefore\ 1 = A(-4)$, and $A = -\frac{1}{4}$.

To find B, put $x=2$; $\therefore\ 5 = B.2.4$, and $B = \frac{5}{8}$.

To find C, put $x=-2$; $\therefore\ 5 = C.-2.-4$, and $C = \frac{5}{8}$.

$$\therefore\ \frac{x^2+1}{x(x^2-4)} = -\frac{1}{4x} + \frac{5}{8(x-2)} + \frac{5}{8(x+2)}.$$

$$\int \frac{x^2+1}{x(x^2-4)}\,dx = -\frac{1}{4}\log x + \frac{5}{8}\log(x-2) + \frac{5}{8}\log(x+2).$$

(ii) $\displaystyle\int \frac{x}{(2-x)(x^2+4x+5)}\,dx.$

Let $$\frac{x}{(2-x)(x^2+4x+5)} \equiv \frac{A}{2-x} + \frac{Bx+C}{x^2+4x+5}.$$

Clearing of fractions, $x \equiv A(x^2+4x+5) + (2-x)(Bx+C)$.

To find A, put $x=2$; $\therefore\ 2 = A(4+8+5)$, and $A = \frac{2}{17}$.

To find B and C, we must compare coefficients. The expressions being identical, the coefficient of any power of x on one side is equal to the coefficient of the same power of x on the other side.

Comparing coefficients of x^2, $0 = A - B$, whence $B = A = \frac{2}{17}$;
comparing constant terms, $0 = 5A + 2C$, whence $C = -\frac{5}{2}A = -\frac{5}{17}$.

Hence the given fraction $\frac{}{17(2-x)} + \frac{2x-5}{17(x^2+4x+5)}$,

and
$$\int \frac{x}{(2-x)(x^2+4x+5)}\,dx = \frac{2}{17}\int \frac{dx}{2-x} + \frac{1}{17}\int \frac{(2x+4)-9}{x^2+4x+5}\,dx$$
$$-\frac{2}{17}\log(2-x) + \frac{1}{17}\int \frac{2x+4}{x^2+4x+5}\,dx - \frac{}{17}\int \frac{dx}{(x+2)^2+1}$$
$$-\tfrac{2}{17}\log(2-x) + \tfrac{1}{17}\log(x^2+4x+5) - \tfrac{9}{17}\tan^{-1}(x+2).$$

(iii)
$$\frac{dx}{(x-1)^2(x^2+1)}.$$

Let
$$\frac{}{(x-1)^2(x^2+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Clearing of fractions, $1 \equiv A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$.

To find B, put $x = 1$; $\therefore\ 1 = B.2$, and $B = \frac{1}{2}$.

Comparing coefficients of x^3, $0 = A + C$;
comparing coefficients of x^2, $0 = -A + B + D - 2C$;
comparing constant terms, $1 = -A + B + D$.

Subtracting the last equation from the preceding one, $-1 = -2C$, and $C = \frac{1}{2}$;

$$\therefore\ A = -C = -\tfrac{1}{2}, \text{ and } D = A - B + 1 = 0.$$

Hence the given fraction
$$= -\frac{1}{2(x-1)} + \frac{1}{2(x-1)^2} + \frac{x}{2(x^2+1)};$$
$$\therefore\ \int \frac{dx}{(x-1)^2(x^2+1)} = -\frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2} + \frac{1}{4}\int \frac{2x}{x^2+1}\,dx$$
$$= -\frac{1}{2}\log(x-1) - \frac{1}{2}\frac{1}{x-1} + \frac{1}{4}\log(x^2+1).$$

Examples XLV.

Integrate the following:

1. $\dfrac{1}{x^2(x-1)}$. 2. $\dfrac{1}{x(x^2+1)}$. 3. $\dfrac{1}{x^3-3x^2+2x}$.

4. $\dfrac{x^2}{(x-1)(x^2+4)}$. 5. $\dfrac{x^2}{(x^2-1)(2x+1)}$. 6. $\dfrac{x^3}{(x-1)^2(x+1)}$.

7. $\dfrac{1}{x^2(x^2+1)}$. 8. $\dfrac{1}{x^4-1}$. 9. $\dfrac{x}{(x^2-1)^2}$.

10. $\dfrac{x^2}{(x^2-1)^2}$. 11. $\dfrac{1}{(x+1)(x^2+2x+2)}$. 12. $\dfrac{1}{x^3-1}$.

13. $\dfrac{x}{x^4+x^2-2}$. 14. $\dfrac{x^3}{x^3+1}$. 15. $\dfrac{1}{x^3(1-x)}$.

16. $\dfrac{x^2}{(x-1)^2(x^2-4)}$. 17. $\dfrac{x^2}{1-x^4}$. 18. $\dfrac{}{x^3+8}$.

19. $\dfrac{1}{x^4+3x^2+2}$. 20. $\dfrac{1-x}{x^2(1+x+x^2)}$ 21. $\dfrac{1}{x^4+4}$.

22. $\dfrac{4x+3}{x^4-5x^2+4}$. 23. $\dfrac{x^2}{1+x^2+x^4}$. 24. $\dfrac{1}{x^2(x-1)(x^2+1)}$.

25. $\dfrac{x^4}{x^3-a^3}$ 26. $\dfrac{x^4}{(x^2-1)(2x^2+9)}$. 27. $\dfrac{x^2}{x^6-1}$.

128. Integration of irrational fractions of the form

$$\frac{px+q}{\sqrt{(ax^2+bx+c)}}.$$

Many irrational expressions can be rationalized by a suitable change of variable, as will be explained later on. We will here consider a fraction whose numerator is constant or of the first degree, and whose denominator is the square root of an expression of the second degree, i.e. of the form $\sqrt{(ax^2+bx+c)}$. It should be noticed that the form $\sqrt{[(ax+b)/(cx+d)]}$ is reduced to the form just mentioned, by multiplying numerator and denominator by $\sqrt{(ax+b)}$.

We must begin by adding to our list of standard forms. It has already been shown (Art. 102) that

$$\int\frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a}; \qquad \text{(i)}$$

and (Art. 104) that

$$\int\frac{dx}{\sqrt{(a^2+x^2)}} = \sinh^{-1}\frac{x}{a}; \qquad \text{(ii)}$$

$$\int\frac{dx}{\sqrt{(x^2-a^2)}} = \cosh^{-1}\frac{x}{a}. \qquad \text{(iii)}$$

It should be noticed that the last two integrals can be expressed in the alternative form

$$\int\frac{dx}{\sqrt{(x^2\pm a^2)}} = \log\left[x+\sqrt{(x^2\pm a^2)}\right]. \qquad \text{(iv)}$$

as follows from an example worked in Art. 98.

It was shown in Art. 94 that $\sinh^{-1}x = \log\left[x+\sqrt{(x^2+1)}\right]$, and that one of the two values of $\cosh^{-1}x = \log\left[x+\sqrt{(x^2-1)}\right]$.

In exactly the same way, the more general results

$$\sinh^{-1}\frac{x}{a} = \log\frac{x+\sqrt{(x^2+a^2)}}{a} \quad \text{and} \quad \cosh^{-1}\frac{x}{a} = \log\frac{x\pm\sqrt{(x^2-a^2)}}{a}$$

can be obtained. Since

$$\log\left\{\left[x+\sqrt{(x^2\pm a^2)}\right]/a\right\} = \log\left[x+\sqrt{(x^2\pm a^2)}\right]-\log a,$$

it follows that the two alternative forms given above for $\displaystyle\int\frac{dx}{\sqrt{(x^2\pm a^2)}}$ differ only by the constant term $\log a$. See Art. 76.

Taking now the fraction mentioned above,

(1) *Let the numerator be* 1.

Divide the expression under the root by the numerical value of the coefficient of x^2, and complete the square of the terms which contain x; the integral reduces to one of the three forms above.

Examples:

$$\int\frac{dx}{\sqrt{(x^2+4x+13)}} = \int\frac{dx}{\sqrt{[(x+2)^2+9]}} = \sinh^{-1}\frac{x+2}{3},\ \text{by (ii)},$$
$$\text{or}\quad \log[x+2+\sqrt{(x^2+4x+13)}],\ \text{by (iv)}.$$

$$\int\frac{dx}{\sqrt{(8-5x-3x^2)}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(\frac{8}{3}-\frac{5}{3}x-x^2)}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{[\frac{121}{36}-(x+\frac{5}{6})^2]}}$$
$$= \frac{1}{\sqrt{3}}\sin^{-1}\frac{x+\frac{5}{6}}{\frac{11}{6}},\ \text{by (i)},\ = \frac{1}{\sqrt{3}}\sin^{-1}\frac{6x+5}{11}.$$

$$\int\frac{dx}{\sqrt{[x(3+2x)]}} = \int\frac{dx}{\sqrt{(2x^2+3x)}} = \frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(x^2+\frac{3}{2}x)}}$$
$$= \frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{[(x+\frac{3}{4})^2-\frac{9}{16}]}} = \frac{1}{\sqrt{2}}\cosh^{-1}\frac{x+\frac{3}{4}}{\frac{3}{4}},\ \text{by (iii)},$$
$$= \frac{1}{\sqrt{2}}\cosh^{-1}\frac{4x+3}{3}\ \text{or}\ \log[x+\tfrac{3}{4}+\sqrt{(x^2+\tfrac{3}{2}x)}].$$

Examples XLVI.

Integrate

$\dfrac{}{\sqrt{(x^2+2x+10)}}$. 2. $\dfrac{}{\sqrt{(x^2+10x-11)}}$. 3. $\dfrac{1}{\sqrt{(7-6x-x^2)}}$.

$\dfrac{1}{\sqrt{[x(4-x)]}}$. 5. $\dfrac{1}{\sqrt{[x(1+x)]}}$. 6. $\dfrac{1}{\sqrt{[(x-3)(x-4)]}}$.

$\dfrac{1}{\sqrt{[(5+x)(x-2)]}}$. 8. $\dfrac{1}{\sqrt{(18x^2-42x+37)}}$. 9. $\dfrac{1}{\sqrt{[x(3-2x)]}}$.

10. $\dfrac{1}{\sqrt{(2x^2-7x+5)}}$. 11. $\dfrac{1}{\sqrt{(8+3x-x^2)}}$. 12. $\dfrac{1}{\sqrt{(9x^2-4ax)}}$.

129. (2) Numerator of the first degree.

Since the d. c. of $x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$, it follows that the d. c. of $u^{\frac{1}{2}}$ with respect to $x = \frac{1}{2}u^{-\frac{1}{2}}\times du/dx$, which may be written $\dfrac{\frac{1}{2}du/dx}{\sqrt{u}}$, a fraction whose denominator is $\sqrt{u}$ and whose numerator is half the d. c. of u.

Conversely the integral of such a fraction is $\sqrt{u}$. (This again is really a particular case of the method of integration by change of

variable, to be considered in Art. 131). Hence, proceeding on the same lines as in Art. 126, to integrate $\frac{px+q}{\sqrt{(ax^2+bx+c)}}$, put the numerator equal to $k\times(\frac{1}{2}$ the d. c. of the expression under the radical sign$)+l$, where, as before, k and l are constants whose values in a numerical case are evident on inspection. The integral breaks up into two parts, of which the former is a fraction of the form just described, whose integral is therefore equal to the denominator, and the latter is of the type considered in the preceding article.

Examples:

$$\int\frac{x}{\sqrt{(4-x^2)}} = \int\frac{\frac{1}{2}(-2x)}{\sqrt{(4-x^2)}}dx = -\sqrt{(4-x^2)}.$$

$$\int\frac{x+1}{\sqrt{[x(x-2)]}} = \int\frac{\frac{1}{2}(2x-2)+2}{\sqrt{(x^2-2x)}}dx$$

$$= \int\frac{\frac{1}{2}(2x-2)}{\sqrt{(x^2-2x)}}dx + 2\int\frac{dx}{\sqrt{(x^2-2x)}}$$

$$= \sqrt{(x^2-2x)}+2\int\frac{dx}{\sqrt{[(x-1)^2-1]}}$$

$$= \sqrt{(x^2-2x)} + 2\cosh^{-1}(x-1).$$

$$\int\sqrt{\frac{x-1}{2x+3}}\,dx = \int\frac{x-1}{\sqrt{(2x^2+x-3)}}dx = \int\frac{\frac{1}{2}\times\frac{1}{2}(4x+1)-\frac{5}{4}}{\sqrt{(2x^2+x-3)}}dx$$

$$= \frac{1}{2}\int\frac{\frac{1}{2}(4x+1)}{\sqrt{(2x^2+x-3)}}dx - \frac{5}{4}\int\frac{dx}{\sqrt{(2x^2+x-3)}}$$

$$= \frac{1}{2}\sqrt{(2x^2+x-3)} - \frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{(x^2+\frac{1}{2}x-\frac{3}{2})}}$$

$$= \frac{1}{2}\sqrt{(2x^2+x-3)} - \frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{[(x+\frac{1}{4})^2-\frac{25}{16}]}}$$

$$= \frac{1}{2}\sqrt{(2x^2+x-3)} - \frac{5}{4\sqrt{2}}\cosh^{-1}\frac{4x+1}{5}.$$

Examples XLVII.

Integrate

1. $\frac{1}{\sqrt{(x^2+5)}}$.
2. $\frac{x+1}{\sqrt{(x^2-1)}}$.
3. $\frac{2x-1}{\sqrt{(4-x^2)}}$.
4. $\sqrt{\left(\frac{x}{1+x}\right)}$.
5. $\frac{1}{\sqrt{(4-3x-x^2)}}$.
6. $\frac{2x+3}{\sqrt{(x^2+5x+6)}}$.
7. $\sqrt{\left(\frac{1-x}{1+x}\right)}$.
$\frac{x+1}{\sqrt{(2x^2+x-3)}}$.
9. $\sqrt{\left(\frac{2+x}{x}\right)}$.
10. $\frac{3x-4}{\sqrt{(3x^2+4x+7)}}$
11. $\sqrt{\left(\frac{3-x}{2+x}\right)}$.
12. $\frac{x}{\sqrt{(5x^2-4x)}}$.

130. Standard forms.

All the standard forms which it is absolutely necessary to remember have now been mentioned, and it will be convenient at this stage to make a list of them. They are:—

$$\int x^n\,dx = \frac{x^{n+1}}{n+1},\ \text{for all values of } n \text{ except } n = -1,$$

and in that case, $\displaystyle\int \frac{1}{x}\,dx = \log x.$

$$e^x\,dx = e^x.$$

$$\sin x\,dx = -\cos x.$$

$$\int \cos x\,dx = \sin x.$$

$$\int \sec^2 x\,dx = \tan x.$$

$$\int \frac{}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

If $x^2 < a^2$, $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\frac{a+x}{a-x} \text{ or } \frac{1}{a}\tanh^{-1}\frac{x}{a}.$

If $x^2 > a^2$, $\displaystyle\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\frac{x-a}{x+a} \text{ or } -\frac{1}{a}\coth^{-1}\frac{x}{a}.$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a}.$$

$$\int \frac{1}{\sqrt{(a^2+x^2)}}\,dx = \log[x+\sqrt{(x^2+a^2)}] \text{ or } \sinh^{-1}\frac{x}{a}.$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \log[x+\sqrt{(x^2-a^2)}] \text{ or } \cosh^{-1}\frac{x}{a}.$$

Also the rules of Art. 75 enable us to write down at once the integral when x is replaced by $ax+b$.

131. Integration by substitution or change of variable.

This is the most frequently used of all the devices for converting expressions into standard forms. Particular cases of it have already been considered in Arts. 75, 125, and 129. It will be seen from the proof below that this method of integration is the converse of the method of differentiating a function of a function (Art. 34).

The theory of the method is as follows:

Let $y = \int f(x)\,dx$; it is required to change the variable from x to u, where u is a given function of x.

Since $$y = \int f(x)\,dx, \quad dy/dx = f(x),$$

$$\therefore \quad \frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = f(x) \times \frac{dx}{du},$$

$$\therefore \quad y = \int f(x) \cdot \frac{dx}{du}\,du,$$

$$\text{i.e.} \quad \int f(x)\,dx = \int f(x)\,\frac{dx}{du}\,du.$$

Conversely, if (interchanging x and u in the result just obtained) an integral is recognized to be of the form $\int f(u)\,\frac{du}{dx}\,dx$, it may be replaced by $\int f(u)\,du$.

The latter is a form of the theorem which is very convenient for use, i.e. the integral of the product of $f(u)$ and du/dx with respect to x is the same as the integral of $f(u)$ with respect to u. The difficulty in practice at first is to determine what function of x should be adopted as u in any particular case, and it is only experience which enables this question to be answered readily. If the theorem is used in the form last mentioned, it must be borne in mind that the substitution adopted must be such that

(*a*) one factor of the given expression supplies the du/dx which has to be introduced, and

(*b*) the rest of the expression is easily expressible in terms of u.

The following examples will illustrate the method.

Examples:

(i) $\int \sin^4 x \cos x\,dx$. Let $\sin x = u$. $\therefore$ $du/dx = \cos x$, and the integral becomes

$$\int u^4 \frac{du}{dx}\,dx = \int u^4\,du = \frac{u^5}{5} = \frac{1}{5}\sin^5 x.$$

(ii) $\int \frac{\sin x}{\cos^n x}\,dx$. Let $\cos x = u$. $du/dx = -\sin x$, and the integral becomes

$$\int -\frac{1}{u^n} \cdot \frac{du}{dx} \cdot dx = -\int \frac{du}{u^n} = -\frac{u^{-n+1}}{-n+1} = \frac{}{(n-1)\,u^{n-1}}$$

$$= \frac{}{(n-1)\cos^{n-1} x} = \frac{}{n-1}\sec^{n-1} x.$$

(iii) $\int x^2 \sqrt{(a^3+x^3)}\, dx.$

Here, since x^2 is (save for a constant factor) the d. c. of a^3+x^3, which occurs under the radical sign, let $a^3+x^3=u$; $\therefore\ 3x^2 = du/dx.$

The integral becomes

$$\int \frac{1}{3}\frac{du}{dx}\sqrt{u}\, dx = \frac{1}{3}\int \sqrt{u}\, du = \frac{1}{3}\cdot\frac{u^{3/2}}{\frac{3}{2}} = \frac{2}{9}(a^3+x^3)^{3/2}.$$

(iv) $\int x(a-bx^2)^n\, dx.$

Let $a-bx^2=u.$ $\quad\therefore\quad -2bx = du/dx,$

and the integral becomes

$$-\frac{1}{2b}\frac{du}{dx}u^n\, dx = -\frac{1}{2b}\int u^n\, du = -\frac{1}{2b}\frac{u^{n+1}}{n+1}$$
$$= -\frac{1}{2b(n+1)}(a-bx^2)^{n+1}.$$

(v) $\int \frac{x^5}{a^6+x^6}\, dx.$

In this case, since the numerator x^5 is $\frac{1}{6}$ of the d. c. of the denominator, let $a^6+x^6=u.$ $\quad\therefore\quad 6x^5 = du/dx,$

and the integral becomes

$$\int \frac{1}{u}\cdot\frac{1}{6}\frac{du}{dx}\, dx = \frac{1}{6}\int\frac{du}{u} = \frac{1}{6}\log u = \frac{1}{6}\log(a^6+x^6) \quad \text{(see Art. 125).}$$

(vi) $\int \frac{x^2}{a^6+x^6}\, dx.$

In this case, since the numerator is only x^2, let $x^3 = u$, and then $3x^2 = du/dx$. The integral becomes

$$\int\frac{1}{a^6+u^2}\cdot\frac{1}{3}\frac{du}{dx}\, dx = \frac{1}{3}\int\frac{du}{a^6+u^2} = \frac{1}{3}\cdot\frac{1}{a^3}\tan^{-1}\frac{u}{a^3} = \frac{1}{3a^3}\tan^{-1}\frac{x^3}{a^3}.$$

Generally, if the function to be integrated is the product of x^{n-1} and some function of x^n or of $a+bx^n$, which is recognized to be of a type whose integral is known, the substitution x^n, or $a+bx^n = u$ will effect the integration.

Examples XLVIII.

Integrate:

1. $\sin^2 x\cos x.$ 2. $\cos^3 x\sin x.$ 3. $x\sqrt{(a^2+x^2)}.$

4. $\frac{\sin x}{\cos^2 x}.$ 5. $\frac{\cos x}{\sin^4 x}.$ 6. $\frac{x}{\sqrt{(x^2-a^2)}}.$

7. $\frac{x^3}{(x^4-1)^2}.$ 8. $\frac{x}{(a^2-x^2)^n}.$ 9. $\frac{x^2}{\sqrt{(a^3-x^3)}}.$

10. $x(a^2+x^2)^n.$ 11. $x^2(a^3-x^3)^n.$ 12. $x^{n-1}(a-bx^n)^2.$

13. $\frac{x^2}{a^3-x^3}.$ 14. $\frac{x}{a^4+x^4}.$ 15. $\frac{x^2}{x^6-a^6}.$

16. $\dfrac{x^3}{\sqrt{(a^4-x^4)}}$.

17. $\dfrac{x}{\sqrt{(a^4-x^4)}}$.

18. $\dfrac{x^3}{\sqrt{(a^8-x^8)}}$

19. $x^2(x^3-2)^4$.

20. $\dfrac{\cos x}{1+\sin^2 x}$.

21. $\dfrac{e^x}{1+e^x}$.

22. $\dfrac{e^x}{1-e^{2x}}$.

23. $\dfrac{\sin x}{1-4\cos^2 x}$

24. $\cos x(1-\sin x)^n$.

25. $\dfrac{\log x}{x}$.

26. $\dfrac{(\log x)^n}{x}$.

27. $\tan x \sec^2 x$.

28. $\tan^n x \sec^2 x$.

29. $\dfrac{\sec^2 x}{1+\tan x}$.

30. $\dfrac{\sec^2 x}{1-\tan^2 x}$.

31. $(1+\log x)^2/x$.

32. $e^x(1-e^x)^n$.

33. $(\cos\sqrt{x})/\sqrt{x}$.

34. $\dfrac{\sin^{-1}x}{\sqrt{(1-x^2)}}$.

35. $\dfrac{x}{\sqrt{(x^4-16)}}$.

36. $\dfrac{\cos x}{\sqrt{(2-\sin^2 x)}}$.

37. $\dfrac{\sin x}{(a-b\cos x)^2}$.

38. $\dfrac{1}{x(1+\log x)^2}$

39. $\dfrac{x}{\sqrt{(6-5x^2-x^4)}}$.

40. $\dfrac{x^2}{x^6+4x^3+5}$.

132. Further examples.

In some cases it is more convenient to proceed as below.

Any algebraical expression involving only the one irrational quantity $\sqrt{(ax+b)}$ can be rationalized by the substitution $ax+b=u^2$ as in the following examples, and then its integral can be found by the methods of Arts. 122–127.

Examples:

(i) $\displaystyle\int \frac{x^2}{\sqrt{(x+2)}}\,dx.$

Denoting the integral by y, we have $\dfrac{dy}{dx}=\dfrac{x^2}{\sqrt{(x+2)}}$.

Let $x+2=u^2$; $\quad\therefore\ dx/du=2u$; and $x=u^2-2$;

$$\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}=\frac{x^2}{\sqrt{(x+2)}}\times 2u=\frac{(u^2-2)^2}{u}\times 2u=2(u^4-4u^2+4).$$

$$\therefore\ y=2\int(u^4-4u^2+4)\,du=2\left(\tfrac{1}{5}u^5-\tfrac{4}{3}u^3+4u\right)=2u\left(\tfrac{1}{5}u^4-\tfrac{4}{3}u^2+4\right)$$
$$=2\sqrt{(x+2)}\left[\tfrac{1}{5}(x+2)^2-\tfrac{4}{3}(x+2)+4\right].$$

(ii) $\displaystyle\int\frac{dx}{3+\sqrt{x}}$.

Here $\dfrac{dy}{dx}=\dfrac{1}{3+\sqrt{x}}$. $\qquad$ Let $x=u^2$; $\qquad\dfrac{dx}{du}=2u$,

and
$$\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}=\frac{1}{3+u}\times 2u.$$

$$\therefore\ y=2\int\frac{u}{3+u}\,du=2\int\left[1-\frac{3}{3+u}\right]du=2[u-3\log(3+u)].$$
$$=2\sqrt{x}-6\log(3+\sqrt{x}).$$

(iii) $\displaystyle\int \frac{dx}{x+\sqrt{(2x-1)}}$.

Here $\displaystyle\frac{dy}{dx}=\frac{1}{x+\sqrt{(2x-1)}}$. Let $2x-1=u^2$.

$$2\frac{dx}{du}=2u, \quad \text{and} \quad x=\tfrac{1}{2}(1+u^2).$$

$$\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}=\frac{1}{\frac{1}{2}(1+u^2)+u}\times u=\frac{2u}{u^2+2u+1}=\frac{2u}{(u+1)^2}.$$

$\therefore \quad \displaystyle y=2\int\frac{u}{(u+1)^2}\,du=2\int\left[\frac{1}{u+1}-\frac{1}{(u+1)^2}\right]du,$ by partial fractions,

$$=2\left[\log(u+1)+\frac{1}{u+1}\right]=2\log[\sqrt{(2x-1)}+1]+\frac{2}{\sqrt{(2x-1)}+1}.$$

Sometimes two substitutions in succession are needed:

(iv) $\displaystyle\int\frac{dx}{(a^2-a}$

$$\frac{}{dx}=\frac{}{(a^2-x^2)^{3/2}}. \quad \text{Let } x=\frac{1}{u}; \quad \frac{dx}{du}$$

$$\frac{dy}{}=\frac{dy}{dx}\cdot\frac{dx}{du}=\frac{}{(a^2-1/u^2)^{3/2}}\times-\frac{1}{u^2}=\frac{}{(a^2u^2-1)^{3/2}},$$

$\therefore \quad \displaystyle y=-\int\frac{u\,du}{(a^2u^2-1)^{3/2}}.$

Now let $a^2u^2=z$; $\therefore \; 2a^2u=dz/du.$

$$y=-\int\frac{\frac{1}{2a^2}\cdot\frac{dz}{du}\cdot du}{(z-1)^{3/2}}=-\frac{1}{2a^2}\int\frac{dz}{(z-1)^{3/2}}=-\frac{1}{2a^2}\int(z-1)^{-3/2}\,dz$$

$$\frac{1}{2a^2}\cdot\frac{(z-1)^{-1/2}}{}=\frac{1}{a^2}\frac{1}{(z-1)^{1/2}}=\frac{1}{a^2}\frac{1}{\sqrt{(a^2u^2-1)}}=\frac{1}{a^2}\frac{}{\sqrt{(a^2-a}}$$

(v) $\displaystyle\int\frac{dx}{x\sqrt{(x^2-a^2)}}$

$\displaystyle\frac{dy}{dx}=\frac{1}{x\sqrt{(x^2-a^2)}}$; making the same substitution as in the preceding example, $\quad x=1/u,\;$ and $\;\therefore\; dx/du=-1/u^2,\;$ we have

$$\frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du}=\frac{1}{1/u\,.\sqrt{(1/u^2-a^2)}}\times-\frac{}{u^2}=\frac{-1}{\sqrt{(1-a^2u^2)}},$$

$$y=-\int\frac{du}{\sqrt{(1-a^2u^2)}}=-\frac{1}{a}\sin^{-1}au=-\frac{1}{a}\sin^{-1}\frac{a}{x}\;\text{ or }\;-\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}.$$

Any expression of the form $\displaystyle\frac{}{(x-k)\sqrt{(ax^2+bx+c)}}$ can be integrated by the substitution used in the preceding example, viz. $x-k=1/u$; the expression is thereby reduced to the form $\displaystyle\frac{1}{\sqrt{(Ax^2+Bx+C)}}$, which has been already considered in Art. 128.

(vi) $\int \frac{dx}{x(ax^n+b)}$.

This may be written $\int \frac{x^{n-1}}{x^n(ax^n+b)}\,dx$, and therefore can be integrated by substituting $x^n = u$, $nx^{n-1} = du/dx$.

The integral becomes $\int \frac{\frac{1}{n}\frac{du}{dx}\,dx}{u(au+b)}$ $\frac{1}{n}\int \frac{1}{b}\left[\frac{1}{u} - \frac{a}{au+b}\right] du$

$$= \frac{1}{nb}[\log u - \log(au+b)] = \frac{1}{nb}\log\frac{x^n}{ax^n+b}.$$

The integral can also be found by substituting $x^n = 1/u$, and the method of finding dx/du should be noticed. Since $x^n = u^{-1}$, it follows that $n\log x = -\log u$; $\therefore$ differentiating with respect to u,

$$n\frac{1}{x}\frac{dx}{du} = -\frac{1}{u}, \text{ and } \frac{dx}{du} = -\frac{x}{nu}.$$

$$\therefore \quad \frac{dy}{du} = \frac{dy}{dx}\times\frac{dx}{du} = \frac{1}{x(a/u+b)}\times -\frac{x}{nu} = -\frac{1}{n(a+bu)}.$$

Whence $\quad y = -\frac{1}{nb}\log(a+bu) = -\frac{1}{nb}\log\left(a+\frac{b}{x^n}\right)$

$$= \frac{1}{nb}\log\frac{x^n}{ax^n+b}, \text{ as before.}$$

Examples XLIX.

Integrate

1. $\frac{x}{\sqrt{(1-x)}}$.
2. $\frac{x^2}{\sqrt{(a-x)}}$.
3. $\frac{\sqrt{(1+x)}}{x}$.
4. $\frac{\sqrt{(x+2)}}{x+6}$.
5. $\frac{1}{\sqrt{x}-1}$.
6. $\frac{\sqrt{x}}{1+\sqrt{x}}$.
7. $\frac{1}{x+\sqrt{(1-x)}}$.
8. $x\sqrt{(x+2)}$.
9. $x^2\sqrt{(ax+b)}$.
10. $\frac{x}{1+\sqrt{x}}$.
11. $\frac{1}{\sqrt{x}(3+x)}$.
12. $\frac{1}{x\sqrt{(a+x)}}$.
13. $\frac{1}{(a^2+x^2)^{3/2}}$.
14. $\frac{1}{(x^2-4)^{3/2}}$.
15. $\frac{1}{x\sqrt{(1+x^2)}}$.
16. $\frac{1}{x\sqrt{(x^2+x+1)}}$.
17. $\frac{1}{x\sqrt{(x^2-1)}}$.
18. $\frac{1}{(x+1)\sqrt{(1+x^2)}}$.
19. $\frac{1}{x\sqrt{(7-5x-2x^2)}}$.
20. $\frac{1}{(x^2+4x+5)^{3/2}}$.
21. $\frac{1}{x^2\sqrt{(1+x)}}$.
22. $\frac{\sqrt{x}}{3+2x}$.
23. $\frac{1}{x\sqrt{(a^2-x^2)}}$.
24. $\frac{3\sqrt{x}}{1-x}$.

25. $\dfrac{1}{x(2x^2+3)}$ 26. $\dfrac{1}{x(x^3+a^3)}$ 27. $\dfrac{1}{x(1-x^n)}$

28. $\dfrac{1}{x(3-2x^4)}$ 29. $\dfrac{1}{x(1+x^2)^2}$ 30. $\dfrac{1}{x(1+x)^{2/3}}$

133. Integration of the circular functions.

We have already, in the list of standard forms,

$$\int \sin x\, dx = -\cos x, \qquad \int \cos x\, dx = \sin x.$$

Also $$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\log \cos x,$$

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \log \sin x \quad \text{(Art. 125).}$$

$$\int \sec x\, dx = \int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{\cos^2 x}\, dx = \int \frac{\cos x}{1-\sin^2 x}\, dx$$

$$= (\text{if } \sin x = u, \text{ and } \therefore \cos x = du/dx) \int \frac{du/dx}{1-u^2}\, dx = \int \frac{du}{1-u^2}$$

$$= \frac{1}{2} \log \frac{1+u}{1-u} = \frac{1}{2} \log \frac{1+\sin x}{1-\sin x}, \text{ which reduces to } \log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

Similarly, $\int \operatorname{cosec} x\, dx$, if $\cos x = u$ and $\therefore \sin x = -\dfrac{du}{dx}$, becomes $\dfrac{1}{2}\log\dfrac{1-\cos x}{1+\cos x}$, which reduces to $\log \tan \dfrac{x}{2}$.

The two latter integrals can also be obtained as follows:

$$\int \operatorname{cosec} x\, dx = \int \frac{1}{\sin x}\, dx = \int \frac{1}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x}\, dx = \int \frac{1}{2 \tan \frac{1}{2}x \cos^2 \frac{1}{2}x}\, dx$$

$$= \int \frac{\frac{1}{2} \sec^2 \frac{1}{2}x}{\tan \frac{1}{2}x}\, dx = \log \tan \tfrac{1}{2}x, \text{ by Art. 125.}$$

Then $\int \sec x\, dx = \int \operatorname{cosec}(\frac{1}{2}\pi + x)\, dx = \log \tan(\frac{1}{4}\pi + \frac{1}{2}x)$.

134. Integration of the squares of the circular functions.

The first two of these occur very frequently, and the results, together with the method of obtaining them, should be carefully noticed.

$$\int \sin^2 x\, dx = \int \tfrac{1}{2}(1-\cos 2x)\, dx = \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x) = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x.$$

$$\int \cos^2 x\, dx = \int \tfrac{1}{2}(1+\cos 2x)\, dx = \tfrac{1}{2}(x + \tfrac{1}{2}\sin 2x) = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x.$$

[Since $\sin^2 x + \cos^2 x = 1$, it follows that the sum of their integrals $= \int 1\, dx = x$, as is obtained by adding the two preceding results.]

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x.$$

$$\int \cot^2 x\, dx = \int (\operatorname{cosec}^2 x - 1)\, dx = -\cot x - x.$$

$$\int \sec^2 x\, dx = \tan x.$$

$$\int \operatorname{cosec}^2 x\, dx = -\cot x.$$

The integral of any function of $\cos x$, $\cot x$, or $\operatorname{cosec} x$ can be deduced from the integral of the corresponding function of $\sin x$, $\tan x$, or $\sec x$ respectively.

E.g. $\int \operatorname{cosec}^2 x dx = \int \sec^2 (\frac{1}{2}\pi + x)\, dx = \tan(\frac{1}{2}\pi + x) = -\cot x.$

135. Further examples of trigonometrical integrals.

A few more examples of trigonometrical integrals, which illustrate some of the various devices which may be adopted, will now be given.

Examples:

(i) $\int \cos^4 x \sin^3 x\, dx.$

Let $\cos x = u$; $\therefore\ -\sin x = du/dx.$

From the $\sin^3 x$, one factor $\sin x$ is taken in order to supply the necessary du/dx, and that leaves $\sin^2 x$, which can be expressed in terms of u (it is equal to $1-u^2$) without introducing irrational expressions.

Hence the integral becomes

$$\int u^4 (1-u^2)\,.\,-\frac{du}{dx}\, dx = -\int (u^4 - u^6)\, du = -\frac{1}{5}u^5 + \frac{1}{7}u^7$$
$$= \tfrac{1}{7}\cos^7 x - \tfrac{1}{5}\cos^5 x.$$

(ii) $\displaystyle\int \frac{\cos^3 x}{\sin^2 x}\, dx.$

In this case, let $\sin x = u$, then $\cos x = du/dx$.

As in the preceding example, the integral now takes the form

$$\int \frac{(1-u^2)}{u^2}\,.\,\frac{du}{dx}\, dx = \int \left(\frac{1}{u^2} - 1\right) du = -\frac{1}{u} - u = -\operatorname{cosec} x - \sin x.$$

The integral $\int \sin^m x \cos^n x\, dx$ can always be obtained as in the last two examples if either m or n (or both) be an odd number. If the index of $\sin x$ be odd, put $\cos x = u$; if the index of $\cos x$ be odd, put $\sin x = u$.

The integral can also be found when $m+n$ is an even negative integer, by the method indicated in the following examples. (See also Art. 141.)

(iii) $\displaystyle\int \frac{\cos^4 x}{\sin^6 x}\, dx = \int \cot^4 x \operatorname{cosec}^2 x\, dx.$

Since $\operatorname{cosec}^2 x = -$ the d. c. of $\cot x$, let $\cot x = u$.

$\therefore$ the integral becomes

$$\int u^4 \left(-\frac{du}{dx}\right) dx = -\int u^4\, du = -\frac{1}{5}u^5 = -\frac{1}{5}\cot^5 x.$$

(iv) $\int \sec^4 x\, dx = \int \sec^2 x \,.\, \sec^2 x\, dx.$

Let $\tan x = u$, $\sec^2 x = du/dx$, the other $\sec^2 x = 1+\tan^2 x = 1+u^2$, and the integral becomes

$$\int (1+u^2)\frac{du}{dx}\,dx = \int (1+u^2)\,du = u+\frac{1}{3}u^3 = \tan x + \frac{1}{3}\tan^3 x.$$

(v) $\displaystyle\int \frac{1}{\sin^2 x\cos^2 x}\,dx = \int \frac{1}{\tan^2 x\cos^4 x}\,dx = \int \frac{\sec^4 x}{\tan^2 x}\,dx$

$\displaystyle= \int \frac{\sec^2 x\,.\,\sec^2 x}{\tan^2 x}\,dx =$ (as in preceding example) $\displaystyle\int \frac{(1+u^2)}{u^2}\cdot\frac{du}{dx}\,dx.$

$= \int (1/u^2+1)\,du = -1/u+u = \tan x - \cot x.$

The product of a sine and a cosine, or of two sines or two cosines, can be integrated at once by expressing it as a sum or difference.

(vi) $\int \sin 2x\cos x\,dx = \int \frac{1}{2}[\sin 3x+\sin x]\,dx = \frac{1}{2}\left(-\frac{1}{3}\cos 3x-\cos x\right).$

(vii) $\int \sin 3x\sin 4x\,dx = \int \frac{1}{2}[\cos x-\cos 7x]\,dx = \frac{1}{2}\left(\sin x-\frac{1}{7}\sin 7x\right).$

It should be noticed that any rational function of $\sin x$ and $\cos x$ can be transformed into a rational algebraical fraction (such as is dealt with in Arts. 122–127) by the substitution $\tan\frac{1}{2}x = u$.

Then $\displaystyle\frac{du}{dx} = \frac{1}{2}\sec^2\tfrac{1}{2}x = \frac{1}{2}(1+u^2)$, and $\displaystyle\frac{dx}{du} = \frac{2}{1+u^2}.$

$$\sin x = 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x = \frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} = \frac{2u}{1+u^2}.$$

$$\cos x = \cos^2\tfrac{1}{2}x-\sin^2\tfrac{1}{2}x = \frac{1-\tan^2\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} = \frac{1-u^2}{1+u^2}.$$

The integrals of Arts. 133 and 134 are all included in this case, although there some of them were obtained by simpler methods. Two other examples are here given.

(viii) $\displaystyle\int \frac{dx}{5+4\cos x}.$

Denoting it by y, we have $\displaystyle\frac{dy}{dx} = \frac{1}{5+4\cos x}.$

$$\frac{dy}{du} = \frac{dy}{dx}\times\frac{dx}{du} = \frac{1}{5+4(1-u^2)/(1+u^2)}\cdot\frac{2}{1+u^2},$$

(making the substitutions just mentioned),

$$= \frac{2}{5+5u^2+4-4u^2} = \frac{2}{9+u^2}.$$

$$\therefore\ y = \int\frac{2}{9+u^2}\,du = \frac{2}{3}\tan^{-1}\frac{u}{3} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right).$$

(ix) $\displaystyle\int \frac{dx}{12+13\sin x}.$

Denoting it by y, $\displaystyle\frac{dy}{dx} = \frac{1}{12+13\sin x},$

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = \frac{1}{12 + 13 \cdot 2u/(1+u^2)} \times \frac{2}{1+u^2} = \frac{2}{12 + 12u^2 + 26u},$$

$$\therefore \quad y = \int \frac{du}{6u^2 + 13u + 6} = \int \frac{du}{(3u+2)(2u+3)} = \frac{1}{5} \int \left(\frac{3}{3u+2} - \frac{2}{2u+3} \right) du$$

(by partial fractions)

$$= \frac{1}{5}\Big[\log(3u+2) - \log(2u+3)\Big] = \frac{1}{5} \log \frac{3\tan\frac{1}{2}x + 2}{2\tan\frac{1}{2}x + 3}.$$

Examples L.

Integrate

1. $\tan 2x$.
2. $\cot mx$.
3. $\sec \frac{1}{2} x$.
4. $\operatorname{cosec} 3x$.
5. $\operatorname{cosec} (x/a)$.
6. $\tan^2 \frac{1}{3} x$.
7. $\operatorname{cosec}^2 nx$.
8. $\sin^3 x$.
9. $\cos^3 x$.
10. $\sin^3 x \cos^4 x$.
11. $\sin^n x \cos^3 x$.
12. $\sec^6 x$.
13. $\dfrac{\sin^3 x}{\cos^4 x}$.
14. $\dfrac{\cos^5 x}{\sin^2 x}$.
15. $\dfrac{\sin^2 x}{\cos^4 x}$.
16. $\cos^4 x$.
17. $\tan^4 x$.
18. $\operatorname{cosec}^4 x$.
19. $\tan^6 x$.
20. $\sec x \operatorname{cosec} x$.
21. $\tan^3 x$.
22. $\sec^3 x \operatorname{cosec} x$.
23. $\sec^2 x \operatorname{cosec}^2 x$.
24. $\cot^5 x$.
25. $\sin^2 x \cos^2 x$.
26. $\sin^8 x / \cos^{10} x$.
27. $\sin^4 x$.
28. $\dfrac{\cos^2 x}{\sin^6 x}$.
29. $\dfrac{1}{1+\cos x}$.
30. $\dfrac{1}{1-\cos x}$.
31. $\dfrac{1}{1+\sin x}$.
32. $\dfrac{1}{1-\sin x}$.
33. $\dfrac{1}{\cos x \sin^2 x}$
34. $\sin 4x \cos x$.
35. $\cos 2x \cos 3x$.
36. $\sin mx \cos nx$.
37. $\sin px \sin qx$.
38. $\sin^2 x \cos 3x$.
39. $\cos^2 x \sin mx$.
40. $\dfrac{\cos^2 x}{\cos 2x}$.
41. $\dfrac{\sin^2 x}{\sin 2x}$.
42. $\dfrac{1}{4 + 5\cos x}$.
43. $\dfrac{1}{13 + 85 \sin x}$.
44. $\dfrac{1}{1 + 8\cos^2 x}$
45. $\dfrac{1}{25 - 24 \sin^2 x}$.
46. $\dfrac{1}{4\cos^2 x + 9 \sin^2 x}$.
47. $\dfrac{1}{1 + \tan x}$.
48. $\dfrac{1}{\sin^7 x \cos^5 x}$.
49. $\dfrac{1}{2 + \sin x}$.
50. $\dfrac{1}{5 - 3\cos x}$

136. Trigonometrical substitutions.

Many algebraical functions which involve the square root of a quadratic expression can be rationalized by a trigonometrical substitution, and their integration is often thereby simplified. E.g. if an expression involves the irrational quantity $\sqrt{(a^2 - x^2)}$, the substitution of $a \sin\theta$ for x changes $\sqrt{(a^2 - x^2)}$ into $\sqrt{[a^2(1 - \sin^2\theta)]}$, i.e. $a\cos\theta$.

The substitution $x = a\cos\theta$ would of course serve equally well. These are legitimate substitutions, because $a\sin\theta$ and $a\cos\theta$ can have all values from $-a$ to $+a$ inclusive, and these constitute all the values of x for which $\sqrt{(a^2-x^2)}$ is real.

Examples:

(i) $\int\sqrt{(a^2-x^2)}\,dx$. Let $x = a\sin\theta$; $\therefore$ $dx/d\theta = a\cos\theta$.

Denoting the integral by y,

$$\frac{dy}{dx} = \sqrt{(a^2-x^2)};\quad \frac{dy}{d\theta} = \frac{dy}{dx}\times\frac{dx}{d\theta} = a\cos\theta\,.\,a\cos\theta = a^2\cos^2\theta.$$

$$\therefore\quad y = a^2\int\cos^2\theta\,d\theta = \tfrac{1}{2}a^2\int(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}a^2(\theta+\tfrac{1}{2}\sin 2\theta)$$

$$= \tfrac{1}{2}a^2\theta + \tfrac{1}{2}a^2\sin\theta\cos\theta = \tfrac{1}{2}a^2\sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{(a^2-x^2)}.$$

(ii) $\displaystyle\int\frac{\sqrt{(a^2-x^2)}}{x^2}\,dx$. Let $x = a\cos\theta$; $\therefore$ $\dfrac{dx}{d\theta} = -a\sin\theta$.

$$\frac{dy}{d\theta} = \frac{dy}{dx}\cdot\frac{dx}{d\theta} = \frac{\sqrt{(a^2-x^2)}}{x^2}\times -a\sin\theta = \frac{a\sin\theta}{a^2\cos^2\theta}\times -a\sin\theta = -\tan^2\theta.$$

$$\therefore\quad y = -\int\tan^2\theta\,.\,d\theta = -\int(\sec^2\theta-1)\,d\theta = -\tan\theta+\theta$$

$$= \cos^{-1}(x/a) - \sqrt{(a^2-x^2)}/x.$$

Similarly, an expression which involves $\sqrt{(a^2+x^2)}$ is rationalized by the substitution $x = a\tan\theta$, which makes $\sqrt{(a^2+x^2)}$ into $\sqrt{[a^2(1+\tan^2\theta)]}$, i.e. $a\sec\theta$. The hyperbolic substitution $x = a\sinh u$ will do equally well in this case: it changes $\sqrt{(a^2+x^2)}$ into $\sqrt{[a^2(1+\sinh^2 u)]}$, i.e. $a\cosh u$ [Art. 92], and may be used if the student is well acquainted with the simpler relations between these functions. Some of these relations are required in the integration, and in restoring the x after integration.

Again, an expression which involves $\sqrt{(x^2-a^2)}$ may be rationalized by putting $x = a\sec\theta$, which makes $\sqrt{(x^2-a^2)}$ into $\sqrt{[a^2(\sec^2\theta-1)]}$, i.e. $a\tan\theta$. The hyperbolic substitution $x = a\cosh u$ will also serve equally well, for it changes $\sqrt{(x^2-a^2)}$ into $\sqrt{[a^2(\cosh^2 u-1)]}$, i.e. $a\sinh u$.

Examples:

(i) $\displaystyle\int\frac{dx}{x^2\sqrt{(4+x^2)}}$. Let $x = 2\tan\theta$; $\therefore$ $\dfrac{dx}{d\theta} = 2\sec^2\theta$.

Denoting the integral by y,

$$\frac{dy}{d\theta} = \frac{dy}{dx}\times\frac{dx}{d\theta} = \frac{1}{x^2\sqrt{(4+x^2)}}\times 2\sec^2\theta = \frac{2\sec^2\theta}{4\tan^2\theta\,.\,2\sec\theta} = \frac{\cos\theta}{4\sin^2\theta}.$$

$\therefore$ $y = \frac{1}{4}\displaystyle\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$, which is found by putting $u = \sin\theta$, $\dfrac{du}{d\theta} = \cos\theta$,

and becomes $$y = \frac{1}{4}\,\frac{-1}{\sin\theta} = -\frac{\sqrt{(x^2+4)}}{4x}.$$

(ii) $\int \sqrt{(x^2+a^2)}\,dx$. Let $x = a\sinh u$; $\therefore \dfrac{dx}{du} = a\cosh u$.

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = \sqrt{(x^2+a^2)} \times a\cosh u = a\cosh u \times a\cosh u$$
$$= a^2\cosh^2 u = \tfrac{1}{2}a^2(1+\cosh 2u).$$

$$y = \tfrac{1}{2}a^2\int(1+\cosh 2u)\,du = \tfrac{1}{2}a^2\,(u+\tfrac{1}{2}\sinh 2u)$$
$$= \tfrac{1}{2}a^2 u + \tfrac{1}{2}a^2\sinh u\cosh u \quad \text{(using the results of Ex. XXXII. 12)}$$
$$= \tfrac{1}{2}a^2\sinh^{-1}(x/a) + \tfrac{1}{2}x\sqrt{(a^2+x^2)}.$$

It should be noticed that the values of the standard integrals

$$\int\frac{dx}{\sqrt{(a^2-x^2)}} \text{ and } \int\frac{dx}{a^2+x^2}$$

can be worked out by this method, by substituting $x = a\sin\theta$ and $x = a\tan\theta$ respectively.

The substitution x (or $x-k$) $= a\tan\theta$ is often useful in dealing with certain types of rational expressions.

E.g. to find $\displaystyle\int\frac{dx}{(x^2-2x+5)^2}$, we may write x^2-2x+5 in the form $(x-1)^2+4$ which, if $x-1 = 2\tan\theta$, becomes $4\tan^2\theta+4$, i.e. $4\sec^2\theta$.

Denoting the integral by y, we have

$$\frac{dy}{d\theta} = \frac{dy}{dx}\cdot\frac{dx}{d\theta} = \frac{1}{(x^2-2x+5)^2} \times 2\sec^2\theta = \frac{2\sec^2\theta}{(4\sec^2\theta)^2} = \tfrac{1}{8}\cos^2\theta = \tfrac{1}{16}(1+\cos 2\theta);$$

$$\therefore \quad y = \tfrac{1}{16}\int(1+\cos 2\theta)\,d\theta = \tfrac{1}{16}(\theta+\tfrac{1}{2}\sin 2\theta) = \tfrac{1}{16}(\theta+\sin\theta\cos\theta)$$
$$= \frac{1}{16}\left[\tan^{-1}\frac{x-1}{2} + \frac{x-1}{\sqrt{[(x-1)^2+4]}}\;\frac{2}{\sqrt{[(x-1)^2+4]}}\right]$$
$$= \frac{1}{16}\left[\tan^{-1}\frac{x-1}{2} + \frac{2(x-1)}{x^2-2x+5}\right].$$

137. A useful substitution.

It should be noticed that the expressions $\sqrt{[(x-\alpha)(\beta-x)]}$, $1/\sqrt{[(x-\alpha)(\beta-x)]}$, and $\sqrt{[(x-\alpha)/(\beta-x)]}$, where $\beta > \alpha$, are all rationalized by the substitution $x = \alpha\cos^2\theta + \beta\sin^2\theta$.

This expression admits of all values from α to β inclusive, and it is just for these values and these values only that the preceding expressions are real.

If this substitution be made,

$x-\alpha$ becomes $\alpha(\cos^2\theta-1)+\beta\sin^2\theta$, i.e. $(\beta-\alpha)\sin^2\theta$.

$\beta-x$ becomes $\beta(1-\sin^2\theta)-\alpha\cos^2\theta$, i.e. $(\beta-\alpha)\cos^2\theta$.

$dx/d\theta = \alpha\,2\cos\theta\,(-\sin\theta)+\beta\,2\sin\theta\cos\theta$, i.e. $2(\beta-\alpha)\sin\theta\cos\theta$.

Two examples are here given.

Examples:

(i) $\int \sqrt{\dfrac{x-a}{2a-x}}\,dx.$

Let $x = a\cos^2\theta + 2a\sin^2\theta$; $\therefore$ $x-a = a\sin^2\theta$, $2a-x = a\cos^2\theta$,

and $dx/d\theta = -2a\sin\theta\cos\theta + 4a\sin\theta\cos\theta = 2a\sin\theta\cos\theta.$

Hence, denoting the integral by y as usual,

$$\frac{dy}{dx} = \sqrt{\frac{x-a}{2a-x}} = \sqrt{\frac{a\sin^2\theta}{a\cos^2\theta}} = \tan\theta;$$

$$\frac{dy}{d\theta} = \frac{dy}{dx} \times \frac{dx}{d\theta} = \tan\theta \,.\, 2a\sin\theta\cos\theta = 2a\sin^2\theta = a(1-\cos 2\theta);$$

$\therefore$ $y = a\int(1-\cos 2\theta)\,d\theta = a(\theta - \frac{1}{2}\sin 2\theta) = a\theta - a\sin\theta\cos\theta.$

Since $x-a = a\sin^2\theta$, $\sin\theta = \sqrt{\dfrac{x-a}{a}}$, and $\theta = \sin^{-1}\sqrt{\dfrac{x-a}{a}}.$

Also $2a-x = a\cos^2\theta$, $\therefore$ $\cos\theta = \sqrt{\dfrac{2a-x}{a}}.$

$$\therefore\ y = a\sin^{-1}\sqrt{\frac{x-a}{a}} - a\,.\sqrt{\frac{x-a}{a}}\,.\sqrt{\frac{2a-x}{a}}$$

$$= a\sin^{-1}\sqrt{\frac{x-a}{a}} - \sqrt{[(x-a)(2a-x)]}.$$

(ii) $\int\sqrt{(7x-10-x^2)}\,dx$, i.e. $\int\sqrt{[(5-x)(x-2)]}\,dx.$

Let $x = 2\cos^2\theta + 5\sin^2\theta$; $\therefore$ $x-2 = 3\sin^2\theta$, $5-x = 3\cos^2\theta$,

$dx/d\theta = 6\sin\theta\cos\theta.$

$$dy/dx = \sqrt{[(5-x)(x-2)]} = \sqrt{(3\cos^2\theta\,.\,3\sin^2\theta)} = 3\sin\theta\cos\theta.$$

$$\therefore\ \frac{dy}{d\theta} = \frac{dy}{dx} \times \frac{dx}{d\theta} = 3\sin\theta\cos\theta \times 6\sin\theta\cos\theta = 18\sin^2\theta\cos^2\theta;$$

and
$$y = \int 18\sin^2\theta\cos^2\theta\,d\theta = \tfrac{9}{2}\int\sin^2 2\theta\,d\theta = \tfrac{9}{4}\int(1-\cos 4\theta)\,d\theta$$
$$= \tfrac{9}{4}(\theta - \tfrac{1}{4}\sin 4\theta) = \tfrac{9}{4}\theta - \tfrac{9}{16}\,.\,2\sin 2\theta\cos 2\theta$$
$$= \tfrac{9}{4}\theta - \tfrac{9}{4}\sin\theta\cos\theta\,(2\cos^2\theta - 1)$$
$$= \frac{9}{4}\sin^{-1}\sqrt{\frac{x-2}{3}} - \frac{9}{4}\sqrt{\frac{x-2}{3}}\,.\sqrt{\left(\frac{5-x}{3}\right)}\left(2\,.\,\frac{5-x}{3} - 1\right)$$
$$= \tfrac{9}{4}\sin^{-1}\sqrt{\{\tfrac{1}{3}(x-2)\}} - \tfrac{1}{4}(7-2x)\sqrt{[(5-x)(x-2)]}.$$

Examples LI.

Integrate:

1. $\sqrt{(9-x^2)}$.
2. $\sqrt{(x^2-a^2)}$.
3. $\sqrt{(x^2+1)}$.
4. $\sqrt{(x^2-4)}$.
5. $\dfrac{\sqrt{(25-x^2)}}{x^2}$.
6. $\dfrac{1}{x^2\sqrt{(1-x^2)}}$.
7. $\dfrac{\sqrt{(1+x^2)}}{x^2}$.
8. $\dfrac{x^2}{\sqrt{(a^2-x^2)}}$.
9. $\dfrac{x^2}{\sqrt{(x^2+9)}}$.
10. $\dfrac{x^2}{\sqrt{(x^2-a^2)}}$.
11. $\dfrac{a^2+x^2}{\sqrt{(a^2-x^2)}}$.
12. $\dfrac{1}{x^2\sqrt{(a^2+x^2)}}$.

13. $\dfrac{x^3}{\sqrt{(1-x^2)}}$.

15. $\dfrac{}{(a^2-x^2)^{3/2}}$.

16. $\dfrac{x^2}{(1-x^2)^{3/2}}$.

17. $\sqrt{\left(\dfrac{x-1}{2-x}\right)}$.

18. $\sqrt{\left(\dfrac{5-x}{x-2}\right)}$.

19. $\sqrt{[(x-3)(7-x)]}$.

20. $\sqrt{[(x+1)(4-x)]}$.

21. $\dfrac{1}{\sqrt{[(x+2)(7-x)]}}$.

22. $\dfrac{}{\sqrt{[(x-\alpha)(\beta-x)]}}$.

23. $\sqrt{[(x-\alpha)(\beta-x)]}$.

24. $\sqrt{[(x-2a)(6a-x)]}$.

25. $\sqrt{\left(\dfrac{x-\alpha}{\beta-x}\right)}$.

26. $\sqrt{\left(\dfrac{4-x}{x+4}\right)}$.

27. $\sqrt{\left(\dfrac{\alpha-x}{x-\beta}\right)}$.

28. $\dfrac{1}{(x^2+1)^2}$.

29. $\dfrac{1}{(x^2+4x+5)^2}$.

30. $\dfrac{1}{(2x^2-6x+45)^2}$.

31. $\dfrac{x}{(x^2+2x+2)^2}$.

32. $\dfrac{x^2}{(x^2+1)^2}$.

138. Integration by parts.

There remains one more important elementary method of integration, known as 'integration by parts'. This is the converse of the rule for finding the differential coefficient of a product of two functions of x.

We have
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx};$$
therefore, integrating each term, we have (save for an arbitrary constant to be added)
$$= \int u\frac{}{dx}\,dx + \int v\frac{du}{dx}\,dx,$$
whence
$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

The integral on the right-hand side is frequently much easier to evaluate than the one on the left. The method is particularly valuable in many cases when the expression to be integrated contains such functions as $\log x$, or an inverse trigonometrical or hyperbolic function. If such a function be taken as the 'u' in the integral on the left-hand side, the du/dx on the right-hand side becomes a simple algebraical function.

Examples:

(i) $\int x^2 \log x\,dx$.

Take $u = \log x$, $dv/dx = x^2$; $\therefore$ $du/dx = 1/x$, $v = \int x^2\,dx = \frac{1}{3}x^3$.

We have
$$\begin{aligned}\int x^2 \log x\,dx &= \log x \times \tfrac{1}{3}x^3 - \int \tfrac{1}{3}x^3 \,.\, x^{-1}\,dx\\ &= \tfrac{1}{3}x^3\log x - \tfrac{1}{3}\int x^2\,dx\\ &= \tfrac{1}{3}x^3\log x - \tfrac{1}{9}x^3.\end{aligned}$$

(ii) $\int x \tan^{-1} x \, dx$.

Take $u = \tan^{-1} x$, $dv/dx = x$; $\therefore$ $du/dx = 1/(1+x^2)$; $v = \frac{1}{2}x^2$.

$$\int x \tan^{-1} x \, dx = \tfrac{1}{2} x^2 \tan^{-1} x - \int \tfrac{1}{2} x^2 \cdot \frac{1}{1+x^2}\, dx$$

$$= \tfrac{1}{2} x^2 \tan^{-1} x - \tfrac{1}{2} \int \left[1 - \frac{1}{1+x^2}\right] dx$$

$$= \tfrac{1}{2} x^2 \tan^{-1} x - \tfrac{1}{2}[x - \tan^{-1} x]$$

$$= \tfrac{1}{2}(x^2+1)\tan^{-1} x - \tfrac{1}{2} x.$$

(iii) $\int \tan^{-1} x \, dx$.

In this case, take $u = \tan^{-1} x$, $dv/dx = 1$;

$$\therefore \quad du/dx = 1/(1+x^2), \quad v = x.$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx = x \tan^{-1} x - \tfrac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \tan^{-1} x - \tfrac{1}{2} \log(1+x^2). \qquad \text{[Art. 125.]}$$

(iv) $\int x \sin x \, dx$.

In this case, if $\sin x$ be taken as u and x as dv/dx, it will be seen that du/dx and v are respectively $\cos x$ and $\frac{1}{2}x^2$, and therefore the integral on the right-hand side, $\int \frac{1}{2} x^2 \cos x \, dx$, is more complicated than the one we started with. Hence take $u = x$, $dv/dx = \sin x$;

$$\therefore \quad du/dx = 1, \quad v = -\cos x.$$

We have therefore

$$\int x \sin x \, dx = -x \cos x - \int(-\cos x)\, dx = -x \cos x + \sin x.$$

(v) $\int x^2 e^{2x} dx$.

In this case (since x^2 becomes simpler when differentiated, and e^{2x} does not become more complicated when integrated)

$$\text{let } u = x^2, \quad dv/dx = e^{2x}; \quad \therefore \quad du/dx = 2x, \quad v = \tfrac{1}{2} e^{2x};$$

$$\therefore \quad \int x^2 e^{2x} dx = \tfrac{1}{2} x^2 e^{2x} - \int \tfrac{1}{2} e^{2x} \times 2x \, dx = \tfrac{1}{2} x^2 e^{2x} - \int x e^{2x} dx.$$

The integral on the right-hand side cannot yet be written down at once, but it is simpler than the one we started with. Integrate it by parts again, taking

$$u = x, \quad dv/dx = e^{2x}; \quad \therefore \quad du/dx = 1, \quad v = \tfrac{1}{2} e^{2x};$$

$$\therefore \quad \int x e^{2x} dx = \tfrac{1}{2} x e^{2x} - \int \tfrac{1}{2} e^{2x} dx = \tfrac{1}{2} x e^{2x} - \tfrac{1}{4} e^{2x}.$$

$\therefore$ substituting in the preceding result, the given integral

$$\int x^2 e^{2x} dx = \tfrac{1}{2} x^2 e^{2x} - [\tfrac{1}{2} x e^{2x} - \tfrac{1}{4} e^{2x}] = \tfrac{1}{4} e^{2x}(2x^2 - 2x + 1).$$

This is a very simple case of a general method known as integration by 'successive reduction'. Many expressions can only be integrated by stages in this manner, the integral obtained at the end of each stage being simpler than the integral at the beginning of the stage, until finally an integral is arrived at whose value is known. Further examples of this method are considered in Art. 140.

Examples LII.

Integrate:

1. $x^4 \log x$.	2. $\sqrt{x} \log x$.	3. $x^m \log x$.	4. $(\log x)/x^3$.
5. $x \cos x$.	6. $x \sin mx$.	7. xe^x.	8. xe^{-ax}.
9. $x^3 \tan^{-1} x$.	10. $x^2 \tan^{-1} x$.	11. $\sin^{-1} x$.	12. $\log x$.
13. $x \sec^2 x$.	14. $x \operatorname{cosec}^2 mx$.	15. $x \sin^{-1} x$.	16. $x \sinh x$.
17. $x \cosh (x/a)$.	18. $\sinh^{-1} x$.	19. $\cosh^{-1} x$.	20. $x^2 \sin x$.
21. $x^2 \cos \frac{1}{2} x$.	22. $x^3 e^x$.	23. $x^2 e^{-x}$.	24. $x^2 \sin 2x$.

139. Two important types.

There are two important types of integrals which can be evaluated by this method.

I. $\int \sqrt{(ax^2+bx+c)}\, dx$.

Beginning with the simpler form $\int \sqrt{(x^2+a^2)}\, dx$, and integrating by parts, take $u = \sqrt{(x^2+a^2)}$, $dv/dx = 1$;

then $$du/dx = x/\sqrt{(x^2+a^2)}, \qquad v = x.$$

$$\therefore \quad \int \sqrt{(x^2+a^2)}\, dx = x\sqrt{(x^2+a^2)} - \int \frac{x^2}{\sqrt{(x^2+a^2)}}\, dx$$

$$= x\sqrt{(x^2+a^2)} - \int \frac{(x^2+a^2)-a^2}{\sqrt{(x^2+a^2)}}\, dx$$

$$= x\sqrt{(x^2+a^2)} - \int \sqrt{(x^2+a^2)}\, dx + \int \frac{a^2}{\sqrt{(x^2+a^2)}}\, dx.$$

The second term on the right is the integral we started with; therefore, transferring it to the left-hand side, we have

$$2\int \sqrt{(x^2+a^2)}\, dx = x\sqrt{(x^2+a^2)} + a^2 \int \frac{dx}{\sqrt{(x^2+a^2)}}$$

$$= x\sqrt{(x^2+a^2)} + a^2 \sinh^{-1}(x/a);$$

$$\therefore \quad \int \sqrt{(x^2+a^2)}\, dx = \tfrac{1}{2} x\sqrt{(x^2+a^2)} + \tfrac{1}{2} a^2 \sinh^{-1}(x/a). \tag{i}$$

Similarly $$\int \sqrt{(a^2-x^2)}\, dx = \tfrac{1}{2} x\sqrt{(a^2-x^2)} + \tfrac{1}{2} a^2 \sin^{-1}(x/a).$$

$$\int \sqrt{(x^2-a^2)}\, dx = \tfrac{1}{2} x\sqrt{(x^2-a^2)} - \tfrac{1}{2} a^2 \cosh^{-1}(x/a).$$

Notice that, in the second line of the working as above, the numerator x^2 is always written as the sum or difference of a^2 and the expression under the radical sign in the denominator.

In the general case $\int \sqrt{(ax^2+bx+c)}\, dx$, if we divide the expression under the root sign by $|a|$ and complete the square of the terms which contain x, the integral reduces to one or other of the three forms mentioned above, and therefore can be evaluated.

E.g. $\int\sqrt{(2x^2+6x+5)}\,dx=\sqrt{2}\int\sqrt{(x^2+3x+\frac{5}{2})}\,dx=\sqrt{2}\int\sqrt{[(x+\frac{3}{2})^2+\frac{1}{4}]}\,dx$, which is the case worked out in full above with x and a replaced by $x+\frac{3}{2}$ and $\frac{1}{2}$ respectively; therefore from (i) the required integral

$$=\tfrac{1}{2}\sqrt{2}\,(x+\tfrac{3}{2})\sqrt{[(x+\tfrac{3}{2})^2+\tfrac{1}{4}]}+\tfrac{1}{2}\sqrt{2}\,.\,\tfrac{1}{4}\sinh^{-1}\frac{x+\frac{3}{2}}{\frac{1}{2}}$$
$$=\tfrac{1}{4}(2x+3)\sqrt{(2x^2+6x+5)}+\tfrac{1}{8}\sqrt{2}\,.\sinh^{-1}(2x+3).$$

It is of course not desirable to attempt to remember results such as (i), but in practice it is most convenient to go through the working for the simpler case as given by (i), and make the substitutions in the result as we have done in the example immediately preceding.

II. $\int e^{ax}\cos bx\,dx$ and $\int e^{ax}\sin bx\,dx$.

These integrals are of importance in the theory of electric currents.

If each integral is evaluated by parts, the other one is obtained, and therefore we obtain two equations to solve for the two integrals.

Starting with the first integral, and taking

$$u=e^{ax},\ dv/dx=\cos bx;\ \text{and}\ \therefore\ du/dx=ae^{ax},\ v=(1/b)\sin bx,$$

we have $\int e^{ax}\cos bx\,dx=(1/b)\,e^{ax}\sin bx-(a/b)\int e^{ax}\sin bx\,dx.$ (i)

Similarly, taking the second integral and again substituting

$$u=e^{ax},\ dv/dx=\sin bx,\ \therefore\ du/dx=ae^{ax},\ v=-(1/b)\cos bx,$$

we get $\int e^{ax}\sin bx\,dx=-(1/b)\,e^{ax}\cos bx+(a/b)\int e^{ax}\cos bx\,dx.$ (ii)

If the value of the former integral be required, we substitute the result (ii) in the last term of (i); if the latter integral be the one whose value is required, we substitute the result (i) in the last term of (ii).

In the former case, we get

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}\sin bx}{b}-\frac{a}{b}\left[-\frac{e^{ax}\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bx\,dx\right]$$
$$=\frac{e^{ax}\sin bx}{b}+\frac{a}{b^2}e^{ax}\cos bx-\frac{a^2}{b^2}\int e^{ax}\cos bx\,dx,$$

whence $$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx=\frac{be^{ax}\sin bx+ae^{ax}\cos bx}{b^2},$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}(b\sin bx+a\cos bx)}{a^2+b^2}.$$

Similarly, $$\int e^{ax}\sin bx\,dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}.$$

Examples LIII.

Integrate:

1. $\sqrt{(x^2-a^2)}$.
2. $\sqrt{(a^2-x^2)}$.
3. $\sqrt{(32+2x^2)}$.
4. $\sqrt{(12-3x^2)}$.
5. $\sqrt{(x^2+2x+5)}$.
6. $\sqrt{(6-5x-x^2)}$.
7. $\sqrt{(3x^2+4x-7)}$.
8. $\sqrt{(8-5x-3x^2)}$.
9. $\sqrt{[x(3x-2)]}$.

10. $\sqrt{[x(5-4x)]}$.	11. $e^{3x}\cos 2x$.	12. $e^{2x}\sin 5x$.
13. $e^{-x}\cos\frac{1}{2}x$.	14. $e^{-ax}\sin ax$.	15. $e^{x}\cos^2 x$.
16. $e^{2x}\sin^2 x$.	17. $\cosh x \sin x$.	18. $\sinh x \cos x$.
19. $\sinh x \sin x$.	20. $e^{-Rt/L}\sin pt$.	21. $e^{-Rt/L}\cos(pt+\epsilon)$.

140. Integration by successive reduction.

A large number of expressions can be integrated only by the method of *successive reduction*, which consists in making the integral depend upon a simpler integral, then again reducing this to one simpler still, and so on until a known form is obtained, as shown in the following examples.

Examples:

(i) $\int x^n e^{ax}\,dx$.

Integrate by parts, taking $u = x^n$, $dv/dx = e^{ax}$;

$$\therefore\quad du/dx = nx^{n-1},\quad \text{and}\quad v = e^{ax}/a.$$

$$\int x^n e^{ax}\,dx = x^n e^{ax}/a - (n/a)\int x^{n-1} e^{ax}\,dx, \tag{i}$$

an integral of the same form as the given integral, but in which the index of x is reduced by unity. By repeating the process, changing n into $n-1$, the integral is made to depend upon $\int x^{n-2} e^{ax}\,dx$, and so on until finally $\int e^{ax}\,dx$, which is e^{ax}/a, is reached. Of course the actual process of integration by parts has only to be carried out once for the general case, and then all the successive steps follow by substituting numerical values for n. Equation (i) gives the 'reduction formula' for the given integral.

Taking the particular case, $n = 4$, $a = 2$, we have

$$\begin{aligned}\int x^4 e^{2x}\,dx &= \tfrac{1}{2}x^4 e^{2x} - \tfrac{4}{2}\int x^3 e^{2x}\,dx, \quad \text{putting } n=4 \text{ in (i)},\\ &= \tfrac{1}{2}x^4 e^{2x} - 2\left[\tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\int x^2 e^{2x}\,dx\right], \quad \text{putting } n=3 \text{ in (i)},\\ &= \tfrac{1}{2}x^4 e^{2x} - x^3 e^{2x} + 3\left[\tfrac{1}{2}x^2 e^{2x} - \tfrac{2}{2}\int xe^{2x}\,dx\right], \quad \text{putting } n=2 \text{ in (i)},\\ &= \tfrac{1}{2}x^4 e^{2x} - x^3 e^{2x} + \tfrac{3}{2}x^2 e^{2x} - 3\left[\tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\,dx\right], \text{putting } n=1 \text{ in (i)},\\ &= \tfrac{1}{2}x^4 e^{2x} - x^3 e^{2x} + \tfrac{3}{2}x^2 e^{2x} - \tfrac{3}{2}xe^{2x} + \tfrac{3}{2}\cdot\tfrac{1}{2}e^{2x}\\ &= \tfrac{1}{4}e^{2x}\left[2x^4 - 4x^3 + 6x^2 - 6x + 3\right].\end{aligned}$$

(ii) $\int x^n \cos ax\,dx$.

Integrate by parts, taking $u = x^n$, $dv/dx = \cos ax$;

$$\therefore\quad du/dx = nx^{n-1},\quad v = (1/a)\sin ax.$$

$$\int x^n \cos ax\,dx = \frac{x}{a}\sin ax - \frac{n}{a}\int x^{n-1}\sin ax\,dx.$$

Similarly $\displaystyle\int x^{n-1}\sin ax\,dx = -\frac{x}{a}\cos ax + \frac{n-1}{a}\int x^{n-2}\cos ax\,dx.$

Each step reduces the index of x by unity, and the trigonometrical factors are $\sin ax$ and $\cos ax$ alternately; the process is continued until finally the integral reduces to either $\int\cos ax\,dx$ (if n be even) or $\int\sin ax\,dx$ (if n be odd).

In the same way $\int x^n \sin ax\,dx$, $x^n \sinh ax\,dx$, and $\int x^n \cosh ax\,dx$ may be found.

141. Evaluation of $\int \sin^m\theta \cos^n\theta \, d\theta$.

This is an integral of frequent occurrence. It has already been mentioned (Art. 135) that if m be odd, the integration is at once effected by taking $\cos\theta = u$, and if n be odd, by taking $\sin\theta = u$, and also that the integral can be found when $m+n$ is an even negative integer. Several examples of the latter case were given in that article. The integration can always be effected in this case by substituting $\tan\theta = u$;

$$\theta = \tan^{-1} u, \quad \frac{d\theta}{du} = \frac{1}{1+u^2}, \quad \sin\theta = \frac{u}{\sqrt{(1+u^2)}}, \quad \cos\theta = \frac{1}{\sqrt{(1+u^2)}}.$$

E.g. if $y = \int \dfrac{d\theta}{\sin^2\theta\cos^2\theta}$, $\dfrac{dy}{d\theta} = \dfrac{1}{\sin^2\theta\cos^2\theta} = \dfrac{(1+u^2)^2}{u^2}$,

$$\frac{dy}{du} = \frac{dy}{d\theta}\cdot\frac{d\theta}{du} = \frac{(1+u^2)^2}{u^2}\cdot\frac{1}{1+u^2} = \frac{1+u^2}{u^2} = \frac{1}{u^2}+1;$$

$$y = \int(1/u^2+1)\,du = -1/u + u = -\cot\theta + \tan\theta.$$

Again, if $y = \int \dfrac{d\theta}{\cos^6\theta}$, $\dfrac{dy}{d\theta} = \dfrac{1}{\cos^6\theta} = (1+u^2)^3$;

$$\frac{dy}{du} = \frac{dy}{d\theta}\cdot\frac{d\theta}{du} = (1+u^2)^3 \cdot \frac{1}{1+u^2} = (1+u^2)^2 = 1+2u^2+u^4.$$

$$y = u + \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5 = \tan\theta + \tfrac{2}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta.$$

Generally, $\displaystyle\int \frac{\sin^m\theta}{\cos^{m+2p}\theta}\,d\theta$ [in which the sum of the indices is $-2p$]

$$= \int \tan^m\theta \sec^{2p}\theta \, d\theta$$
$$= \int \tan^m\theta \,(1+\tan^2\theta)^{p-1}\sec^2\theta \, d\theta,$$

which, on substituting $\tan\theta = u$, becomes

$$\int u^m (1+u^2)^{p-1}\,du.$$

This can be expanded by the Binomial Theorem and integrated at once if p be a positive integer.

If, in the given integral, $n = -m$ [m positive], so that the integral becomes $\int \tan^m\theta\,d\theta$, we may proceed as follows:

$$\int \tan^m\theta\,d\theta = \int \tan^{m-2}\theta\,.\tan^2\theta\,d\theta = \int \tan^{m-2}\theta\,(\sec^2\theta - 1)\,d\theta$$
$$= \int \tan^{m-2}\theta \sec^2\theta\,d\theta - \int \tan^{m-2}\theta\,d\theta$$
$$= (\tan^{m-1}\theta)/(m-1) - \int \tan^{m-2}\theta\,d\theta$$
$$= \frac{\tan^{m-1}\theta}{m-1} - \left[\frac{\tan^{m-3}\theta}{m-3} - \int \tan^{m-4}\theta\,d\theta\right].$$

Proceeding thus, the integral is eventually reduced either to $\int d\theta$ (if m be even) or to $\int \tan\theta\,d\theta$ (if m be odd).

If $m = -n$ [n positive] the integral becomes $\int \cot^n\theta\,d\theta$, which can be found in exactly similar manner.

If the integral does not belong to any of these cases, i. e. if m and n are both even and $m+n$ is positive, then its value can be found by successive reduction as follows:

In the first place, since the d.c. of $\sin^{m+1}\theta = (m+1)\sin^m\theta\cos\theta$, it follows that $\int \sin^m\theta\cos\theta\,d\theta = (\sin^{m+1}\theta)/(m+1)$.

Now $\int \sin^m\theta\cos^n\theta\,d\theta$ may be written in the form

$$\int \cos^{n-1}\theta\,.\,\sin^m\theta\cos\theta\,d\theta.$$

Integrate by parts, taking $u = \cos^{n-1}\theta$, $dv/d\theta = \sin^m\theta\cos\theta$;

$\therefore\;\; du/d\theta = -(n-1)\cos^{n-2}\theta\sin\theta,\quad v = (\sin^{m+1}\theta)/(m+1)$;

$\therefore\;\; \int \sin^m\theta\cos^n\theta\,d\theta$

$$= \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} - \int\frac{\sin^{m+1}\theta}{m+1}\times -(n-1)\cos^{n-2}\theta\sin\theta\,d\theta$$

$$= \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int\sin^m\theta\,(1-\cos^2\theta)\cos^{n-2}\theta\,d\theta$$

$$= \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int\sin^m\theta\cos^{n-2}\theta\,d\theta - \frac{n-1}{m+1}\int\sin^m\theta\cos^n\theta\,d\theta.$$

Bringing the last term on to the left-hand side, we have

$$\left(1+\frac{n-1}{m+1}\right)\int\sin^m\theta\cos^n\theta\,d\theta$$
$$= \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int\sin^m\theta\cos^{n-2}\theta\,d\theta;$$

$\therefore$ dividing by the coefficient on the left, i.e. $(m+n)/(m+1)$, we have

$$\int\sin^m\theta\cos^n\theta\,d\theta = \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+n} + \frac{n-1}{m+n}\int\sin^m\theta\cos^{n-2}\theta\,d\theta,$$

in which the integral on the right-hand side is of the same form as the given integral, but the index of $\cos\theta$ is reduced by 2.

In a similar manner, by taking $u = \sin^{m-1}\theta$, $dv/d\theta = \cos^n\theta\sin\theta$, the integral may be made to depend on a similar integral in which the index of $\sin\theta$ is reduced by 2. The process may be repeated, reducing the index of either $\sin\theta$ or $\cos\theta$ by 2 at each step until finally the integral is reduced to $\int d\theta$, i.e. θ.

This method is quite general, and can be used for all values of m and n.

If m be odd and n even, the integral ultimately depends upon $\int\sin\theta\,d\theta$, i.e. $-\cos\theta$.

If m be even and n odd, the integral ultimately depends upon $\int\cos\theta\,d\theta$, i.e. $\sin\theta$.

If both m and n be odd, the integral ultimately depends upon $\int \sin\theta\cos\theta\, d\theta$, i. e. $\frac{1}{2}\sin^2\theta$.

The cases when either m or n is zero, i. e. $\int \sin^m\theta\, d\theta$ and $\int \cos^n\theta\, d\theta$, are included in the general case.

These facts are of importance when the definite integral of $\sin^m\theta\cos^n\theta$ is considered (Art. 149).

These integrals are particularly important when m and n are both positive integers, but the preceding investigation holds for all values of m and n except when $m+n=0$. The method then fails, for $m+n$ occurs in the denominators of the terms on the right-hand side. In this case, however, the integral becomes either $\int \tan^m\theta\, d\theta$ or $\int \cot^m\theta\, d\theta$, for which reduction formulae have been obtained in the earlier part of this article.

If n be negative, $n-2$ is numerically greater than n, and the integral on the right-hand side is more complicated than the one on the left; in this case the formula can be reversed. Similarly if m be negative.

142. Another method of obtaining reduction formulae.

The various reduction formulae of the type considered in the previous article can be obtained by differentiation.

If we denote $\int \sin^m\theta\cos^n\theta\, d\theta$ by $I_{m,\,n}$, then $I_{m,\,n}$ can be connected by a reduction formula with any one of the six integrals $I_{m,\,n-2}$, $I_{m-2,\,n}$, $I_{m,\,n+2}$, $I_{m+2,\,n}$, $I_{m-2,\,n+2}$, $I_{m+2,\,n-2}$.

The required formula is obtained by differentiating $\sin^p\theta\cos^q\theta$, where p exceeds by one the *smaller* of the two indices of $\sin\theta$, and q exceeds by one the *smaller* of the two indices of $\cos\theta$ in the two integrals which are to be connected. For instance, the formula worked out above connects $I_{m,\,n}$ and $I_{m,\,n-2}$. The index of $\sin\theta$ is m in both cases, and the smaller of the indices of $\cos\theta$ is $n-2$; therefore we differentiate $\sin^{m+1}\theta\cos^{n-1}\theta$.

We have

$$\begin{aligned}
&\frac{d}{d\theta}(\sin^{m+1}\theta\cos^{n-1}\theta)\\
&= \sin^{m+1}\theta\,.\,(n-1)\cos^{n-2}\theta\,(-\sin\theta)+\cos^{n-1}\theta\,.\,(m+1)\sin^m\theta\cos\theta\\
&= -(n-1)\cos^{n-2}\theta\sin^m\theta\,(1-\cos^2\theta)+(m+1)\sin^m\theta\cos^n\theta\\
&= -(n-1)\cos^{n-2}\theta\sin^m\theta+(n-1)\sin^m\theta\cos^n\theta+(m+1)\sin^m\theta\cos^n\theta\\
&= -(n-1)\cos^{n-2}\theta\sin^m\theta+(m+n)\sin^m\theta\cos^n\theta.
\end{aligned}$$

Integrating, we get $\sin^{m+1}\theta\cos^{n-1}\theta$

$$= -(n-1)\int \cos^{n-2}\theta\sin^m\theta\, d\theta+(m+n)\int \sin^m\theta\cos^n\theta\, d\theta,$$

i.e. $$\int \sin^m\theta\cos^n\theta\, d\theta = \frac{\sin^{m+1}\theta\cos^{n-1}\theta}{m+n}+\frac{n-1}{m+n}\int \sin^m\theta\cos^{n-2}\theta\, d\theta,$$

as before.

Similarly, the relation between $I_{m,n}$ and any other of the six integrals mentioned above can be obtained.

If m and n are both $+$, the relations between $I_{m,n}$ and either $I_{m,n-2}$ or $I_{m-2,n}$ simplify the integral; the former reduces the index of $\cos\theta$ by 2, and the latter reduces the index of $\sin\theta$ by 2.

If m is $+$ and n $-$, the relation between $I_{m,n}$ and $I_{m-2,n+2}$ reduces both indices by 2.

If m is $-$ and n $+$, the relation between $I_{m,n}$ and $I_{m+2,n-2}$ reduces both indices by 2.

If m and n are both $-$, the relation between $I_{m,n}$ and $I_{m,n+2}$ reduces the index of $\cos\theta$ by 2, and the relation between $I_{m,n}$ and $I_{m+2,n}$ reduces the index of $\sin\theta$ by 2.

Examples LIV.

Integrate with respect to x:

1. $x^3 e^{ax}$.
2. $x^4 e^{-x}$.
3. $x^3 \sin 2x$.
4. $x^3 \cos x$.
5. $x^4 \sin x$.
6. $x^2 (\log x)^2$.
7. $x^3 (\log x)^2$.
8. $x^2 \cosh x$.
9. $x^3 \sinh x$.

Integrate with respect to θ:

10. $\tan^4\theta$.
11. $\cot^6\theta$.
12. $\tan^8\theta$.
13. $\dfrac{\sin^3\theta}{\cos^5\theta}$.
14. $\dfrac{1}{\sin\theta\cos^3\theta}$.
15. $\dfrac{1}{\sin^4\theta\cos^4\theta}$.
16. $\sqrt{(\operatorname{cosec}\theta\sec^3\theta)}$.
17. $\sin^6\theta$.
18. $\sin^2\theta\cos^4\theta$.
19. $\cos^4\theta$.
20. $1/\cos^3\theta$.
21. $1/\sin^4\theta$.
22. $\operatorname{cosec}^2\theta\sec\theta$.
23. Obtain the formula connecting $I_{m,n}$ and $I_{m-2,n}$.
24. Find $\int\sin^6\theta\cos^2\theta\,d\theta$ in terms of $\int\sin^4\theta\cos^2\theta\,d\theta$.
25. Obtain the formula connecting $I_{m,n}$ and $I_{m-2,n+2}$.
26. Find $\displaystyle\int\frac{\sin^6\theta}{\cos^4\theta}d\theta$ in terms of $\displaystyle\int\frac{\sin^4\theta}{\cos^2\theta}d\theta$.
27. Obtain the formula connecting $I_{m,n}$ and $I_{m+2,n-2}$.
28. Find $\displaystyle\int\frac{\cos^5\theta}{\sin^3\theta}d\theta$ in terms of $\displaystyle\int\frac{\cos^3\theta}{\sin\theta}d\theta$.
29. Obtain the formula connecting $I_{m,n}$ and $I_{m,n+2}$.
30. Find $\displaystyle\int\frac{d\theta}{\sin\theta\cos^3\theta}$ in terms of $\displaystyle\int\frac{d\theta}{\sin\theta\cos\theta}$.
31. Obtain the formula connecting $I_{m,n}$ and $I_{m+2,n}$.
32. Find $\displaystyle\int\frac{d\theta}{\sin^4\theta\cos^2\theta}$ in terms of $\displaystyle\int\frac{d\theta}{\sin^2\theta\cos^2\theta}$

Miscellaneous Examples for Practice in Integration. LV.

Integrate:

1. $\dfrac{1}{1-4x}$.
2. $\dfrac{1}{1-4x^2}$.
3. $\dfrac{1}{\sqrt{(1-4x)}}$.
4. $(1-4x)^n$.
5. $x\sqrt{(1-4x^2)}$.
6. $\sqrt{(1-4x^2)}$.
7. $\dfrac{x}{\sqrt{(1-4x^2)}}$.
8. $\dfrac{1}{\sqrt{(1-4x^2)}}$.
9. $\dfrac{1}{(1-4x)^2}$.
10. $x(1-4x^2)^n$.
11. $\sqrt[n]{(1-4x)}$.
12. $x^2\sqrt{(1-4x^3)}$.
13. $\dfrac{x^2}{\sqrt{(1-4x^2)}}$.
14. $\dfrac{x}{\sqrt{(1-4x)}}$.
15. $\dfrac{\;}{1-4x}$.
16. $\dfrac{x^3}{\sqrt{(1-4x^4)}}$.
17. $\dfrac{x}{\sqrt{(1-4x^4)}}$.
18. $x(1-4x)^n$.
19. $\dfrac{x}{(1-4x)^2}$.
20. $\dfrac{x^2}{(1-4x)^2}$.
21. $\dfrac{x^2}{1-4x^2}$.
22. $x^2\sqrt{(1-4x^2)}$.
23. $x^2(1-4x)^2$.
24. $x^2\sqrt{(1-4x)}$.
25. $\dfrac{1}{(1-4x^2)^{3/2}}$.
26. $\dfrac{x}{(1-4x^2)^{3/2}}$.
27. $\dfrac{x}{(1-4x^2)^n}$.
28. $\dfrac{1}{\sqrt{(1-4x)^3}}$.
29. $\dfrac{x^2}{\sqrt{(1-4x)}}$.
30. $\dfrac{x^3}{(1-4x^2)^2}$.
31. $\dfrac{1}{x(1-4x)}$.
32. $\dfrac{1}{x(1-4x^2)}$.
33. $\dfrac{1}{x(1-4x)^2}$.
34. $\dfrac{1}{x^2(1-4x)}$.
35. $\dfrac{1}{x^2(1-4x^2)}$.
36. $\dfrac{1-4x}{x(1-4x^2)}$.
37. $\dfrac{1-4x^2}{x(1-4x)}$.
38. $\dfrac{x^3}{\sqrt{(1-4x^2)}}$.
39. $\dfrac{x}{(1-4x^2)^2}$.
40. $x\sqrt{(1-4x)}$.
41. $\sin^2 ax$.
42. $\cos^3 \frac{1}{2}x$.
43. $\sin 2x \cos 2x$.
44. $\sin x \cos^4 x$.
45. $\sin^3 x \cos x$.
46. $\sin^3 x \cos^3 x$.
47. $\sin^2 x \cos^2 x$.
48. $\sin x \cos 2x$.
49. $\cos x \cos 2x$.
50. $\sin x \sin 2x$.
51. $\sin^2 x \cos^3 x$.
52. $\tan^2 2x$.
53. $\cot^2 \frac{1}{2}x$.
54. $\sec^3 x$.
55. $\tan^4 x$.
56. $\tan x \sec^2 x$.
57. $\tan x \sec^3 x$.
58. $\cot x \operatorname{cosec} x$.
59. $\operatorname{cosec} 2x$.
60. $\sec 2x$.
61. $x \sin nx$.
62. $x \cos \frac{1}{2}x$.
63. $x \sec^2 mx$.
64. $x \tan^2 x$.
65. $x^2 \sin x$.
66. xe^{-2x}.
67. $x^2 e^x$.
68. xe^{x^2}.
69. $(a+bx)e^x$.
70. $x^6 \log x$.
71. $x \log(1+x)$.
72. $(\log x)/x$.
73. $x^{-n} \log x$.
74. $\log(a-x)$.
75. $x \log(1+x^2)$.
76. $e^{-x} \sin 5x$.
77. $e^{3x} \cos 3x$.
78. $e^x \sin x \cos x$.
79. $e^{-x} \sin^2 x$.
80. $x^3 \tan^{-1} x$.
81. $x \cos^{-1} x$.
82. $\sec^{-1} x$.
83. $x \operatorname{cosec}^{-1} x$.
84. $\tanh ax$.
85. $\dfrac{\sin x - \cos x}{\sin x + \cos x}$.
86. $\dfrac{1}{x(x^2+1)}$.
87. $\dfrac{1}{x(x^3+2)}$.
88. $\dfrac{x}{x^2+1}$.
89. $\dfrac{x}{\sqrt{(x^2+1)}}$.
90. $\dfrac{1}{x\sqrt{(x^2+1)}}$.

91. $\dfrac{1}{x^2\sqrt{(x^2+1)}}$.

92. $\dfrac{x^2}{x^2+1}$.

93. $\dfrac{x}{\sqrt{(x^2+1)}}$.

94. $\sqrt{(x^2+1)}/x^2$.

95. $x\sqrt{(x^2+1)}$.

96. $(x^2+1)^{3/2}$.

97. $\dfrac{1}{(x^2+1)^2}$.

98. $\dfrac{1}{(x^2+1)^{3/2}}$.

99. $\dfrac{x}{(x^2+1)^{3/2}}$.

100. $x(x^2+1)^{3/2}$.

101. $x\sqrt{(1-x)}$.

102. $x\sqrt{(x-1)}$.

103. $\dfrac{x}{\sqrt{(1-x)}}$.

104. $\dfrac{x+1}{\sqrt{(x-1)}}$.

105. $\sqrt{\left(\dfrac{x}{1-x}\right)}$.

106. $\sqrt{[x(x-1)]}$.

107. $\sqrt{[x(1-x)]}$.

108. $\sqrt{[(x-1)/x]}$.

109. $\dfrac{1}{\sqrt{[x(1-x)]}}$.

110. $\sqrt{[x(1+x)]}$.

111. $\dfrac{\sin x}{1+\cos x}$.

112. $\dfrac{1}{1+\cos x}$.

113. $\dfrac{\cos x}{1+\cos x}$.

114. $\dfrac{\sin^2 x}{1+\cos x}$.

115. $\dfrac{\cos^2 x}{1+\cos x}$

116. $\dfrac{\sin x}{(1+\cos x)^2}$.

117. $\dfrac{1}{(1+\cos x)^2}$.

118. $\dfrac{\cos x}{(1+\cos x)^2}$

119. $\dfrac{\cos^2 x}{(1+\cos x)^2}$.

120. $\dfrac{\sin^2 x}{(1+\cos x)^2}$.

121. $\dfrac{e^x}{e^x+1}$.

122. $\dfrac{1}{e^x+1}$.

123. $\dfrac{x}{e^x+1}$.

124. $\operatorname{sech} x$.

125. $\operatorname{cosech} x$.

126. $x\cosh x$.

127. $x\sinh \frac{1}{2}x$.

128. $\cosh^2 x$.

129. $\sinh^2 x$.

130. $\sinh^{-1} x$.

131. $x\cosh^{-1} x$.

132. $\sin ax \sinh ax$.

133. $\cos^2 x \sinh x$.

134. $\sin 2x \cosh 3x$.

135. $\cos mx \cosh nx$.

136. $\dfrac{1}{x^2+6x+109}$.

137. $\dfrac{x+3}{x^2+6x+109}$.

138. $\dfrac{x-3}{x^2+6x+109}$.

139. $\dfrac{x^2}{x^2+6x+109}$.

140. $\dfrac{1}{\sqrt{(x^2+6x+109)}}$.

141. $\dfrac{x+3}{\sqrt{(x^2+6x+109)}}$.

142. $x/\sqrt{(x^2+6x+109)}$.

143. $\sqrt{(x^2+6x+109)}$.

144. $x\sqrt{(x^2+6x+109)}$.

145. $\dfrac{1}{x(x^2+6x+109)}$.

146. $\dfrac{1}{x\sqrt{(x^2+6x+109)}}$.

147. $\dfrac{1}{x(1+x^3)}$.

148. $\dfrac{1}{1+x^3}$.

149. $\dfrac{x}{1+x^3}$.

150. $\dfrac{x^3}{(1+x^2)^3}$

CHAPTER XV

DEFINITE INTEGRALS

143. Integration as a summation.

Let $f(x)$ be a function of x which is finite and continuous from $x=a$ to $x=b$, both inclusive. Let $b>a$, and let the interval $b-a$ be divided into n intervals

$$x_1-a, \quad x_2-x_1, \quad x_3-x_2, \ldots x_{n-1}-x_{n-2}, \quad b-x_{n-1}.$$

Then the value of the sum

$$(x_1-a)f(a)+(x_2-x_1)f(x_1)+(x_3-x_2)f(x_2)+\ \ldots\ +(b-x_{n-1})f(x_{n-1})$$

[which may be written $\sum_{x=a}^{x=b} f(x)\,\delta x$] tends, when the intervals are all indefinitely diminished, to a limit, which is called the *definite integral* of $f(x)$ with respect to x from $x=a$ to $x=b$. This is written

$$f(x)\,dx.$$

The value of the given expression is evidently finite whatever the value of n, for if M be the maximum value of $f(x)$ in the given interval, the sum

$$< M\left[(x_1-a)+(x_2-x_1)+\ \ldots\ +(b-x_{n-1})\right],$$

i.e. $< M(b-a)$, which is finite, since M, b, a are all finite.

The definite integral is here defined as the limiting value of the sum of a series. The calculation of the limiting value from this definition is complicated even in the case of quite simple functions, and in most cases would be quite impossible.

For instance, take the very simple function x^2, and let each of the intervals x_1-a, x_2-x_1, ... be equal to h; so that

$$x_1=a+h, \quad x_2=x_1+h=a+2h, \quad x_3=a+3h, \ldots x_{n-1}=a+(n-1)h,$$

and

$$b-a=nh.$$

Then, from the above definition,

$$\int_a^b x^2\,dx = \underset{n\to\infty}{\mathrm{Lt}}\ \{ha^2 + h(a+h)^2 + h(a+2h)^2 + \dots + h[a+(n-1)h]^2\}$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\ h[a^2 + a^2 + 2ah + h^2 + a^2 + 4ah + 2^2h^2 + \dots + a^2 + 2(n-1)ah + (n-1)^2h^2]$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\ h[na^2 + 2ah(1+2+\dots+n-1) + h^2\{1+2^2+\dots+(n-1)^2\}]$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\ [nha^2 + 2ah^2 \times \tfrac{1}{2}(n-1)n + h^3 \times \tfrac{1}{6}(n-1)n(2n-1)]$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\ \left[(b-a)a^2 + ah^2n^2\left(1-\frac{1}{n}\right) + \frac{1}{6}h^3n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]$$

$$= (b-a)a^2 + a(b-a)^2 + \tfrac{1}{6}(b-a)^3 \,.\, 2$$

$$= a^2b - a^3 + ab^2 - 2a^2b + a^3 + \tfrac{1}{3}b^3 - b^2a + ba^2 - \tfrac{1}{3}a^3$$

$$= \tfrac{1}{3}(b^3 - a^3).$$

The values of the definite integrals of a few very simple functions may be calculated in this way, but it will be seen that the method of the next article saves an enormous amount of labour.

We now proceed to show how the value of the definite integral can be deduced at once from that of the corresponding indefinite integral.

144. Relation between definite and indefinite integrals.

This can be obtained either geometrically or analytically.

1. *Geometrically.*

Let A, X_1, X_2, ... X_{n-1}, B (Fig. 93) be the points on the axis of x whose abscissae are a, x_1, x_2, ... x_{n-1}, b respectively, and let the

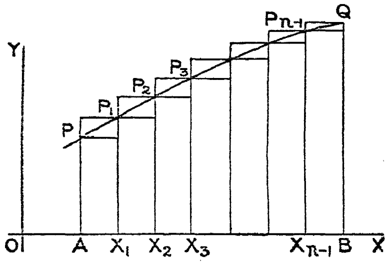


Fig. 93.

ordinates of A, X_1, X_2, ... X_{n-1}, B cut the graph of $y = f(x)$ in P, P_1, P_2, ... P_{n-1}, Q.

Then AP, X_1P_1, X_2P_2, ... $X_{n-1}P_{n-1}$ represent the values of $f(a)$, $f(x_1)$, $f(x_2)$, ... $f(x_{n-1})$ respectively.

Therefore

$$(x_1-a)f(a)+(x_2-x_1)f(x_1)+\dots+(b-x_{n-1})f(x_{n-1}) \quad \text{(i)}$$
$$=AX_1 . AP+X_1X_2 . X_1P_1+\dots+X_{n-1}B . X_{n-1}P_{n-1}$$
$$=\text{the sum of the rectangles } PX_1,\ P_1X_2,\ \dots P_{n-1}B.$$

The difference between this sum and the area $APQB <$ the sum of the small rectangles PP_1, P_1P_2, ... $P_{n-1}Q$, and if α be the greatest of the bases, this sum is less than $\alpha \times$ the sum of their heights, i.e. $< \alpha(BQ-AP)$, and this $\to 0$ when $\alpha \to 0$, since BQ and AP are finite. Hence the area $APQB$ is the limiting value of the sum of the rectangles, and therefore represents the limit of (i). But it was shown in Art. 80 that the area $APQB=F(b)-F(a)$, where $F'(x)=f(x)$.

$\therefore \quad (x_1-a)f(a)+(x_2-x_1)f(x_1)+\dots+(b-x_{n-1})f(x_{n-1})$

tends to the limit $F(b)-F(a)$;

i.e. $$\int_a^b f(x)\,dx=F(b)-F(a),$$

where $F(x)$ is the indefinite integral of $f(x)$.

2. *Analytically.*

Let $F'(x)=f(x)$, i.e. let $F(x)$ be the function whose d. c. is $f(x)$. Then from the definition of a d. c. (Art. 26),

$$\operatorname*{Lt}_{\delta x\to 0}\frac{F(x+\delta x)-F(x)}{\delta x}=F'(x)=f(x);$$

$$\therefore \quad \frac{F(x+\delta x)-F(x)}{\delta x}=f(x)+\epsilon, \text{ where } \epsilon\to 0 \text{ as } \delta x\to 0 \text{ [Art. 24]},$$

i.e. $$F(x+\delta x)-F(x)=\delta x . f(x)+\epsilon\,\delta x.$$

Take $x=a$, $\delta x=x_1-a$,

then $$F(x_1)-F(a)=(x_1-a)f(a)+\epsilon_1(x_1-a).$$

Take $x=x_1$, $\delta x=x_2-x_1$,

then $$F(x_2)-F(x_1)=(x_2-x_1)f(x_1)+\epsilon_2(x_2-x_1).$$

Take $x=x_2$, $\delta x=x_3-x_2$,

then $$F(x_3)-F(x_2)=(x_3-x_2)f(x_2)+\epsilon_3(x_3-x_2).$$

. . . .

Take $x=x_{n-1}$, $\delta x=b-x_{n-1}$,

then $$F(b)-F(x_{n-1})=(b-x_{n-1})f(x_{n-1})+\epsilon_n(b-x_{n-1}).$$

Adding together all these results,

$$F(b)-F(a)=(x_1-a)f(a)+(x_2-x_1)f(x_1)+\dots+(b-x_{n-1})f(x_{n-1})$$
$$+\epsilon_1(x_1-a)+\epsilon_2(x_2-x_1)+\dots+\epsilon_n(b-x_{n-1}).$$

Let η be the (numerically) greatest of the quantities $\epsilon_1, \epsilon_2, \dots \epsilon_n$; then the expression in the last line

$$< \eta(x_1-a)+\eta(x_2-x_1)+ \dots +\eta(b-x_{n-1}),$$

i.e. $$< \eta(x_1-a+x_2-x_1+ \dots +b-x_{n-1}),$$

i.e. $$< \eta(b-a).$$

All the numbers $\epsilon_1, \epsilon_2, \epsilon_3 \dots \epsilon_n$, and therefore η, which is one of them, tend to zero as the intervals $x_1-a, x_2-x_1, \dots, b-x_{n-1}$ are indefinitely diminished; hence, since a and b are finite, $\eta(b-a)$ tends to the limit zero, and therefore

$$F(b)-F(a) = \text{Lt}\,[(x_1-a)f(a)+(x_2-x_1)f(x_1)+ \dots +(b-x_{n-1})f(x_{n-1})]$$
$$= \int_a^b f(x)\,dx.$$

This gives the same rule for evaluating a definite integral in general as was obtained in Arts. 80 and 81 for the particular cases of areas and volumes, viz.:

The value of $\int_a^b f(x)\,dx$ *is obtained by substituting* (i) b, (ii) a *in the indefinite integral of* $f(x)$, *and subtracting the latter result from the former.*

For instance, in the example just worked out in full from the definition of a definite integral, we have at once, by this rule,

$$\int_a^b x^2\,dx = \left[\frac{1}{3}x^3\right]_a^b = \frac{1}{3}b^3 - \frac{1}{3}a^3.$$

We have now connected the two different points of view from which an integral may be regarded (as given in Arts. 71 and 143), so that the value of a definite integral can be deduced at once from that of the corresponding indefinite integral obtained by the methods of Chapters IX and XIV.

Further examples are:

$$\int_0 \frac{dx}{4+5x}\cdot \quad \left[\frac{1}{5}\log(4+5x)\right]_0^1 = \frac{1}{5}(\log 9 - \log 4) = \frac{1}{5}\log\frac{9}{4}.$$

$$\int_0 dx \quad \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0) = \frac{1}{2}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{8}.$$

$$\int_0^a \sqrt{(a^2-x^2)}\,dx = \left[\frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a \text{ (Art. 139)}$$
$$= (0+\tfrac{1}{2}a^2\sin^{-1}1) - 0 = \tfrac{1}{4}\pi a^2.$$

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot x\,dx = \left[\log\sin x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = \log 1 - \log\frac{1}{\sqrt{2}} = \log\sqrt{2} = \frac{1}{2}\log 2.$$

$$\int_0^{\pi/b} e^{ax}\cos bx\,dx = \left[\frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]_0^{\pi/b} \text{ (Art. 139)}$$
$$= \frac{1}{a^2+b^2}\left[e^{a\pi/b}(-a+0) - 1(a+0)\right] = -\frac{a}{a^2+b^2}(e^{a\pi/b}+1).$$

In the preceding investigation, the values of the functions at the commencements of the successive intervals have been taken, but this is not essential; it is sufficient to take the values of the function at any points within the intervals. It can be shown that the limiting value to which the corresponding sum tends is the same in this case as when the values at the beginnings of the intervals are taken.

In the geometrical proof above, the successive terms of the corresponding series (i) will then be represented by rectangles which are intermediate between the inner rectangles $PX_1, P_1X_2, \ldots P_{n-1}B$ and the outer rectangles $P_1A, P_2X_1, \ldots QX_{n-1}$ respectively, and both sets of rectangles, and therefore also any intermediate set, tend to the same limiting value, the area $APQB$.

145. **Exceptions.**

The condition has been laid down above that the function $f(x)$ is to be continuous for all values of x from a to b inclusive, and it has been supposed that a and b are finite. We are therefore not yet at liberty to apply the preceding result if these conditions are not satisfied, e.g. we cannot, as yet, evaluate such expressions as

$$\int_{-1}^{+1}\frac{1}{x^2}dx,\quad \int_0^1 \frac{dx}{\sqrt{(1-x^2)}},\quad \text{or} \int_0^\infty \frac{dx}{a^2+x^2};$$

because, in the first case, the function $1/x^2$ is discontinuous for the value $x=0$, which is within the range of integration; in the second case, the function becomes infinite at one end of the range, when $x=1$; and in the last case, one end of the range of integration is at infinity.

Such cases will be considered later (Art. 148), and it will then be seen that the first of these three integrals has no value, whereas the other two have finite values.

It should be noticed that an indefinite integral may be regarded as a definite integral taken between some arbitrarily fixed value a and a variable value x, the arbitrary constant of integration being the value of the integral function when $x=a$, with the sign changed, i.e. $\int f'(x)\,dx$, which has hitherto been written in the form $f(x)+C$, may be regarded as

$$\int_a^x f'(x)\,dx,\quad \text{i.e.} \left[f(x)\right]_a^x = f(x) - f(a),$$

which is the same result as before with C replaced by $-f(a)$.

The following set of examples will serve as exercises for a revision of the various methods of integration which have been considered.

Examples LVI.

Find the values of the following:

1. $\int_1^4 x^4\,dx,\quad \int_2^3 x^{-2}\,dx,\quad \int_4^9 x^{-\frac{1}{2}}\,dx,\quad \int_0^{10} (x^2-x+1)\,dx,\quad \int_{-1}^1 (1-x^2)^2\,dx.$

2. $\int_0^1 (3x+2)^2\quad \int_{-2}^{-1}\frac{dx}{(x-1)^3},\quad \int_0^a (x+a)^n\,dx,\quad \int_0^1\frac{dx}{\sqrt{(1+x)}},\quad \int_0^a \sqrt{(4a+5x)}\,dx.$

3. $\int_1^4 \frac{dx}{x},\quad \int_0^a \frac{dx}{x+a},\quad \int_{-1}^2 \frac{dx}{x+2}.$

4. $\int_0^{\frac{1}{2}\pi} \sin x\,dx,\quad \int_0^\pi \cos x\,dx,\quad \int_{-\pi}^{\pi} \sin\tfrac{1}{2}x\,dx.$

5. $\int_0^{\frac{1}{2}\pi} \sin^2 x\,dx,\quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2 x\,dx,\quad \int_0^{\frac{1}{4}\pi} \sec^2 x\,dx.$

6. $\int_0^1 \frac{dx}{x^2+1},\quad \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)}},\quad \int_0^1 \frac{dx}{\sqrt{(1+x^2)}},\quad \int_2^3 \frac{dx}{\sqrt{(x^2-1)}}.$

7. $\int_0^1 \sqrt{(1-x^2)}\,dx,\quad \int_0^a \sqrt{(x^2+a^2)}\,dx,\quad \int_1^2 \sqrt{(x^2-1)}\,dx.$

8. $\int_0^1 \frac{x}{\sqrt{(1+x^2)}}\,dx,\quad \int_1^2 \frac{dx}{x(2+x)},\quad \int_0^1 \frac{dx}{x^2+2x+2}.$

9. $\int_0^{\frac{1}{2}\pi} \frac{\sin\theta}{1+\cos^2\theta}\,d\theta,\quad \int_0^{\frac{1}{2}\pi} \sin^3\theta\,d\theta,\quad \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \cot\theta\,d\theta.$

10. $\int_1^2 x\log x\,dx,\quad \int_1^4 x^2\log x\,dx,\quad \int_a^b \log x\,dx.$

11. $\int_0 \frac{dx}{1+\cos x},\quad \int_{-\pi}^{-\frac{1}{2}\pi} \frac{dx}{1-\cos x}.$

12. $\int_0^1 \sin^{-1} x\,dx,\quad \int_0^1 \tan^{-1} x\,dx.$

13. $\int_0^\pi e^{2x}\sin x\,dx,\quad \int_0^{\frac{1}{2}\pi} e^{-x}\cos x\,dx.$

14. $\int_0^1 x^2 e^x\,dx,\quad \int_0^1 \cosh x\,dx.$

15. $\int_0^a x\sqrt{(a^2-x^2)}\,dx,\quad \int_0^a \frac{x}{\sqrt{(a^2+x^2)}}\,dx.$

16. $\int_0^\pi x\sin x\,dx,\quad \int_0^\pi x^2\sin x\,dx.$

17. $\int_0^\pi \cos^3\theta\,d\theta,\quad \int_0^\pi \cos^2\theta\sin\theta\,d\theta.$

18. $\int_0^1 \frac{dx}{\sqrt{(x^2+4x+3)}},\quad \int_0^1 \sqrt{(x^2+4x+3)}\,dx.$

19. $\int_0^a \frac{x^2-a^2}{x^2+a^2}\,dx,\quad \int_0^1 \frac{x^3}{x+2}\,dx.$

20. $\int_0 \sec x\,dx,\quad \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \operatorname{cosec} x\,dx.$

21. $\int_0^{\frac{1}{2}\pi} \frac{dx}{5+4\cos x},\quad \int^{\frac{1}{2}\pi} \frac{dx}{4+5\cos x}.$

22. $\int_0^{\frac{1}{4}\pi} \tan^4 x\,dx,\quad \int_0^{\frac{1}{2}\pi} \sin^4 x\,dx.$

23. $\int_0^{\frac{1}{2}\pi} \sin^2 x\cos^3 x\,dx,\quad \int_0^\pi \sin^2 x\cos^3 x\,dx.$

24. $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \tan x\,dx.$

25. $\int_1^2 \frac{dx}{x^2(x+1)},\quad \int_0^1 \frac{dx}{(x+1)^2(x+2)}.$

26. $\int_0^{\frac{1}{2}\pi} x\sin^2 x\,dx,\quad \int_{\frac{1}{2}\pi}^{\pi} x\cos^2 x\,dx.$

27. $\int_0 \sin 4x \cos 2x \, dx, \quad \int_0^{\frac{1}{2}\pi} \sin 2mx \cos 2nx \, dx.$ 28. $\int_0^{\frac{1}{4}\pi} \sec^4 x \, dx.$

$\int_0^1 x \tan^{-1} x \, dx, \quad \int_0^1 \cos^{-1} x \, dx.$ 30. $\int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}.$

31. $\int_0^1 \sqrt{\left(\frac{1-x}{1+x}\right)} \, dx.$ 32. $\int_0^1 \frac{dx}{1 + 2x \cos \alpha + x^2}$

33. $\int_0^9 \frac{dx}{1+\sqrt{x}}$ 34. $\int_0^1 \frac{dx}{e^x + e^{-x}}.$ 35. $\frac{\sin x}{} \, dx.$

36. $\int_0^1 \frac{dx}{x^3+1}.$ 37. $\int_0^2 \frac{x \, dx}{(x+1)(x^2+4)}$ 38. $\int_a^{2a} \frac{dx}{\sqrt{(2ax+x^2)}}.$

39. $\int_0^1 \frac{dx}{\sqrt{(x^2+2x+2)}}, \quad \int_{-a}^{a} \frac{dx}{\sqrt{(a^2+x^2)}}.$ 40. $\int_0^{\frac{1}{4}\pi} x \tan^2 x \, dx.$

41. $\int_0^1 \frac{dx}{x^2+2x+5}, \quad \int_0^1 \frac{x \, dx}{x^2+5x+6}.$ 42. $\int_0^2 \frac{dx}{\sqrt{(3+2x-x^2)}}.$

Evaluate from the definition of Art. 143 :

43. $\int_0^1 x \, dx.$ 44. $\int_1^2 (3x^2 - 4x) \, dx.$ 45. $\int_0^{\frac{1}{2}\pi} \sin x \, dx.$

46. $\int e^x \, dx.$ 47. $\int_0^{\frac{1}{2}\pi} \cos 2x \, dx.$ 48. $\int_a x^3 \, dx.$

146. General properties of definite integrals.

Let $F(x)$ be the indefinite integral of $f(x)$.

I. It is at once evident, since $\int_a^b f(x) \, dx = F(b) - F(a),$

and $\int_b^a f(x) \, dx = F(a) - F(b),$

that *an interchange of the 'limits'* a *and* b *changes the sign of the definite integral.*

II. $$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx,$$

since the latter expression $= F(c) - F(a) + F(b) - F(c)$
$= F(b) - F(a).$

In a similar manner an integral may be divided up into the sum of any number of parts.

III. $\int_{-a}^{a} f(x) \, dx = 0$ *or* $2\int_0^a f(x) \, dx,$ *according as* $f(x)$ *is an odd or an even function of* x.

For $$\int_{-a}^{a} f(x) \, dx = F(a) - F(-a).$$

If $F(x)$ be an even function of x, i.e. if $f(x)$ be an odd function of x,*

$$F(-a)=F(a), \quad \text{and} \quad \int_{-a}^{a} f(x)\,dx = 0.$$

If $F(x)$ be an odd function of x, i.e. if $f(x)$ be an even function of x,*

$$F(-a)=-F(a), \quad \text{and} \quad \int_{-a}^{a} f(x)\,dx = 2F(a) = 2\int_0^a f(x)\,dx,$$

for in this case, $\int_0^a f(x)\,dx = F(a)-F(0) = F(a)$, since, $F(x)$ being an odd function of x, it follows that $F(0)$ is zero [Art. 5].

E. g.
$$\int_{-1}^{+1} x^3\,dx = 0, \qquad \int_{-1}^{+1} x^4\,dx = 2\int_0^1 x^4\,dx.$$

$$\int_{-a}^{a} \frac{x^2}{a^2+x^2}\,dx = 2\int_0^a \frac{x^2}{a^2+x^2}\,dx\,; \quad \int_{-a}^{a} \frac{x^3}{a^2+x^2}\,dx = 0.$$

This result also follows directly from the definition in Art. 143, if the two halves of the range from $-a$ to 0 and 0 to a be divided into equal intervals. For, if $f(x)$ be an odd function, the terms of the series obtained from negative values of x are equal in magnitude and opposite in sign to the terms obtained from the corresponding positive values of x; hence the terms of the series cancel out in pairs and the sum is zero. If $f(x)$ be an even function, the terms obtained from negative values of x have the same magnitude and sign as the terms obtained from the corresponding positive values of x; hence the terms occur in equal pairs, and their sum is twice the sum obtained from the positive values of x alone. This theorem is especially useful in dealing with the integrals of certain trigonometrical functions.

E. g.
$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin x\,dx = 0\,; \qquad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos x\,dx = 2\int_0^{\frac{1}{2}\pi} \cos x\,dx.$$

$$\int_{-\pi}^{\pi} \sin^3 x\,dx = 0\,; \qquad \int_{-\pi}^{\pi} \sin^2 x\,dx = 2\int_0^{\pi} \sin^2 x\,dx.$$

$$\int_{-\pi}^{\pi} \sin^3 x\cos^2 x\,dx = 0\,; \qquad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^2 x\cos^3 x\,dx = 2\int_0^{\frac{1}{2}\pi} \sin^2 x\cos^3 x\,dx,$$

since, in each line, the first function given is odd and the second even.

* If $F(x)$ be an even function of x, $F(x) = F(-x)$.

$\therefore$ Differentiating, $F'(x) = F'(-x)\times -1$, i.e. $f(x) = -f(-x)$; hence $f(x)$ is an odd function of x.

Similarly, if $F(x)$ be an odd function of x, $F(-x) = -F(x)$;

$$\therefore \quad -F'(-x) = -F'(x), \quad \text{i.e.} \quad f(-x) = f(x)\,;$$

hence $f(x)$ is an even function of x.

IV. $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

For, on putting $x = a-z$, $\int f(x)\,dx$ becomes

$$\int f(a-z)\frac{dx}{dz}\,dz, \quad \text{i.e.} \quad -\int f(a-z)\,dz, \quad \text{since} \quad \frac{dx}{dz} = -1.$$

Also when $x = 0$, $z = a$, and when $x = a$, $z = 0$;

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a-z)\,dz = \int_0^a f(a-z)\,dz = \int_0^a f(a-x)\,dx,$$

since the value of the definite integral depends only upon the values of the limits, so that it does not matter whether the variable be denoted by x or z.

In particular, $\int_0^{\frac{1}{2}\pi} f(\sin x)\,dx = \int_0^{\frac{1}{2}\pi} f[\sin(\frac{1}{2}\pi - x)]\,dx = \int_0^{\frac{1}{2}\pi} f(\cos x)\,dx.$

Also $\int_0^1 x(1-x)^n\,dx = \int_0^1 (1-x)x^n\,dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$

(except when $n = -1$ or -2) $\qquad = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}.$

V. $\int_0^{2a} f(x)\,dx = 0$ *or* $2\int_0^a f(x)\,dx$, *according as* $f(2a-x) = -f(x)$ *or* $+f(x)$.

For $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx.$$

In the last integral, let $x = 2a-z$; then when $x = a$, $z = a$, and when $x = 2a$, $z = 0$.

$$\int_a^{2a} f(x)\,dx \text{ becomes } -\int_a^0 f(2a-z)\,dz, \quad \text{i.e.} \quad \int_0^a f(2a-x)\,dx.$$

Hence

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = \int_0^a [f(x)+f(2a-x)]\,dx,$$

which is equal to 0 if $f(2a-x) = -f(x)$, and to $2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$. This is especially useful in dealing with trigonometrical integrals, since $\sin(\pi - x) = \sin x$, and $\cos(\pi - x) = -\cos x$.

E.g. $$\int_0^{\pi} \sin^n x \cos^2 x\,dx = 2\int_0^{\frac{1}{2}\pi} \sin^n x \cos^2 x\,dx,$$

but $$\int_0^{\pi} \sin^n x \cos^3 x\,dx = 0.$$

This result also follows, like Theorem III, directly from the definition of a definite integral, if the two halves of the range be divided into equal

intervals. For, in the first of the two examples just given, the expression to be integrated takes the same series of values (in the reverse order) between $\frac{1}{2}\pi$ and π as between 0 and $\frac{1}{2}\pi$; hence the integral from 0 to π is twice the integral from 0 to $\frac{1}{2}\pi$. In the second example, the values of the expression to be integrated between $\frac{1}{2}\pi$ and π are equal in magnitude and opposite in sign to the values between 0 and $\frac{1}{2}\pi$; hence the terms cancel out in pairs, and the integral is zero.

VI. If G be the greatest value and L the least value of $f(x)$ within the range of integration a to b ($b > a$), then the value of $\int_a^b f(x)\,dx$ lies between $L(b-a)$ and $G(b-a)$.

This follows at once from the definition of Art. 143, for, since G is the maximum value of $f(x)$, the sum there given is less than the sum obtained by replacing every value of $f(x)$ by G,

i.e. $\quad < G[x_1-a+x_2-x_1+\ldots+b-x_{n-1}]$, i.e. $< G(b-a)$.

Similarly, it is greater than $L(b-a)$.

If $f(x)$ be a continuous function of x, then as x increases from a to b, $f(x)$ must pass through every value intermediate between L and G; hence the value of the definite integral is equal to $b-a$ multiplied by the value of $f(x)$ for some value of x between a and b. This value may, as in Art. 117, be denoted by $a+\theta(b-a)$ where $0<\theta<1$; hence

$$\int_a^b f(x)\,dx = (b-a)f[a+\theta(b-a)].$$

E.g., since the maximum and minimum values of $\sqrt{(5+\sin^2 x)}$ are $\sqrt{5}$ and $\sqrt{6}$, it follows that $\int_0^{\frac{1}{2}\pi}\sqrt{(5+\sin^2 x)}\,dx$ is between $\sqrt{5}\cdot\frac{1}{2}\pi$ and $\sqrt{6}\cdot\frac{1}{2}\pi$, i.e. between $1{\cdot}118\,\pi$ and $1{\cdot}225\,\pi$.

147. Geometrical proofs.

If the definite integral be represented by an area, the preceding results are all easily seen to be true from geometrical considerations.

Theorem I simply states that the area from AP to BQ is the same as the area from BQ to AP; the change of sign is due to the fact that the intervals AB and BA are measured in opposite directions.

Theorem II states that the area from AP to BQ is equal to the sum of the areas from AP to CR and from CR to BQ (Fig. 94 (i)).

In Theorem III, if $f(x)$ is an even function of x, the graph of $f(x)$ is symmetrical about the axis of y; if $f(x)$ is an odd function of x, the graph is symmetrical about the origin (Art. 10). The theorem follows from the facts that in the first case the area $APQB$

is double the area $ORQB$ (Fig. 94 (ii)), and in the second case, the areas AOP and BOQ are equal, and, since they are on opposite sides of the axis of x, opposite in sign (Fig. 94 (iii)).

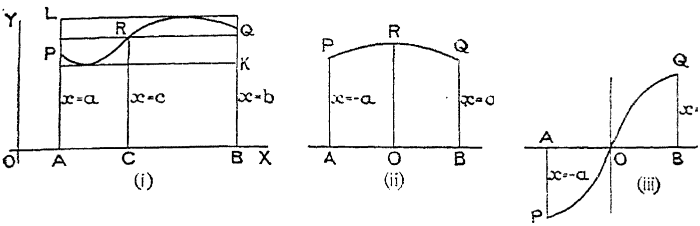


Fig. 94.

Theorem IV is equivalent to the statement that, if in figure (ii) the origin O be moved to the point B, where $x=a$, and the direction of the axis of x be reversed, the same area $OBQR$ is obtained, provided the range 0 to a is the same.

Theorem V follows from figures (ii) and (iii) in the same way as Theorem III, by taking A as the origin and O, B as the points $x=a$, $x=2a$ respectively. The curve is symmetrical about OR if $f(2a-x)=f(x)$, and then the area $APQB$ is double the area $APRO$; the curve is symmetrical about the point O if $f(2a-x)=-f(x)$, and then the area $AOP=-$the area OBQ.

Theorem VI follows from the fact that the area $PABQ$ in fig. (i) is greater than the rectangle AK contained by AB and the minimum ordinate of the curve, and less than the rectangle BL contained by AB and the maximum ordinate. The final form in which the theorem is given is equivalent to the statement that there is some intermediate point R, such that the area $PABQ$ is equal to the rectangle contained by AB and the ordinate CR.

Examples LVII.

Express as integrals from 0 to $\frac{1}{2}\pi$ the following:

1. $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^2 x \cos^2 x \, dx.$ 2. $\int_0^{\pi} \sin^3 x \cos^2 x \, dx.$ 3. $\int_{-\pi}^{\pi} \sin^6 x \, dx.$

4. $\int_0^{2\pi} \cos^8 x \, dx.$ 5. $\int_{-2\pi}^{2\pi} \sin^4 x \cos^4 x \, dx.$ 6. $\int_0^{4\pi} \sin^4 x \, dx.$

7. $\int_{\pi}^{0} \frac{\sin^2 x}{1+\cos^2 x} \, dx.$ 8. $\int_{-2\pi}^{0} \cos^4 x \, dx.$ 9. $\int_{\frac{1}{2}\pi} \sin^2 x \cos^2 x \, dx.$

10. $\int_{-\frac{1}{2}\pi}^{\pi} \sin^8 x \, dx.$ 11. $\int_{\pi}^{2\pi} \cos^{10} x \, dx.$ 12. $\int_{\frac{1}{2}\pi}^{2\pi} \sin^4 x \, dx.$

Find the value of:

13. $\displaystyle\int_0^1 x(1-x)^{3/2}\,dx.$ 14. $\displaystyle\int_0^a x(a-x)^n\,dx.$ 15. $\displaystyle\int_0^2 x^2\sqrt{(2-x)}\,dx.$

16. $\displaystyle\int_{-1} \frac{x^5}{a^2+x^2}\,dx.$ 17. $\displaystyle\int_{-\alpha}^{\alpha} \frac{\sin^3 x}{1+\cos^2 x}\,dx.$ 18. $\displaystyle\int^{\frac{1}{3}\pi} \tan^5 x\,dx.$

19. $\displaystyle\int^1 x^2(1-x)^{10}\,dx.$ 20. $\displaystyle\int_{-a}^{a} x^2\sqrt{(a-x)}\,dx.$ 21. $\displaystyle\int_{-a}^{a} x^3\sqrt{(a^2-x^2)}\,dx.$

22. $\displaystyle\int^{\pi} \sin^3 x\cos^4 x\,dx.$ 23. $\displaystyle\int^{\frac{1}{2}\pi} \sin x\cos^3 x\,dx.$ 24. $\displaystyle\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \sec^2 x\,dx.$

25. $\displaystyle\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \sin x\sec^4 x\,dx.$ 26. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x-\sin x}{\cos x+\sin x}\,dx.$

27. From the fact that, within the range of integration, $0<x^n<x^2$ if $n>2$, deduce two values between which $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^n)}}$ must lie.

Between what two values must the following three integrals lie?

28. $\displaystyle\int_0^{\frac{1}{4}\pi} (4-\cos^2 x)^{1/2}\,dx.$ 29. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(1-\frac{1}{8}\sin^2\theta)}}.$ 30. $\displaystyle\int^1 \frac{2+x^n}{1+x^n}\,dx.$

31. Prove that $\displaystyle\int_a^b f(x)\,dx=\int_0^{b-a} f(x+a)\,dx.$

32. Prove that $\displaystyle\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx.$

33. Prove that $\displaystyle\int_a^b f(mx)\,dx=\frac{1}{m}\int_{ma}^{mb} f(x)\,dx.$

34. Prove that $\displaystyle\int_1^2 e^{-x^2}\,dx<\int_1^2 xe^{-x^2}\,dx$, and hence find a number below which it must lie.

35. Prove that $\displaystyle\int_0^1 x^n(1-x)^m\,dx=\int_0^1 x^m(1-x)^n\,dx.$

148. Extension of Theorem of Art. 144.

It has been assumed in Art. 144 in evaluating $\displaystyle\int_a^b f(x)\,dx$ that the extreme values of the range of integration are both finite, and that the function $f(x)$ is continuous, and therefore finite, throughout that range. Cases frequently occur in which these conditions are not satisfied, and it remains to consider how far the results of that article may be extended to such cases.

It has been shown that

$$\int_a^b f(x)\,dx=F(b)-F(a),\quad \text{where } F'(x)=f(x).$$

If, as $b \longrightarrow \infty$, $F(b)$ tends to a finite limit L, then $L - F(a)$ is defined as the value of $\int_a^\infty f(x)\,dx$.

Similarly, if as $a \longrightarrow -\infty$, $F(a)$ tends to a finite limit L', then $F(b) - L'$ is defined as the value of $\int_{-\infty}^b f(x)\,dx$.

Examples:

(i) $\int_a^b x^{-2}\,dx$ (where a and b are positive, so that the value 0 for which the function $1/x^2$ is discontinuous is not within the range of integration)

$$= \Big[-1/x\Big]_a^b = 1/a - 1/b.$$

As $b \rightarrow \infty$, $1/b \rightarrow 0$; hence $\int_a^\infty x^{-2}\,dx = 1/a$.

(ii) $\int_a^b \frac{1}{x}\,dx = \Big[\log x\Big]_a^b = \log b - \log a.$

As $b \rightarrow \infty$, $\log b$ also $\rightarrow \infty$, hence $\int_a^b \frac{1}{x}\,dx$ has no value.

This example shows that the condition that $f(x) \rightarrow 0$ as $x \rightarrow \infty$ is not a sufficient condition that $\int^\infty f(x)\,dx$ may have a value.

(iii) $\int_0^b \frac{dx}{a^2+x^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^b = \frac{1}{a}\tan^{-1}\frac{b}{a}.$

As $b \rightarrow \infty$, $\tan^{-1}\frac{b}{a} \rightarrow \frac{\pi}{2}$; $\quad \frac{dx}{a^2+x^2} = \frac{\pi}{2a}.$

(iv) $\int_0^\theta e^{-ax}\cos bx\,dx\ [a+] = \left[\frac{e^{-ax}}{a^2+b^2}(b\sin bx - a\cos bx)\right]_0^\theta$ (Art. 139)

$$\frac{e^{-a\theta}}{a^2+b^2}(b\sin b\theta - a\cos l\quad) + \frac{-}{a^2+b^2}.$$

When $\theta \rightarrow \infty$, $\sin b\theta$ and $\cos b\theta$ are finite since they cannot be greater than 1, and $e^{-a\theta}$, i.e. $1/e^{a\theta}$, $\rightarrow 0$.

$$e^{-ax}\cos bx\,dx \quad \frac{a}{a^2+b^2}.$$

Next, suppose that $f(x)$ becomes infinite at one of the extremities of the range of integration. Let $f(x) = \infty$ when $x = b$.

Taking the integral $\int_a^{b-\epsilon} f(x)\,dx$, $f(x)$ is finite throughout this range of integration. If, as $\epsilon \longrightarrow 0$, $\int_a^{b-\epsilon} f(x)\,dx$ tends to a finite limit L, this value L is defined as the value of $\int_a^b f(x)\,dx$.

Similarly, if $f(x) = \infty$ when $x = a$ and if, as $\epsilon \to 0$, $\int_{a+\epsilon}^{b} f(x)\,dx$ tends to a finite limit L', then L' is defined as the value of $\int_a^b f(x)\,dx$.

In practice it is usually at an end of the range of integration that $f(x)$ becomes infinite. If $f(x)$ becomes infinite for a single value $x = c$ within the range of integration, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

and each of the latter integrals may be treated in the manner just described.

Similarly, if $f(x)$ becomes infinite at any (finite) number of points within the range of integration, the integral may be split up into a number of integrals in which the infinite values occur at extremities of the ranges of integration.

Examples :

(i) $\int_0^a x^{-\frac{1}{2}}\,dx$; $\quad x^{-\frac{1}{2}}$ is infinite when $x = 0$.

$$\int_\epsilon^a x^{-\frac{1}{2}}\,dx = \left[2x^{\frac{1}{2}}\right]_\epsilon^a = 2a^{\frac{1}{2}} - 2\epsilon^{\frac{1}{2}}. \quad \text{As } \epsilon \to 0, \ \epsilon^{\frac{1}{2}} \to 0.$$

$$\int_0^a x^{-\frac{1}{2}}\,dx = 2a^{\frac{1}{2}}.$$

(ii) $\int_1^a \frac{dx}{(x-1)^2}$; $\quad \frac{1}{(x-1)^2}$ is infinite when $x = 1$.

$$\int_{1+\epsilon}^a \frac{dx}{(x-1)^2} = \left[-\frac{1}{x-1}\right]_{1+\epsilon}^a = -\frac{1}{a-1} + \frac{1}{\epsilon}.$$

$1/\epsilon \to \infty$ as $\epsilon \to 0$, therefore this integral has no value.

(iii) $\int_0^4 \frac{dx}{\sqrt[3]{(x-3)}}$; $\quad \frac{1}{\sqrt[3]{(x-3)}}$ becomes infinite when $x = 3$, which is within the range of integration.

Hence we write $\displaystyle\int_0^4 \frac{dx}{\sqrt[3]{(x-3)}} = \int_0^3 \frac{dx}{\sqrt[3]{(x-3)}} + \int_3^4 \frac{dx}{\sqrt[3]{(x-3)}}$.

Now $\displaystyle\int_0^{3-\epsilon} \frac{dx}{\sqrt[3]{(x-3)}} = \left[\frac{3}{2}(x-3)^{2/3}\right]_0^{3-\epsilon} = \frac{3}{2}(-\epsilon)^{2/3} - \frac{3}{2}(-3)^{2/3}$,

which tends to the limit $-\frac{3}{2}(-3)^{2/3}$ as $\epsilon \to 0$.

Also $\displaystyle\int_{3+\epsilon}^4 \frac{dx}{\sqrt[3]{(x-3)}} = \left[\frac{3}{2}(x-3)^{2/3}\right]_{3+\epsilon}^4 = \frac{3}{2}(1) - \frac{3}{2}(\epsilon)^{2/3}$,

which tends to the limit $\frac{3}{2}$ as $\epsilon \to 0$.

Hence $\displaystyle\int_0^4 \frac{dx}{\sqrt[3]{(x-3)}} = -\frac{3}{2}(-3)^{2/3} + \frac{3}{2} = \frac{3}{2}(1 - \sqrt[3]{9})$.

Geometrical illustration.

To illustrate the matter geometrically, consider $\int_a^b \frac{1}{x^2}\,dx$.

If P and Q (Fig. 95) be the two points on the graph of $y = 1/x^2$ whose

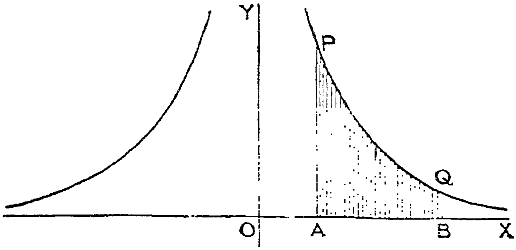


Fig. 95.

abscissae are a and b, $\int_a^b \frac{1}{x^2}\,dx$ represents the area $PABQ$.

Now $\int_a^b \frac{1}{x^2}\,dx = \frac{1}{a} - \frac{1}{b}$, which, as $b \to \infty$, tends to the limit $\frac{1}{a}$;

$$\int_a^\infty \frac{1}{x^2}\,dx = \frac{1}{a}.$$

When $a = 0$, $\frac{1}{x^2}$ is infinite; and as $a \to 0$, $\frac{1}{a} \to \infty$; $\therefore \int_0^b \frac{1}{x^2}\,dx$ has no value.

Hence the area $APQB$ tends to a definite limit as the ordinate BQ recedes to an infinite distance (AP remaining fixed), but has no limit as the ordinate AP approaches the axis of y; as great an area as we please can be obtained by taking AP sufficiently near to the axis of y.

Examples LVIII.

Find, when they exist, the values of the following:

1. $\int_1^\infty \frac{dx}{x^3}$.
2. $\int_{-\infty}^{-2} \frac{dx}{x^{4/3}}$.
3. $\int_1^\infty \frac{dx}{\sqrt{x}}$.
4. $\int_0^1 \frac{dx}{x^{2/3}}$.
5. $\int_0^1 \frac{dx}{x}$.
6. $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$.
7. $\int_0^4 \frac{dx}{(x-1)^2}$.
8. $\int_{-1}^8 \frac{dx}{\sqrt[3]{(7-x)}}$.
9. $\int_0^1 \frac{dx}{\sqrt{(1-x^2)}}$
10. $\int_{-\infty}^\infty \frac{dx}{a^2+b^2x^2}$.
11. $\int_0^1 \frac{1+x}{1-x}\,dx$.
12. $\frac{x}{1+x^2}\,dx$.
13. $\int_0^\infty e^{-x}\sin x\,dx$.
14. $\int_0^1 \frac{dx}{\sqrt{[x(1-x)]}}$.
15. $\int_0^\infty$
16. $\int_0^a \frac{x\,dx}{\sqrt{(a^2-x^2)}}$.
17. $\int_0 \log x\,dx$.
18. $\int_1^\infty \frac{dx}{x^2+4x+13}$

19. $\int_0^1 \frac{dx}{x(2+x)}$. 20. $\int_1^\infty \frac{dx}{x(1+x)}$. 21. $\int_1^\infty \frac{dx}{x^2(1+x)}$

22. $\int \frac{dx}{x^2(1+x^2)}$. 23. $\int_\alpha^\beta \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}$. 24. $\int_{-\infty}^\infty \frac{dx}{\cosh x}$.

25. Prove that, if n be positive,

$$x^n e^{-x}\,dx = n\int_0^\infty x^{n-1}e^{-x}\,dx.$$

Hence evaluate the former integral if n be an integer.

149. An important definite integral: $\int^{\frac{1}{2}\pi} \sin^m\theta\cos^n\theta\,d\theta$.

One of the most important definite integrals is $\int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^n\theta\,d\theta$, when m and n are positive integers.

It has been shown (Art. 141) that

$$\int \sin^m\theta\cos^n\theta\,d\theta = \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+n} + \frac{n-1}{m+n}\int \sin^m\theta\cos^{n-2}\theta\,d\theta.$$

The evaluation of the above definite integral is rendered very simple by the fact that the first term on the right-hand side vanishes for both the values 0 and $\frac{1}{2}\pi$, $\cos\theta$ being 0 when $\theta=\frac{1}{2}\pi$, and $\sin\theta$ when $\theta=0$.

$$\therefore \quad \int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^n\theta\,d\theta = \frac{n-1}{m+n}\int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^{n-2}\theta\,d\theta,$$

in which the index of $\cos\theta$ is reduced by 2.

By Art. 146 IV, the integral is unchanged when $\frac{1}{2}\pi-\theta$ is substituted for θ;

$$\therefore \quad \int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^n\theta\,d\theta = \int_0^{\frac{1}{2}\pi} \sin^n\theta\cos^m\theta\,d\theta$$

$$= \frac{m-1}{m+n}\int_0^{\frac{1}{2}\pi} \sin^n\theta\cos^{m-2}\theta\,d\theta = \frac{m-1}{m+n}\int_0^{\frac{1}{2}\pi} \sin^{m-2}\theta\cos^n\theta\,d\theta$$

(again replacing θ by $\frac{1}{2}\pi-\theta$), in which the index of $\sin\theta$ is reduced by 2. This result may also be obtained from the reduction formula connecting $I_{m,\,n}$ and $I_{m-2,\,n}$ (Art. 142).

The integrals on the right-hand side can be reduced a stage further by using these results with $n-2$ for n, and $m-2$ for m respectively.

$$\therefore \quad \int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^n\theta\,d\theta = \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^{n-4}\theta\,d\theta,$$

$$\text{or } \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\int_0^{\frac{1}{2}\pi} \sin^{m-4}\theta\cos^n\theta\,d\theta.$$

By repeated use of these results, the numerical factors in the numerator will be $n-1$, $n-3$, $n-5$, ... down to either 2 or 1 (according as n is odd or even) as the index of $\cos\theta$ is gradually reduced, and $m-1$, $m-3$, $m-5$, ... down to either 2 or 1 (according as m is odd or even) as the index of $\sin\theta$ is gradually reduced; at any stage the last factor in the denominator exceeds by 2 the sum of the remaining indices of $\sin\theta$ and $\cos\theta$.

If one index is odd and the other even, $m+n$ is odd and the integral reduces to either $\int_0^{\frac{1}{2}\pi} \sin\theta\, d\theta$ or $\int_0^{\frac{1}{2}\pi} \cos\theta\, d\theta$, either of which is equal to 1; the last factor in the denominator is 3.

If both indices are odd, $m+n$ is even, and the integral reduces to $\int_0^{\frac{1}{2}\pi} \sin\theta\cos\theta\, d\theta$, i.e. $\frac{1}{2}\int_0 \sin 2\theta\, d\theta$, which is equal to $\frac{1}{2}$. The last factor in the denominator of the coefficient of $\int_0^{\frac{1}{2}\pi} \sin\theta\cos\theta\, d\theta$ is 4; hence, in the final result, the last factor is 2.

If both indices are even, $m+n$ is even, and the integral reduces to $\int_0^{\frac{1}{2}\pi} d\theta$, which is $\frac{1}{2}\pi$, and the last factor in the preceding denominator is 2.

Hence we have the following simple rule for writing down the value of the integral:

$$\int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^n\theta\, d\theta = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots},$$ followed by the factor $\frac{1}{2}\pi$ only when m and n are both even.* All the three sets of factors descend 2 at a time to either 1 or 2 according as the first factor of the set is odd or even.

The value of the integral when the limits are multiples of $\frac{1}{2}\pi$ can be obtained from the preceding case by the aid of the theorems of Art. 146.

Examples:

$$\int^{\frac{1}{2}\pi} \sin^5\theta\cos^3\theta\, d\theta = \frac{4\,.\,2\,.\,2}{6\,.\,4\,.\,2} = \frac{1}{24}.$$

$$\int_0^{\frac{1}{2}\pi} \sin^6\theta\cos^3\theta\, d\theta = \frac{5\,.\,3\,.\,1\,.\,2}{9\,.\,7\,.\,5\,.\,3\,.\,1} = \frac{\quad}{63}.$$

$$\int_0^{\frac{1}{2}\pi} \sin^4\theta\cos^6\theta\, d\theta = \frac{3\,.\,1\,.\,5\,.\,3\,.\,1}{10\,.\,8\,.\,6\,.\,4\,.\,2}\cdot\frac{\pi}{2} = \frac{3\pi}{512}.$$

* 0 counts as an even number.

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^2\theta\cos^3\theta\, d\theta = 2\int_0^{\frac{1}{2}\pi} \sin^2\theta\cos^3\theta\, d\theta = 2\cdot\frac{1.2}{5.3.1} = \frac{4}{15}.$$

$$\int_0^{2\pi} \sin^8\theta\, d\theta = 4\int_0^{\frac{1}{2}\pi} \sin^8\theta\, d\theta = 4\cdot\frac{7.5.3.1}{8.6.4.2}\cdot\frac{\pi}{2} = \frac{35\pi}{64}.$$

150. Change of limits of integration.

It has been seen that many algebraical expressions which contain irrational functions can be integrated by trigonometrical substitutions; in these cases, the transformation back from the angle θ to the original variable x is often troublesome, but in the corresponding definite integrals this may be avoided, since the value of the definite integral depends only upon the limits, and the limits for x may be replaced by the corresponding limits for θ.

Examples:

(i) $\int_0^a \sqrt{(a^2-x^2)}\, dx$ may be found by substituting $x = a\sin\theta$.

As x increases from 0 to a, $\sin\theta$ increases from 0 to 1, and therefore θ from 0 to $\frac{1}{2}\pi$; hence

$$\int_0^a \sqrt{(a^2-x^2)}\, dx = a^2\int_0^{\frac{1}{2}\pi} \cos^2\theta\, d\theta = a^2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\pi = \tfrac{1}{4}\pi a^2,$$

by the rule of the preceding article.

(ii) $\int_\alpha^\beta \sqrt{[(x-\alpha)(\beta-x)]}\, dx$ may be obtained by the substitution

$$x = \alpha\cos^2\theta + \beta\sin^2\theta. \quad \text{[Art. 137.]}$$

$\int\sqrt{[(x-\alpha)(\beta-x)]}\, dx$ becomes $\int(\beta-\alpha)\cos\theta\sin\theta\,.\,2(\beta-\alpha)\sin\theta\cos\theta\, d\theta$.

When $x=\alpha$, $\sin\theta = 0$ [since $x-\alpha = (\beta-\alpha)\sin^2\theta$]; $\therefore$ $\theta = 0$.

When $x=\beta$, $\cos\theta = 0$ [since $\beta-x = (\beta-\alpha)\cos^2\theta$]; $\therefore$ $\theta = \frac{1}{2}\pi$.

$$\therefore \text{ the given definite integral} = 2(\beta-\alpha)^2\int_0^{\frac{1}{2}\pi} \sin^2\theta\cos^2\theta\, d\theta$$

$$= 2(\beta-\alpha)^2\cdot\frac{1.1}{4.2}\cdot\frac{\pi}{2}$$

$$= \tfrac{1}{8}\pi(\beta-\alpha)^2.$$

151. Reduction of algebraical expressions to preceding form.

The integrals of many other algebraical expressions, rational as well as irrational, can, by trigonometrical substitutions, be reduced to the form $\int\sin^m\theta\cos^n\theta\, d\theta$ with multiples of $\frac{1}{2}\pi$ as limits, in which case their values can be written down by the preceding rule.

The following examples show types of expressions which can be so reduced and the methods of reducing them.

Examples:

(i) $\displaystyle\int_0^a x^2(a^2-x^2)^{3/2}\,dx.$

If $x = a\sin\theta$, we get $\displaystyle\int_0^{\frac{1}{2}\pi} a^2\sin^2\theta\,.\,(a^2\cos^2\theta)^{3/2}\,.\,a\cos\theta\,d\theta$

$$= a^6\int_0^{\frac{1}{2}\pi}\sin^2\theta\cos^4\theta\,d\theta = a^6\,.\,\frac{1\,.\,3\,.\,1}{6\,.\,4\,.\,2}\,.\,\frac{\pi}{2} = \frac{\pi a^6}{32}$$

(ii) $\displaystyle\int_0^a x^2(a-x)^{3/2}\,dx.$

Here the substitution $x = a\sin\theta$ will not rationalize the expression, but $x = a\sin^2\theta$ will, and the limits for θ will be 0 and $\frac{1}{2}\pi$.

The integral $= \displaystyle\int^{\frac{1}{2}\pi} a^2\sin^4\theta\,(a\cos^2\theta)^{3/2}\,.\,2a\sin\theta\cos\theta\,d\theta$

$$= 2a^{9/2}\int_0^{\frac{1}{2}\pi}\sin^5\theta\cos^4\theta\,d\theta = 2a^{9/2}\,.\,\frac{4\,.\,2\,.\,3\,.\,1}{9\,.\,7\,.\,5\,.\,3\,.\,1} = \frac{16}{315}$$

(iii) $\displaystyle\int_{-\infty}^{\infty}\frac{x^4}{(a^2+x^2)^3}\,dx.$

In this case, if $x = a\tan\theta$, $a^2+x^2 = a^2\sec^2\theta$, $dx/d\theta = a\sec^2\theta$.

When $x = +\infty$, $\theta = +\frac{1}{2}\pi$; when $x = -\infty$, $\theta = -\frac{1}{2}\pi$;

$$\therefore \text{ the given integral} = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{a^4\tan^4}{a^6\sec^6\theta}\,a\sec^2\theta\,d\theta = \frac{1}{a}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\tan^4\theta}{\sec^4\theta}\,d\theta$$

$$= \frac{1}{a}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin^4\ d\theta = \frac{2}{a}\int_0^{\frac{1}{2}\pi}\sin^4\theta\,d\theta$$

$$= \frac{2}{a}\,.\,\frac{3\,.\,1}{4\,.\,2}\,.\,\frac{\pi}{2} = \frac{3\pi}{8a}.$$

Examples LIX.

Find the values of the following:

1. $\displaystyle\int_0^{\frac{1}{2}\pi}\sin^4\theta\cos^4\theta\,d\theta.$ 2. $\displaystyle\int_0^{\frac{1}{2}\pi}\sin^3\theta\cos^7\theta\,d\theta.$ 3. $\displaystyle\int_0^{\frac{1}{2}\pi}\sin^5\theta\cos^2\theta\,d\theta.$

4. $\displaystyle\int_0^{\frac{1}{2}\pi}\cos^{10}\theta\,d\theta.$ 5. $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin^2\theta\cos^7\theta\,d\theta.$ 6. $\displaystyle\int_0^{\pi}\sin^7\theta\,d\theta.$

7. $\displaystyle\int_0^{2\pi}\sin^2\theta\cos^6\theta\,d\theta.$ 8. $\displaystyle\int_0^{\pi}\cos^6\theta\sin\theta\,d\theta.$ 9. $\displaystyle\int_{-2\pi}^{2\pi}\cos^4\theta\,d\theta.$

10. $\displaystyle\int_0^{\frac{1}{2}\pi}\sin^6\theta\cos^5\theta\,d\theta.$ 11. $\displaystyle\int_{-a}^{a}\sqrt{(a^2-x^2)}\,dx.$ 12. $\displaystyle\int_0^{4}(16-x^2)^{3/2}\,dx.$

13. $\displaystyle\int_0^{a} x^3(a^2-x^2)^{1/2}\,dx.$ 14. $\displaystyle\int_0^{a} x^2(a^2-x^2)^{7/2}\,dx.$ 15. $\displaystyle\int_0^{2} x^4(2-x)^{1/2}\,dx.$

16. $\displaystyle\int_0^{\infty}\frac{x^3}{(a^2+x^2)^4}\,dx.$ 17. $\displaystyle\int_0^{\sqrt{3/2}}(3-4x^2)^{5/2}\,dx.$ 18. $\displaystyle\int_{-\infty}^{\infty}\frac{}{(1+x^2)^{5/2}}\,dx.$

19. $\int_0^a x^3 (a-x)^3\, dx.$ 20. $\int_0^1 x^{3/2} (1-x)^{1/2}\, dx.$ 21. $\int_0 \quad dx.$

22. $\int_0 \frac{x^3}{\sqrt{(1-x)}}\, dx.$ 23. $\int^a \sqrt{(ax-x^2)}\, dx.$ 24. $\int_0 \frac{x^2}{\sqrt{(2\,ax-x^2)}}\, dx.$

25. $\int_0^1 x^2 \sqrt{(x-x^2)}\, dx.$ 26. $\int_0^{\frac{3}{2}} x^3 \sqrt{(9-4x^2)}\, dx.$ 27. $\int_0^\infty \frac{x^3}{(a^2+x^2)^5}\, dx.$

28. $\int_0^\infty \frac{x}{(a+x)^4}\, dx.$ 29. $\int_\alpha^\beta \frac{dx}{\sqrt{[(x-\alpha)(\beta-x)]}}.$ 30. $\int_\alpha^\beta \sqrt{\left(\frac{x-\alpha}{\beta-x}\right)}dx.$

31. $\int_2^5 \sqrt{(7x-10-x^2)}\, dx.$ 32. $\int_{-1}^4 \sqrt{(3\,x-x^2+4)}\, dx.$

33. $\int_a^{3a} \sqrt{\left(\frac{x-a}{3a-x}\right)}\, dx.$ 34. $\int_{-a}^a \sqrt{\left(\frac{a+x}{a-x}\right)}\, dx.$

35. $\int^3 x\sqrt{(4x-3-x^2)}\, dx.$ 36. $\int_2^7 x\sqrt{\left(\frac{7-x}{x-2}\right)}\, dx.$

CHAPTER XVI

GEOMETRICAL APPLICATIONS

AREAS

152. Areas of Curves.

Some simple cases of the determination of areas (which is sometimes referred to as *quadrature*) have already been considered in Chapter IX. The following method, in which an area is regarded as the limit of a sum, yields the same result as the method of Art. 79.

Let AH, BK (Fig. 96) be the ordinates of two points A and B on a curve; and let HK be divided into very small equal parts, each δx.

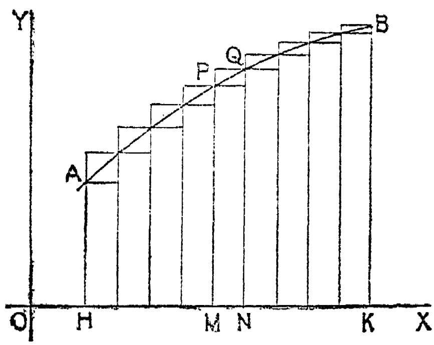


Fig. 96.

Let MP, NQ be the ordinates at two successive points of division, M and N. Complete the rectangles PN, QM; draw all the ordinates, and complete the rectangles in the same way.

The difference between the sum of the inner rectangles, $\Sigma(PN)$, and the sum of the outer rectangles, $\Sigma(QM)$, is equal to the sum of the small rectangles, $\Sigma(PQ)$, and this is equal to the length of their common base, δx, multiplied by the sum of their heights, i.e. to $\delta x(BK-AH)$, which can be made as small as we please by taking δx sufficiently small. Hence the two sums of rectangles have a common limit, and this is the area of the figure $HABK$.

Therefore the area $HABK$

$$= \operatorname*{L}t\,\Sigma(PN) = \operatorname*{L}t_{\delta x\to 0} \sum_{x=a}^{x=b} (y\,\delta x), \quad \text{if } OH = a \text{ and } OK = b,$$

$$= \int_a^b y\,dx \quad \text{(by the definition of Art. 143)}$$

$$= \int_a^b f(x)\,dx, \quad \text{if } y = f(x) \text{ be the equation of the curve } AB.$$

This gives the same rule as was obtained in Art. 80.

Since y is + or − according as the point (x, y) is above or below the axis of x, it follows that, if δx be taken positive, the value obtained for the area is + or − according as the area is above or below the axis of x.

This accounts for results such as the following: If, to find the area between the graph of $y = \sin x$ and the axis of x from $x = 0$ to $x = 2\pi$, we take $\int_0^{2\pi} \sin x\,dx$, we get $\Big[-\cos x\Big]_0^{2\pi}$, which is equal to 0.

It is obvious from the graph that the area from $x = 0$ to $x = \pi$ is above the axis of x, and the area from $x = \pi$ to $x = 2\pi$ below the axis of x. The preceding integral gives the sum of these areas with opposite signs, and therefore merely indicates that the areas above and below the axis are equal in magnitude. The area of each part is numerically

$$\int_0^{\pi} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi} = 2.$$

In such cases the points where the curve cuts the axis should be found, and the areas on opposite sides of the axis determined separately.

In some cases the area required has to be divided into several parts, as shown in the following example:

Example. Find the area between the circle $x^2 + y^2 = 4a^2$ *and the ellipse* $x^2 + 5y^2 = 16a^2$.

Let P (Fig. 97) be a point of intersection of the circle and the ellipse.

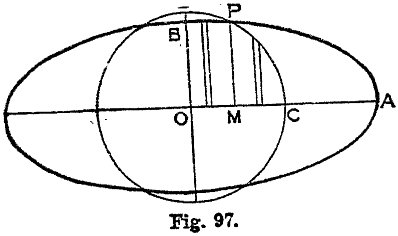


Fig. 97.

The required area $A = 4 \times \text{area } OBPC = 4 \times \text{area } OBPM + 4 \times \text{area } MPC$.

The coordinates of P are obtained by solving the equations of the curves. This gives $x = \pm a$, $y = \pm\sqrt{3}\,a$. C is the point $(2a, 0)$.

The ordinate of the circle is $\sqrt{(4a^2 - x^2)}$,

and the ordinate of the ellipse is $\sqrt{(16a^2 - x^2)}/\sqrt{5}$.

Hence $A = 4\int_0^{a} \sqrt{\tfrac{1}{5}}\cdot\sqrt{(16\,a^2 - x^2)}\,dx + 4\int_a^{2a} \sqrt{(4\,a^2 - x^2)}\,dx$

$$= 4\sqrt{\tfrac{1}{5}}\left[\tfrac{1}{2}x\sqrt{(16\,a^2-x^2)} + 8\,a^2\sin^{-1}\frac{x}{4\,a}\right]_0^a$$

$$+4\left[\tfrac{1}{2}x\sqrt{(4\,a^2-x^2)} + 2\,a^2\sin^{-1}\frac{x}{2\,a}\right]_a^{2a} \text{ (Art. 139)}$$

$$= 4\sqrt{\tfrac{1}{5}}\left[\tfrac{1}{2}a^2\sqrt{15} + 8\,a^2\sin^{-1}\tfrac{1}{4}\right] + 4\left[2\,a^2\,.\,\tfrac{1}{2}\pi - \tfrac{1}{2}a^2\sqrt{3} - 2\,a^2\,.\,\tfrac{1}{6}\pi\right]$$

$$= a^2\left(32\sqrt{\tfrac{1}{5}}\,.\,\sin^{-1}\tfrac{1}{4} + \tfrac{8}{3}\pi\right)$$

$$= 12\,a^2, \text{ approximately.}$$

153. Area of cycloid.

As an example of the determination of an area when the coordinates are each expressed in terms of a third variable, we will take the case of the cycloid (Art. 50), and find the area between one arch of the curve and the axis of x.

The coordinates of a point on the cycloid are $x = a\,(\theta - \sin\theta)$, $y = a\,(1-\cos\theta)$, where θ is the angle turned through by a fixed radius.

$\int y\,dx = \int y\,\dfrac{dx}{d\theta}\,d\theta$ (Art. 131), and in describing one arch, the angle turned through is $2\,\pi$.

Hence the required area

$$= \int_0^{2\pi} y\,\frac{dx}{d\theta}\,d\theta = \int_0^{2\pi} a(1-\cos\theta)\,.\,a(1-\cos\theta)\,d\theta = a^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta.$$

Now, by Theorem V, Art. 146,

$$\int_0^{2\pi}\cos\theta\,d\theta = 0, \text{ and } \int_0^{2\pi}\cos^2\theta\,d\theta = 4\int_0^{\frac{1}{2}\pi}\cos^2\theta\,d\theta = 4\,.\,\tfrac{1}{2}\,.\,\tfrac{1}{2}\pi = \pi.$$

Therefore the required area $= a^2\,[2\,\pi+\pi] = 3\,\pi a^2$, i.e. three times the area of the rolling circle.

154. Area of a closed oval curve.

Let AH and BK (Fig. 98) be tangents to the curve parallel to the

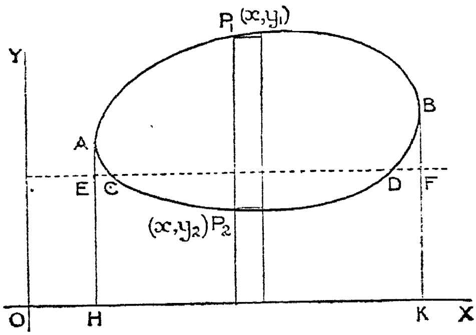


Fig. 98.

y-axis. Any intermediate ordinate will cut the curve in two points $P_1(x, y_1)$ and $P_2(x, y_2)$; let y_1 be $> y_2$.

Then, if $OH=a$, $OK=b$, the area $HAP_1BK=\int_a^b y_1\,dx$,

and the area $HAP_2BK=\int_a^b y_2\,dx$;

therefore, by subtraction, the area $AP_1BP_2=\int_a^b (y_1-y_2)\,dx$.

Or, parallel ordinates may be drawn dividing the area into strips perpendicular to the x-axis, and the area is the limit of the sum of the rectangles $P_1P_2\times\delta x$, i.e. the limit of $\Sigma(y_1-y_2)\,\delta x$ taken between $x=a$ and $x=b$, i.e. $\int_a^b (y_1-y_2)\,dx$.

It is easily seen that this expression gives the whole area, whether the curve cuts the x-axis or not.

In Fig. 98 it does not; if, however, the axis of x, as shown by the dotted line $ECDF$, cuts the curve (below A and B) in C and D, and the ordinates AH and BK in E and F, then $\int_a^b y_1\,dx$ gives the area EAP_1BF, and $\int_a^b y_2\,dx$ gives the areas ECA and DFB (which are above the x-axis) with $+$ sign, and the area CP_2D (which is below the x-axis) with $-$ sign; therefore, on subtracting, the common areas ECA and DFB disappear, and there remains $CAP_1BD-(-CDP_2)$, i.e. the area AP_1BP_2 of the closed curve.

The same result follows more readily from the facts that the area is the limit of $\Sigma(P_1P_2\times\delta x)$, and that no upward or downward movement of the x-axis affects the length P_1P_2, which is y_1-y_2.

It should be carefully noticed that the limits are the values of x for which y_1 and y_2 are equal.

The following example illustrates the application of the method:

Example. Find the area of the curve $3x^2-10xy+10y^2=2$.

The two values of y which correspond to any particular value of x are found by solving the equation as a quadratic for y in terms of x.

Thus $$10y^2-10xy+3x^2-2=0,$$

whence $$y=\tfrac{1}{20}\{10x\pm\sqrt{[20(4-x^2)]}\}.$$

These two values are y_1 and y_2, hence

$$y_1-y_2=\tfrac{1}{10}\sqrt{[20(4-x^2)]}=\tfrac{1}{5}\sqrt{[5(4-x^2)]}.$$

The limits are the values of x for which $y_1 - y_2 = 0$, and therefore are given by $x^2 = 4$, i.e. $x = \pm 2$.

Hence the area of the curve

$$= \int_{-2}^{2} \tfrac{1}{5}\sqrt{[5(4-x^2)]}\,dx$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tfrac{1}{5}\sqrt{5}\,.\,2\cos\theta\,.\,2\cos\theta\,d\theta \qquad (\text{if } x = 2\sin\theta)$$

$$= \tfrac{4}{5}\sqrt{5}\times 2\int_0^{\frac{1}{2}\pi}\cos^2\theta\,d\theta$$

$$= \tfrac{8}{5}\sqrt{5}\times\tfrac{1}{2}\,.\,\tfrac{1}{2}\pi = 2\pi/\sqrt{5}.$$

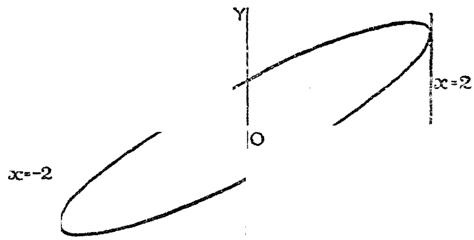


Fig. 99.

The curve is an ellipse whose centre is the origin, and is shown in Fig. 99. $y = \pm 1$ when $x = \pm 2$, and $y = \pm 1/\sqrt{5} = \pm\cdot 45$ when $x = 0$. Also when $y = 0$, $3x^2 = 2$ and $x = \pm\cdot 82$.

Examples LX.

1. Find the area of the curve $y^2 = x^2(4-x^2)$.
2. Find the area of $y^2 = x^4(9-x^2)$.
3. Obtain the area between the curve $y^2 = x^3/(2a-x)$ and its asymptote. This curve is called the *cissoid*.
4. Find the area of the curve $x = a\cos^3\theta$, $y = b\sin^3\theta$.
5. Find the area of the ellipse $y^2 = (x-2)(9-2x)$.
6. Show that the area between the curve $xy^2 = a^2(a-x)$ and its asymptote is equal to the area of a circle of radius a. This curve is called the *witch of Agnesi*.
7. Find the area of each loop of the curve $x = a\sin 2\theta$, $y = b\cos\theta$.
8. Find the area of the loop of $ay^2 = 4x^2(a-x)$.
9. Find the area cut off from the parabola $y^2 = 16x$ by the straight line $y = 3x$.
10. Also the area between the two parabolas $y^2 = 20x$ and $x^2 = 16y$.
11. Draw the curve $(a-x)y^2 = x^2(a+x)$; and find (i) the area of the loop, (ii) the area between the curve and its asymptote. This curve is called the *strophoid*.
12. The rectangular hyperbola $x^2 - y^2 = 3a^2$ divides the circle $x^2 + y^2 = 5a^2$ into 3 parts; find the area of each part.

13. Find the area of the oval of the curve $ay^2 = (x-a)(x-5a)^2$, and find its ratio to the area of the circumscribing rectangle with sides parallel to the axes.

14. Prove that the area between the catenary $y = a\cosh(x/a)$, the axes, and any ordinate PN is double the area of the triangle PNL, where NL is the perpendicular from N to the tangent at P.

15. Find the area cut off from the hyperbola $x^2/a^2 - y^2/b^2 = 1$ by the double ordinate $x = 2a$. Find also the area between the curve and the lines which join the ends of the double ordinate to the origin.

16. Find the area of the ellipse $x^2 + 4y^2 - 6x + 8y + 9 = 0$.

17. Also of $9x^2 + 16y^2 - 90x - 64y - 119 = 0$.

18. Find the area of the curve $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$.

19. Find the area common to the ellipses $x^2/a^2 + y^2/b^2 = 1$ and $x^2/b^2 + y^2/a^2 = 1$.

20. Obtain the area of the ellipse whose equation is $(y-x)^2 = (x-1)(3-x)$.

21. Find the area of the curve $x^{2/5} + y^{2/5} = a^{2/5}$.
[Put $x = a\cos^5\theta$, $\therefore$ $y = a\sin^5\theta$].

22. Draw the curve * $(x^2 + 4a^2)y = 8a^3$, and find the area between the curve and its asymptote.

23. Find the area of the curve $x = 2a\cos\theta + a\cos 2\theta$, $y = 2a\sin\theta + a\sin 2\theta$.

24. Find the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

25. If $y = f(x)$ be a closed curve, show that $y = bf(x/a)$ is also a closed curve, whose area is ab times the area of the former curve.

155. Approximate integration.

The preceding method of finding an area requires (i) that we know the equation of the curve, and (ii) that the equation may give y as an integrable function of x. In practical work, a curve is often plotted from a number of isolated observations or drawn by some mechanical device, so that the equation of the curve is not known. Various methods have been devised for finding an approximate value for the area of such a curve. Moreover, since the integral of any function may be represented graphically by an area bounded by the graph of the function and the axis of x, these methods may be used to find an approximate value for the integral of a function whose graph can be drawn, but which does not yield to any of the ordinary methods of integration.

For instance, if the value of $\int_0^1 (1+x^3)^{-\frac{1}{2}}\,dx$ be required, we cannot find the indefinite integral of $1/\sqrt{(1+x^3)}$ in terms of elementary functions, and therefore cannot find the exact value of the definite integral by the method of Art. 144, but by plotting the curve $y^2 = 1/(1+x^3)$ carefully, and using one of the following methods to find approximately the area between the curve, the axes, and the ordinate $x = 1$, we obtain an approximate value for $\int_0^1 y\,dx$, i.e. for the given definite integral $\int_0^1 (1+x^3)^{-\frac{1}{2}}\,dx$.

* This is the curve in Question 6, with x and y interchanged, and a replaced by $2a$.

The simplest and most obvious way of approximating to the area between a curve and the axis of x is to draw a number of equidistant ordinates, and then, by joining their extremities, obtain a series of trapeziums whose areas can be found by elementary geometry; the sum of these areas will be less or greater than the actual area according as the curve is concave downwards or upwards. By drawing tangents at the ends of alternate ordinates and producing these tangents to meet the consecutive ordinates on either side, we obtain a number of trapeziums, the sum of whose areas is greater or less than the actual area according as the curve is concave downwards or upwards. Hence the actual area is intermediate in value between the sums of the areas of these two sets of trapeziums. The more ordinates are drawn, the more accurate is the approximation obtained, and the error involved in the approximation is obviously less than the difference between the two sums.

A more accurate approximation than is possible by this method is obtained by the rule known as 'Simpson's Rule'.

156. Simpson's Rule.

Let y_1, y_2, y_3 be the ordinates of three points A, B, and C (Fig. 100), whose abscissae are $a-h$, a, and $a+h$, and let h be small.

Suppose $y=f(x)$ to be the equation of the curve (the form of the function $f(x)$ may not be known); let P be any point on the curve between A and C, whose coordinates are (x, y).

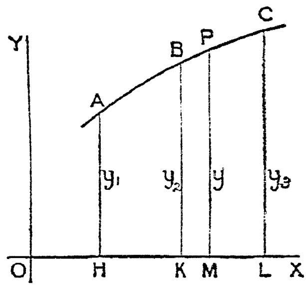


Fig. 100.

$$\text{The area } HACL = \int_{a-h}^{a+h} y\,dx$$
$$= \int_{a-h}^{a+h} f(x)\,dx.$$

Let $x=a+z$; then as x increases from $a-h$ to $a+h$, z increases from $-h$ to h;

$$\therefore \text{ the area } = \int_{-h}^{h} f(a+z)\,dz.$$

By the extended Mean-Value Theorem (Art. 119),

$$f(a+z)=f(a)+zf'(a)+\tfrac{1}{2}z^2 f''(a+\theta z). \tag{i}$$

If z is very small, the last term will only differ by a very small quantity* from $\frac{1}{2}z^2 f''(a)$.

* If $f(x)$ is a quadratic function of x, say px^2+qx+r, $f''(x)$ has the constant value $2p$ for all values of x, and hence in this case the following expression for the area is exact, and not merely an approximation.

An approximation can also be made to the value of this integral by expanding e^{-x^2} and retaining a few terms only; since x is not greater than 1, the terms will diminish fairly rapidly. E.g. if we neglect terms after the 6th,

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left[1 - x^2 + \frac{x^4}{2\,!} - \frac{x^6}{3\,!} + \frac{x^8}{4\,!} - \frac{x^{10}}{5\,!}\right] dx$$
$$= \left[x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5 - \tfrac{1}{42}x^7 + \tfrac{1}{216}x^9 - \tfrac{1}{1320}x^{11}\right]_0^1$$
$$= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \tfrac{1}{1320}$$
$$= \cdot 7467 \ldots.$$

Since the terms decrease and are alternately + and −, the error in this result is certainly

$$< \int_0^1 \frac{x^{12}}{6!}\,dx < \left[\frac{1}{6\,!}\cdot\frac{x^{13}}{13}\right]_0^1, \text{ i.e. } < \tfrac{1}{9360}, \text{ which is } \cdot 0001.$$

The above method may also be used to obtain approximate values for such numbers as π and e. For instance, since

$$\int_0^1 \frac{dx}{1+x^2} = \tfrac{1}{4}\pi, \quad \text{and} \quad \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)}} = \tfrac{1}{6}\pi,$$

we can, by dividing the ranges of integration into equal intervals, and treating the corresponding values of the functions as above, obtain approximate values for $\frac{1}{4}\pi$ and $\frac{1}{6}\pi$.

Also since $\displaystyle\int_1^2 \frac{dx}{x} = \log_e 2$, we can in a similar manner obtain an approximate value for $\log_e 2$, whence e can be found, since

$$\log_e 2 = \log_{10} 2/\log_{10} e \text{ (Art. 91).}$$

This gives $\log_{10} e$, and e is then found from a table of common logarithms.

157. Mean values.

If the range $b-a$ be divided into n equal intervals, the values of a function $f(x)$ when $x = a,\ a+h,\ a+2h, \ldots a+(n-1)h$, are

$$f(a),\ f(a+h),\ f(a+2h),\ \ldots\ f\{a+(n-1)h\} \text{ respectively,}$$

and the arithmetic mean of these values is

$$[f(a)+f(a+h)+ \ldots +f\{a+(n-1)h\}]/n.$$

Since $nh = b-a$, this may be written

$$h[f(a)+f(a+h)+ \ldots +f\{a+(n-1)h\}]/(b-a).$$

The limit to which this arithmetic mean tends, as n is indefinitely increased, is called the *mean value* of the function over the range $x = a$ to $x = b$. By Art. 143, this limit is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Geometrically, $\int_a^b f(x)\,dx$ is the area $HABK$ (Fig. 101), and $b-a$ is HK; therefore the mean value is represented by the height of the rectangle on the base HK, which has the same area as $HABK$. It will evidently be equal to the value of the function at some point M within the range. (See Theorem VI, Art. 146.)

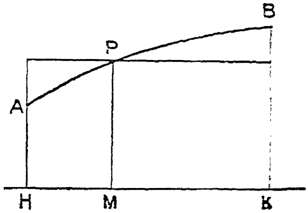


Fig. 101.

Examples:

(i) *The mean value of* $\sin x$ *between* $x=0$ *and* $x=\pi$

$$=\frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi} = \cdot 6366.$$

(ii) *The mean value of* $\sin^2 x$ *between* $x=0$ *and* $x=$

$$=\frac{1}{\pi}\int_0^\pi \sin^2 x\,dx = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\sin^2 x\,dx = \frac{2}{\pi}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{1}{2}.$$

The latter integral is important in the theory of alternating currents in Electricity.

If the quantity whose mean value is required can be expressed as a function of one or other of several variables, it is important to notice which is the variable to which equal increments are given.

(iii) *A particle is projected vertically upwards with velocity of* 80 *feet per second; find the mean value of the velocity up to the highest point.*

The velocity may be considered as a function of the time, or of the distance from the starting point.

(a) *For equal increments of time.* The time to the highest point is $2\frac{1}{2}$ seconds, and $v=80-32t$ gives the velocity at any instant.

$$\therefore \text{ the mean velocity } = \frac{1}{2\cdot 5}\int_0^{2\cdot 5}(80-32t)\,dt = \tfrac{2}{5}\Big[80t-16t^2\Big]_0^{2\cdot 5}$$
$$= \tfrac{2}{5}[200-100] = 40 \text{ ft. secs.};$$

which is obvious, *a priori*, since the velocity decreases uniformly as the time increases.

(b) *For equal increments of distance.* The total distance is 100 feet, and $v^2 = 6400-64s$ gives the velocity at the height s above the point of projection.

$$\therefore \text{ the mean velocity } = \tfrac{1}{100}\int_0^{100}\sqrt{(6400-64s)}\,ds = \tfrac{8}{100}\int_0^{100}\sqrt{(100-s)}\,ds$$
$$= \tfrac{8}{100}\Big[-\tfrac{2}{3}(100-s)^{3/2}\Big]_0^{100} = \tfrac{8}{100}\cdot\tfrac{2}{3}\cdot 1000 = 53\tfrac{1}{3} \text{ ft. secs.}$$

(iv) *A quantity of steam expands so that it follows the law* $pv^{\cdot 8} = 1000$, p *being measured in pounds weight per square inch; find the mean pressure as* v *increases from* 2 *to* 5 *cubic inches.*

Here $p = 1000\,v^{-\frac{4}{5}}$, and the increase in volume $= 3$ cubic inches;

$$\text{the mean pressure} = \tfrac{1}{3}\int_2^5 1000\,v^{-\frac{4}{5}}\,dv = \tfrac{1000}{3}\Big[5\,v^{\frac{1}{5}}\Big]_2^5$$

$$= \tfrac{5000}{3}\left[5^{\frac{1}{5}} - 2^{\frac{1}{5}}\right] = 385 \text{ lb. wt. per sq. inch, nearly.}$$

Examples LXI.

Find approximate values for the following definite integrals 1–4:

1. $\int_0^1 \sqrt{(1+x^3)}\,dx.$ 2. $\int_1^2 e^{x^2}\,dx.$

3. $\int_0^{\frac{1}{2}\pi} \log(1+\sin x)\,dx.$ 4. $\int_0^{\frac{1}{2}\pi} \frac{\sin x}{x}\,dx.$

5. Find an approximate value for π by applying Simpson's Rule to integral $\int_0^1 \frac{dx}{1+x^2}$ (see end of Art. 156).

6. Find an approximate value for π from $\int_0 \frac{dx}{\sqrt{(1-x^2)}}$.

7. Find an approximate value for $\log_e 2$, and thence for e, from $\int_1^2 x^{-1}\,dx.$

8. Find the mean value of $\sqrt{(4+3x)}$ from $x=1$ to $x=5$.

9. Also of $1/x$ from $x=1$ to $x=10$.

10. In simple harmonic motion, $s = a\cos nt$. Find the mean value of the velocity during one quarter of a complete oscillation (i) for equal intervals of time, (ii) for equal intervals of distance.
 Find also the mean values of the acceleration.

11. Show that, in simple harmonic motion, the mean kinetic energy, with respect to the time, is half the maximum kinetic energy.

12. Find the mean value of the ordinate of a semicircle of radius r when taken at equal intervals, measured (i) along the diameter, (ii) along the arc.

13. A quantity of steam expands and follows the law $pv^{1\cdot 2} = 500$; find the mean value of the pressure as v increases from 3 to 8.

14. Find the mean value of $10\sin 250\,t$ as t increases from 0 to $\frac{1}{100}\pi$.

15. Find the mean value of C^2 where $C = 10\sin 4t$, when $4t$ increases by 2π.

16. Also where $C = a\cos(pt+\alpha)$ when $pt+\alpha$ increases by 2π.

17. Find the mean distance of points on the circumference of a circle from a fixed point on the circumference.

18. A number a is divided into two parts; find the mean value of their product.

19. Find the mean value of the ordinates of the parabola $y^2 = 4ax$ from $x=0$ to $x=4a$.

20. Find the mean value of the positive ordinates of the ellipse $x^2/a^2+y^2/b^2=1$.

21. The radius of a circle rotates uniformly about the centre; find the mean value of the ordinate of its extremity.

22. Find the area of a curve in which successive ordinates at intervals of ·2 inch are 3·5, 3·2, 2·8, 2·9, 3·3, 3·6, 4 inches.

23. Equidistant ordinates of a curve are 4·2, 4·55, 4·9, 5·17, 4·8 inches; estimate the area between the extreme ordinates, which are 3 inches apart.

24. Use Simpson's Rule to find the area between $xy=12$, the axis of x, $x=1$, $x=4$; and compare the result with the area found by integration.

25. Find, if $k=·2$, the value of $\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}$ to 5 figures.

VOLUMES

158. Volumes of solids of revolution.

The volume of a solid of revolution was defined in Art. 14 (4), and some simple cases have already been considered in Art. 81.

As, in Fig. 96, the area $AHKB$ is the limit of the sum of all the rectangles such as PN, so, if the curve APB rotates about the axis of x and thereby forms a solid of revolution, the volume of this solid is the limit of the sum of the cylinders generated by the rotation of these rectangles, i.e. the volume

$$=\text{Lt}\,\Sigma\,(\pi\,PM^2\,.\,MN)=\text{Lt}\sum_{x=a}^{x=b}\pi\,y^2\,\delta x=\int_a^b\pi\,y^2\,dx.$$

Examples:

(i) *Find the volume formed by the rotation of one arch of a cycloid about*

The volume

$$=\int\pi\,y^2\,dx=\int_0^{2\pi}\pi\,y^2\frac{dx}{d\theta}\,d\theta=\int_0^{2\pi}\pi a^2(1-\cos\theta)^2\,a(1-\cos\theta)\,d\theta$$

$$=\pi a^3\int_0^{2\pi}(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\,d\theta.$$

Now

$$\int_0^{2\pi}\cos\theta\,d\theta=0;\quad\int_0^{2\pi}\cos^2\theta\,d\theta=4\int_0^{\frac{1}{2}\pi}\cos^2\theta\,d\theta=\pi;\quad\int_0^{2\pi}\cos^3\theta\,d\theta=0;$$

(Theorem V, Art. 146.)

$\therefore$ the required volume $=\pi a^3[2\pi-0+3\pi-0]=5\pi^2a^3$.

The volume of a solid of revolution may be found in a similar manner, if the curve rotates about a straight line parallel to one of the axes of coordinates.

(ii) *Find the volume generated by the rotation of the figure bounded by a quadrant of a circle and the tangents at its extremities about one of these tangents.*

It is more important that the equation of the rotating circle be written in its simplest possible form, $x^2+y^2 = a^2$, than that the axis of the solid should be one of the axes of coordinates; hence let its centre O be taken as origin.

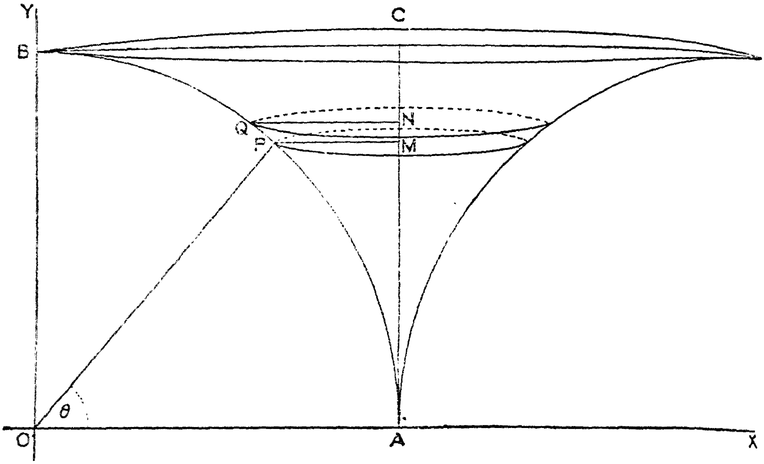


Fig. 102.

Let AC (Fig. 102) be the tangent about which the quadrant rotates; and let MP, NQ be the perpendiculars to this tangent at the ends of the small interval MN.

Then the volume required = the limit of the sum of the cylinders formed by the rotation of rectangles PN about AC

$$= \text{Lt}\, \Sigma\, \pi\, PM^2 \,.\, MN = \text{Lt}\, \Sigma\, \pi\, (a-x$$

$$= \int_0^a \pi\,(a-x)^2\, dy.$$

Let $x = a\cos\theta$, $y = a\sin\theta$; then the limits for θ are 0 and $\frac{1}{2}\pi$.

$$\therefore \text{ the volume} = \int_0^{\frac{1}{2}\pi} \pi a^2 (1-\cos\theta)^2\, a\cos\theta\, d\theta$$

$$= \pi a^3 \int_0^{\frac{1}{2}\pi} [\cos\theta - 2\cos^2\theta + \cos^3\theta]\, d\theta$$

$$= \pi a^3 [1 - 2\,.\,\tfrac{1}{2}\,.\,\tfrac{1}{2}\pi + \tfrac{2}{3}]$$

$$= \tfrac{5}{3}\pi a^3 - \tfrac{1}{2}\pi^2 a^3$$

$$= \tfrac{1}{6}\pi a^3 (10 - 3\pi).$$

159. Volume of any solid.

If the solid be divided up by planes perpendicular to the axis of x at distance δx apart, and if A be the area of the section by the plane which is at distance x from the origin, $A\,\delta x$ is the volume of the cylinder of base A and thickness δx, and the volume of the solid will be the limiting value of $\Sigma A\,\delta x$, i.e. $\int A\,dx$ taken between proper limits.

In Art. 158, $A = \pi y^2$; in some cases, A can be found in terms of x by an integration, and then a second integration will give the required volume. As examples of this method, we will take the following:

Examples:

(i) *Find the volume of a cone of height* h *standing on an elliptical base whose semi-axes are* a *and* b.

The equation of this ellipse, referred to its axes as axes of coordinates, is $x^2/a^2 + y^2/b^2 = 1$ [p. 19]. Its area $= 4\int_0^a y\,dx$. Taking the coordinates of a point on the ellipse in the form $x = a\cos\theta$, $y = b\sin\theta$ [Art. 50], the limits for θ are $\frac{1}{2}\pi$ and 0, hence the area

$$= 4\int_{\frac{1}{2}\pi}^{0} b\sin\theta \times -a\sin\theta\,d\theta = 4ab\int_0^{\frac{1}{2}\pi}\sin^2\theta\,d\theta = 4\,ab\,.\,\tfrac{1}{2}\,.\,\tfrac{1}{2}\,\pi = \pi\,ab.$$

Next, taking the perpendicular from the vertex of the cone to its base as

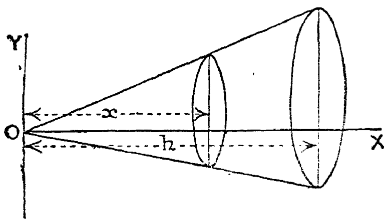


Fig. 103.

axis of x, the area A of a section perpendicular to OX (Fig. 103) at distance x from O is to the area of the base as $x^2 : h^2$; i.e. $A = \pi\,ab\,x^2/h^2$.

$\therefore$ the volume of the cone

$$\int_0^h A\,dx = \frac{\pi\,ab}{h^2}\int_0^h x^2\,dx = \frac{\pi\,ab}{h^2}\cdot \qquad = \tfrac{1}{3}\pi\,abh.$$

(ii) *Find the volume of an ellipsoid*, i.e. a solid figure such that the section by any plane parallel to the plane XOY, or perpendicular to either OX or OY, is an ellipse.

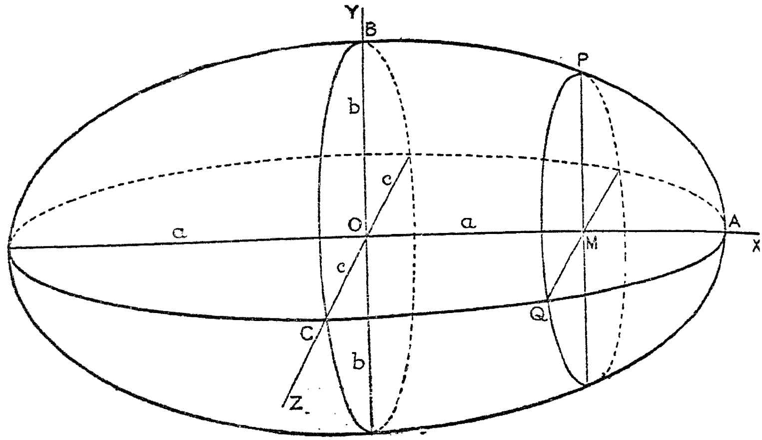


Fig. 104.

Let a, b be the semi-axes OA, OB (Fig. 104) of the section by the plane XOY; a, c the semi-axes OA, OC of the section by the plane through OX perpendicular to OY; therefore b, c are the semi-axes of the section by the plane through OY perpendicular to OX. a, b, c are called the axes of the ellipsoid.

Consider the section PQ by a plane perpendicular to OX at distance x from O. The area of this section is, by the preceding example, $\pi PM \cdot QM$.

Since P is a point on the ellipse AB, it follows that $x^2/a^2 + MP^2/b^2 = 1$;

$$\therefore \quad MP = b\sqrt{(a^2 - x^2)}/a.$$

Since Q is a point on the ellipse AC, it follows that $x^2/a^2 + MQ^2/c^2 = 1$;

$$\therefore \quad MQ = c\sqrt{(a^2 - x^2)}/a;$$

hence the area of the section $PQ = \pi bc(a^2 - x^2)/a^2$.

Therefore the volume of the ellipsoid

$$= \int_{-a}^{a} \frac{\pi bc}{a^2}(a^2 - x^2)\,dx$$

$$= \frac{2\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx = \frac{2\pi bc}{a^2}(a^3 - \tfrac{1}{3}a^3) = \tfrac{4}{3}\pi abc.$$

In some cases an approximation may be made to a volume by the use of Simpson's Rule. If, in Art. 156, y_1, y_2, ... denote the areas of the sections A_1, A_2, ... of a solid at equidistant intervals, the application of the rule will give an approximate value for $\int A\,dx$, the volume of the solid between the extreme sections.

If three sections only are taken, viz. the extreme sections A_1, A_3 and the section A_2 midway between them, Simpson's Rule gives the volume as $\frac{1}{3}h(A_1 + 4A_2 + A_3)$, a rule which is sometimes used in Mensuration.

(iii) *Five equidistant sections of a solid are circles of circumferences* 36. 42, 46, 50, *and* 52 *inches, their common distance apart being* 6 *inches; find the volume between the extreme sections.*

If r be the radius of the first section, $2\pi r = 36$; $\therefore$ $r = 18/\pi$, and the area $\pi r^2 = 324/\pi$ sq. inches. Similarly, the areas of the other sections are $441/\pi$, $529/\pi$, $625/\pi$, $676/\pi$ sq. inches.

$\therefore$ the volume $= \frac{1}{3}.6[324+676+2.529+4(441+625)]/\pi$
$= 2 \times 6322/\pi = 4025$ cubic inches, nearly.

Examples LXII.

1. The loop of the curve $ay^2 = x^2(a-x)$ rotates about the axis of x; find the volume of the solid formed.
2. Find the total volume generated by the rotation of the curve $a^2y^2 = x^2(a^2-x^2)$ about the axis of x.
3. A segment of a parabola cut off by a double ordinate perpendicular to its axis rotates about the tangent at the vertex; find the volume generated.
4. The same segment rotates about the double ordinate; find the volume generated, and its ratio to the volume of the circumscribing cylinder.
5. Find the volume formed when one semi-undulation of the curve $y = b\sin(x/a)$ rotates about the axis of x.
6. From an extremity B of the latus rectum of the parabola $y^2 = 4ax$, BK is drawn perpendicular to the axis of y; find the volume formed by the rotation of OBK about BK.
7. Find the volume of the solid generated by the rotation of the curve $xy^2 = a^2(a-x)$ about its asymptote.
8. The arc of a quadrant of a circle rotates about its chord; find the volume formed.
9. Find the volume formed by the rotation of $(a-x)y^2 = x^3$ about its asymptote.
10. The curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ rotates about the axis of x; find the volume formed.
11. One-half of one arch of a cycloid rotates about the tangent at the highest point; find the volume generated.
12. Find the volume formed when one arch of a cycloid rotates about its maximum ordinate.
13. Find the volume formed when the figure bounded by the axis of x, the catenary $y = c\cosh(x/c)$, and the ordinates $x = \pm c$ rotates about the axis of x.
14. The common part of the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$ rotates about the axis of x; find the volume of the solid formed.
15. The curve * $x = a(\log\cot\frac{1}{2}\theta - \cos\theta)$, $y = a\sin\theta$ rotates about the axis of x, which is an asymptote; find the volume generated.

* This curve is called the *tractrix*. It is the path of a heavy particle A drawn along a rough horizontal plane by a string AB, when the end B of the string is made to move in a straight line which does not pass through A. (See Ex. XVII. 13.)

16. A circle of radius r rotates about a tangent; find the volume of the resulting solid.

17. Find the volume formed when the figure bounded by the curve $y = a \sin (x/b)$, the axis of y and the line $y = a$ rotates about the axis of y.

18. Show that the volume formed by rotating $y = e^{-x}$ (from $x = 0$ to $x = \infty$) about the axis of y is four times the volume formed by rotating it about the axis of x.

19. Find the volume obtained by rotating the figure bounded by $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the axes about one of the axes.

20. Find the volume obtained by rotating the oval part of the curve $x^2 y^2 = a(x-a)(x-b)^2$ about the axis of x.

21. The circle $(x-a)^2 + (y-b)^2 = r^2$, $(r < b)$, rotates about the axis of x; find the volume of the solid ring thereby formed.

22. Prove that the volume of a cone or pyramid of height h, which stands on a base of area A, is $\frac{1}{3}Ah$.

23. Five equidistant sections of a barrel are circles of circumferences 80, 90, 96, 90, and 80 inches respectively, their common distance apart being 1 foot; find the volume of the barrel.

24. The bounding sections of a solid are ellipses (perpendicular to its axis) with semi-axes 4, 6 inches and 10, 12 inches respectively, the middle section is an ellipse with semi-axes 8, 10 inches, and its length is 9 inches. Find its volume.

25. A square with a semicircle described upon one of its sides rotates about the opposite side; find the volume generated.

26. The smaller of the two portions into which an ellipse is divided by its latus rectum rotates about that latus rectum; find the volume generated.

27. Find the volume generated by the rotation about the line $x = 4$ of the figure bounded by this line and the curve $y^2 = x^3$.

28. A sector of a circle, radius r, of angle 60° rotates about its middle radius; find the volume formed.

29. An isosceles triangle rotates about an axis through its vertex parallel to its base; find the volume generated.

30. A quadrant of a circle rotates about a line through the centre of the circle, parallel to the chord which joins the extremities of its arc; find the volume generated.

LENGTHS OF CURVES

160. Lengths of curves.

The length of an arc of a curve has already been defined in Art. 14 as the limit of the perimeter of an inscribed polygon, when the sides are all indefinitely diminished. The process of finding the length of a curve is often referred to as the *rectification* of the curve. Some simple examples were worked out in Art. 82 from the fact that

$$ds/dx = \sqrt{[1 + (dy/dx)^2]}.$$

If P and Q be two consecutive angular points of the inscribed polygon, whose coordinates are (x, y) and $(x+\delta x, y+\delta y)$,

$$PQ=\sqrt{[(\delta x)^2+(\delta y)^2]}=\delta x\sqrt{[1+(\delta y/\delta x)^2]}.$$

Hence the length s of the arc between two points whose abscissae are a and b

$$=\mathrm{Lt}\,\Sigma(PQ)=\mathrm{Lt}\sum_{x=a}^{x=b}\sqrt{\left[1+\left(\frac{\delta y}{\delta x}\right)^2\right]}\,\delta x=\int_a^b\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx,$$

since it follows from the mean-value theorem (Art. 116) that $\delta y/\delta x$ is equal to the value of dy/dx at some point between P and Q, and it was mentioned in Art. 144 that, in the definition of the definite integral of a function, it was sufficient to take the values of the function at any points within the successive intervals.

Similarly, the length of the arc may be expressed as

$$\int_{a'}^{b'}\sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}\,dy,$$

where a' and b' are the ordinates of the extreme points of the arc.

If the values of the coordinates x and y are expressed in terms of a third variable θ, it follows in the same way that

$$\frac{PQ}{\delta\theta}=\sqrt{\left[\left(\frac{\delta x}{\delta\theta}\right)^2+\left(\frac{\delta y}{\delta\theta}\right)^2\right]},$$

and therefore the length of the arc is equal to

$$\int\sqrt{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right]}\,d\theta$$

taken between suitable limits for θ.

In only comparatively few cases can the integration be effected in finite terms of such functions as have hitherto been considered. Even in the case of the ellipse, the resulting integral can only be completely evaluated by introducing and investigating the properties of a new class of functions known as *elliptic functions*. The integral obtained for the length of an arc of an ellipse is called an *elliptic integral* (the name being due to the fact that the rectification of the ellipse was the first problem in which such integrals presented themselves).

In some cases an approximation to the length of an arc may be made by the use of Simpson's Rule by taking equidistant values of ds/dx or ds/dy.

Examples:

(i) *Find the length of the arc of the parabola* $y^2 = 4ax$ *from the vertex to any point* (x_1, y_1) *on the curve.*

In this case it is best to take y as the independent variable.

Since $y^2 = 4ax$, we have $2y = 4a\, dx/dy$; $\therefore$ $dx/dy = y/2a$.

$$\therefore\quad s = \int_0^{y_1} \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}\, dy = \int_0^{y_1}\sqrt{\left(1+\frac{y^2}{4a^2}\right)}\, dy = \frac{1}{2a}\int_0^{y_1}\sqrt{(4a^2+y^2)}\, dy$$

$$= \frac{1}{2a}\left[\tfrac{1}{2} y\sqrt{(4a^2+y^2)} + \tfrac{1}{2}\,.\,4a^2 \sinh^{-1}\frac{y_1}{2a}\right]_0^{y_1} \quad \text{(Art. 139)}$$

$$= \frac{1}{4a}\left[y_1\sqrt{(4a^2+y_1^2)} + 4a^2\sinh^{-1}\frac{y_1}{2a}\right], \quad \text{since } \sinh^{-1} 0 = 0.$$

For instance, the length of the arc from the vertex to an extremity of the latus rectum, where y_1 is equal to $2a$,

$$= \frac{1}{4a}\left[2a\sqrt{(8a^2)} + 4a^2\sinh^{-1} 1\right] = a(\sqrt{2} + \sinh^{-1} 1)$$

$$= a(1{\cdot}414.. + {\cdot}881..) = 2{\cdot}295\ldots a.$$

(ii) *Find the length of the arc of a quadrant of an ellipse.*

The coordinates of any point on the ellipse can be expressed in the form $x = a\cos\phi$, $y = b\sin\phi$ (Art. 50). Therefore

$$(ds/d\phi)^2 = (dx/d\phi)^2 + (dy/d\phi)^2 = a^2\sin^2\phi + b^2\cos^2\phi$$
$$= a^2 - (a^2-b^2)\cos^2\phi = a^2 - a^2e^2\cos^2\phi \text{ (p. 19)};$$

$\therefore$ $s = \int a(1-e^2\cos^2\phi)^{\frac{1}{2}}\, d\phi$ between suitable limits.

Measuring s from the end A of the major axis where $\phi = 0$, s increases with ϕ; therefore $ds/d\phi$ is $+$. At the end B of the minor axis, $\phi = \frac{1}{2}\pi$; hence the values of ϕ at the extremities of a quadrant are 0 and $\frac{1}{2}\pi$, and the

length of the arc $\qquad = a\int_0^{\frac{1}{2}\pi}(1-e^2\cos^2\phi)^{1/2}\, d\phi.$

This integral cannot be found in terms of functions hitherto considered, but an approximate value can be obtained by expanding $(1-e^2\cos^2\phi)^{1/2}$ by the binomial theorem and retaining a few terms only. If the series converges rapidly, a good approximation is easily obtained.

$$(1-e^2\cos^2\phi)^{\frac{1}{2}} = 1 - \tfrac{1}{2}e^2\cos^2\phi + \frac{\frac{1}{2}\,.\,-\frac{1}{2}}{2\,!}e^4\cos^4\phi - \frac{\frac{1}{2}\,.\,-\frac{1}{2}\,.\,-\frac{3}{2}}{3\,!}e^6\cos^6\phi + \ldots$$

$$= 1 - \tfrac{1}{2}e^2\cos^2\phi - \tfrac{1}{8}e^4\cos^4\phi - \tfrac{1}{16}e^6\cos^6\phi - \ldots.$$

This series satisfies the conditions under which an infinite series can be integrated term by term. *Assuming this fact*, we obtain, by integrating each term between 0 and $\frac{1}{2}\pi$, the length of the arc

$$= a\left[\tfrac{1}{2}\pi - \tfrac{1}{2}e^2\,.\,\tfrac{1}{2}\,.\,\tfrac{1}{2}\pi - \tfrac{1}{8}e^4\,.\,\frac{3\,.\,1}{4\,.\,2}\,.\,\tfrac{1}{2}\pi - \tfrac{1}{16}e^6\,.\,\frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}\,.\,\tfrac{1}{2}\pi - \ldots\right]$$

$$= \tfrac{1}{2}\pi a\left(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \tfrac{5}{256}e^6 - \ldots\right).$$

In the ellipse, e is always < 1, and the terms diminish rapidly. E.g. if the semi-axes are 6 and 10 inches, $e^2 = 1 - b^2/a^2 = {\cdot}64$, and the length of the arc

$$= \tfrac{1}{2}\pi\,.\,10\,[1 - {\cdot}16 - {\cdot}0192 - {\cdot}0051] = 5\pi \times {\cdot}8157$$
$$= 12{\cdot}8 \text{ inches approximately.}$$

Examples LXIII.

1. Find by integration the length of the circumference of a circle.
2. Find the length of the arc of the parabola $y^2 = 4x$ from the vertex to the point $(9, 6)$.
3. Find the length of the arc of the curve $ay^2 = x^3$ from the origin to the point whose abscissa is $\frac{7}{3}a$.
4. If s be the length of the arc of the catenary $y = c\cosh(x/c)$ from the vertex to the point (x, y), show that $s^2 = y^2 - c^2$.
5. Find the length of the curve $y = e^x$ from $y = \frac{3}{4}$ to $y = \frac{4}{3}$.
6. Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$.
7. Prove that the area between the catenary $y = c\cosh(x/c)$, the axis of x and the ordinates of two points on the curve is equal to cs, where s is the length of the arc intercepted between the two points.
8. Express the length of one semi-undulation of the curve $y = b\sin(x/a)$ as a definite integral.
9. Find the length of the loop of the curve $3ay^2 = x(x-a)^2$.
10. Calculate the perimeter of an ellipse whose major axis is 15 inches in length and whose eccentricity is $\frac{1}{5}$.
11. The axes of an ellipse are 10 and 20 inches; find its perimeter.
12. Show that in the curve $x^{2/3} + y^{2/3} = a^{2/3}$, if s be the length of the arc measured from the axis of y, $s^3 \propto x^2$.
13. Find the length of the curve $y = \log\cos x$ from $x = 0$ to $x = \frac{1}{3}\pi$.
14. Find the total length of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$.
15. The eccentricity e of an ellipse is small; prove that the perimeter is $2\pi a(1 - \frac{1}{4}e^2)$, nearly.
16. Find the length of the curve $x = 2a\cos\theta - a\cos 2\theta$, $y = 2a\sin\theta - a\sin 2\theta$ from $\theta = \pi$ to $\theta = \alpha$.
17. Find the length of the curve $y = \log[(e^x+1)/(e^x-1)]$ from $x = a$ to $x = 2a$.
18. A curve is given by the equations $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$; find the length of the arc from $\theta = 0$ to $\theta = \alpha$.
19. Find, by Simpson's Rule, the perimeter of the ellipse in Ex. (ii), Art. 160.
20. Find approximately the length of the arc of the hyperbola $xy = 12$ from $x = 1$ to $x = 4$.

AREAS OF SURFACES

161. Areas of surfaces of solids of revolution.

If in Fig. 96 (p. 287) the curve AB rotates about the axis of x, the straight line PQ generates a frustum of a cone; the sum of the areas of all these frusta tends, when $\delta x \to 0$, to a limiting value which is

defined as the area of the curved surface of the solid (Art. 14). It has been shown (p. 44) that the area described by PQ

$$= PQ \times \text{circumference of circle described by middle point of } PQ$$

$$= PQ \, . \, 2\pi \, (y + \tfrac{1}{2}\delta y).$$

$\therefore$ the area of the surface formed by the rotation of AB

$$= \text{Lt} \, \Sigma \, PQ \, . \, 2\pi \, (y + \tfrac{1}{2}\delta y)$$

$$= \text{Lt} \, \Sigma \, (PQ/\delta s) \, 2\pi \, (y + \tfrac{1}{2}\delta y) \, \delta s$$

$$= \int_{s_1}^{s_2} 2\pi y \, ds, \text{ since } PQ/\delta s \longrightarrow 1 \text{ and } y + \tfrac{1}{2}\delta y \longrightarrow y,$$

where s_1 and s_2 are the lengths of the arc measured from some fixed point on the curve to A and B respectively.

i. e. area of surface $= \displaystyle\int_a^b 2\pi y \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \, dx$ (Art. 82).

If it is more convenient, this may be expressed as

$$\int 2\pi y \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]} \, dy$$

between suitable limits.

As with volumes, the area of the surface can be found in a similar manner, if the curve rotates about a line parallel to one of the axes. In some cases, too, Simpson's Rule may be used, circumferences being taken of sections at equal distances measured along the arc of the generating curve.

Sometimes it is more convenient to express both y and s in terms of some other variable θ.

Examples:

(i) *Find the area of the curved surface formed by the rotation of a quadrant of a circle about the tangent at one extremity of it.*

Referring to Fig. 102, we have, since $PQ = a\,\delta\theta$, the area of the surface

$$= \text{Lt} \, \Sigma \, (2\pi MP \, . \, PQ) = \text{Lt} \, \Sigma \, 2\pi \, (a - x) \, a \, \delta\theta$$

$$\int_0^{\frac{1}{2}\pi} 2\pi a \, (1 - \cos\theta) \, a d\theta = 2\pi a^2 \int_0^{\frac{1}{2}\pi} (1 - \cos\theta) \, d\theta$$

$$= 2\pi a^2 \, (\tfrac{1}{2}\pi - 1) = \pi \, (\pi - 2) \, a^2.$$

(ii) *Find the area of the whole surface of a sphere, and of the area intercepted between two parallel planes.*

Let the sphere be formed by the rotation of the circle $x^2 + y^2 = r^2$ about the axis of x. If θ be the inclination of the radius OP (Fig. 105) to the axis of x, the coordinates of P are $(r\cos\theta,\ r\sin\theta)$, and the length of the arc s from A to P is $r\theta$.

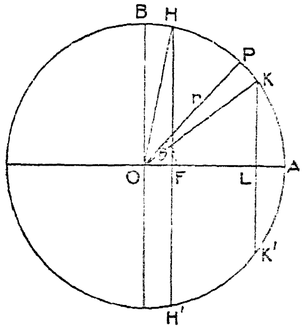


Fig. 105.

The whole surface is twice the surface generated by the rotation of AB

$$= 2\int_0^{\frac{1}{2}\pi} 2\pi y \frac{ds}{d\theta}\, d\theta$$

$$= 4\pi\int_0^{\frac{1}{2}\pi} r\sin\theta \,.\, r\, d\theta = 4\pi r^2 \int_0^{\frac{1}{2}\pi} \sin\theta\, d\theta$$

$$= 4\pi r^2.$$

If the area intercepted between two parallel planes HH' and KK' be required, and if α and β be the inclinations of OK and OH to the axis of x, this area

$$= \int_\alpha^\beta 2\pi y \frac{ds}{d\theta}\, d\theta = 2\pi r^2 \int_\alpha^\beta \sin\theta\, d\theta = 2\pi r^2(\cos\alpha - \cos\beta)$$

$$= 2\pi r(r\cos\alpha - r\cos\beta) = 2\pi r(OL - OF) = 2\pi r\,.\,FL.$$

This is equal to the area intercepted by the same two planes on the cylinder with axis OA circumscribing the sphere. (See also Art. 14.)

(iii) *Find the area of the surface of the solid formed by the rotation of one arch of a cycloid about its base.*

In the cycloid, $y = a(1 - \cos\theta)$, $ds/d\theta = 2a\sin\frac{1}{2}\theta$ (Art. 82);

$$\therefore \text{ the area required} = \int_0^{2\pi} 2\pi y \frac{ds}{d\theta}\, d\theta = \int_0^{2\pi} 2\pi a(1-\cos\theta)\,.\,2a\sin\tfrac{1}{2}\theta\, d\theta$$

$$= 4\pi a^2 \int_0^{2\pi} 2\sin^2\tfrac{1}{2}\theta\,.\,\sin\tfrac{1}{2}\theta\, d\theta = 8\pi a^2 \int_0^{2\pi} \sin^3\tfrac{1}{2}\theta\, d\theta.$$

Let $\frac{1}{2}\theta = \phi$; the limits for ϕ are then 0 and π, and $d\theta/d\phi = 2$.

$$\therefore \text{ the area} = 8\pi a^2\,.\,2\int_0^\pi \sin^3\phi\, d\phi = 32\pi a^2 \int_0^{\frac{1}{2}\pi} \sin^3\phi\, d\phi$$

$$= 32\pi a^2\,.\,\tfrac{2}{3} = \tfrac{64}{3}\pi a^2.$$

Examples LXIV.

1. Find the area of the surface of the solid formed by the rotation about the axis of x of the parabola $y^2 = 16x$ from $x = 5$ to $x = 12$.
2. Find the area of the curved surface of the belt of a sphere of radius 1 foot between two parallel planes at distances 3 and 9 inches from the centre.
3. Find the area of the surface generated by rotating one arch of a cycloid about the tangent at its vertex (i.e. the middle point of the arch).

4. Find the area of the surface generated by the rotation of the cycloid about its axis.

5. Obtain the superficial area of the solid formed by rotating about the axis of y the curve $ay^2 = x^3$ from $x = 0$ to $x = 4a$.

6. Find the area of the surface obtained by rotating a circle of radius r about a straight line in its plane at a distance a $(> r)$ from its centre.

7. Find the area of the surface generated by rotating a quadrant of a circle about the tangent at its middle point.

8. The arc of a quadrant of a circle rotates about its chord; find the area of the surface thereby formed.

9. Find the area of the surface formed by the rotation of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about one of the axes.

10. The arc of the catenary $y = c\cosh(x/c)$ from $x = 0$ to $x = c$ rotates about the axis of y; find the area of the surface formed.

11. Find the area of the surface of the prolate spheroid obtained by rotating the ellipse $x^2/a^2 + y^2/b^2 = 1$ about its major axis.
First prove that $(ds/d\theta)^2 = a^2(1 - e^2\sin^2\theta)$, where $(a\sin\theta,\ b\cos\theta)$ are the coordinates of a point on the ellipse, and integrate by putting $e\sin\theta = \sin\phi$.

12. Find the area of the surface of the oblate spheroid formed by rotating the ellipse $x^2/a^2 + y^2/b^2 = 1$ about its minor axis.

13. The arc of the parabola $y^2 = 4ax$ cut off by the latus rectum rotates about the tangent at the vertex; find the area of the surface described.

14. Find the area of the surface produced by rotating about the axis of x the arc of the rectangular hyperbola $y^2 = x^2 + 2a^2$ from $x = 0$ to $x = a$.

15. The arc of the catenary $y = c\cosh(x/c)$ between $x = -c$ and $x = c$ revolves about the axis of x; find the area of the surface generated.

16. Find the area of the surface formed by the rotation about the axis of x of the loop of the curve $3ay^2 = x(x-a)^2$.

17. The part of the curve $y = e^x$ from $x = 0$ to $x = -\infty$ rotates about the axis of x; find the surface described.

18. The curve $x = a(\log\cot\theta - \cos 2\theta)$, $y = a\sin 2\theta$ rotates about the axis of x, which is an asymptote of the curve; find the area of the surface generated.

19. Find, by Simpson's Rule, the area of the surface of the solid described in Ex. LXII. 23, the common distance of 1 foot being measured along the arc.

20. Find, with a similar modification, the surface of the solid in Art. 159, Ex. (iii).

CHAPTER XVII

POLAR EQUATIONS

162. Plotting of curves from polar equations.

If the equation of a curve be given in rectangular coordinates, it can be transformed into polar coordinates by making the substitutions

$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2+y^2=r^2 \quad \text{(p. 23).}$$

In the case of several important curves, the polar equation is much simpler than the Cartesian equation.

Examples:

(i) *The Lemniscate.*

The Cartesian equation of a well-known curve called the Lemniscate of Bernouilli is $(x^2+y^2)^2 = a^2(x^2-y^2)$. It would not be very easy to plot the curve or develop its properties from this equation, but, transforming to polars by aid of the above substitutions, we get

$$(r^2)^2 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$$

i.e.
$$r^2 = a^2(\cos^2\theta - \sin^2\theta) = a^2\cos 2\theta.$$

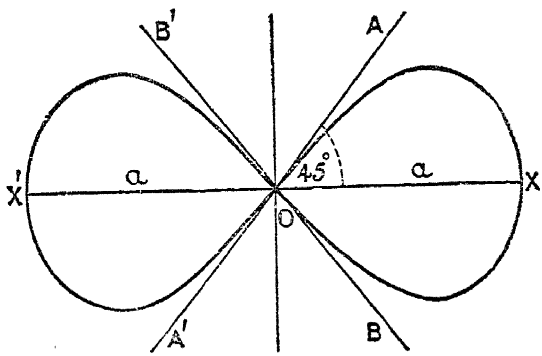


Fig. 106.

By the aid of this equation the curve is easily drawn, and the lemniscate affords a good illustration of the way in which the form of a curve is deduced from its polar equation.

In the first place, a change in the sign of θ does not alter the equation, since $\cos(-\alpha) = \cos\alpha$; this shows that the curve is symmetrical about the initial line OX. Again, if θ is increased by π, $\cos 2\theta$ becomes $\cos(2\theta + 2\pi)$, which is the same as $\cos 2\theta$; since the equation is unchanged when the radius vector makes half a complete revolution, it follows that the curve is symmetrical about the origin. Hence it only remains to plot it from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. When $\theta = 0$, r is numerically equal to a, and as θ increases from 0 to $\frac{1}{4}\pi$, r decreases from a to 0; as θ increases from $\frac{1}{4}\pi$ to $\frac{1}{2}\pi$, $\cos 2\theta$ is $-$; therefore r^2 is $-$, and r is imaginary.

Hence, if the angles between the rectangular axes be bisected by the straight lines AOA' and BOB', the curve consists of two equal ovals in the angles AOB and $A'OB'$ which are bisected by XX' (Fig. 106).

(ii) *The Cardioid.*

This curve has the equation $r = a(1 + \cos\theta)$, and is of importance in Optics. As in the preceding example, it is symmetrical about the initial line OX. As θ increases from 0 to $\frac{1}{2}\pi$, r decreases from $2a$ to a; as θ increases from $\frac{1}{2}\pi$ to π, r continues to decrease from a to 0. Hence its shape is as indicated in Fig. 107.

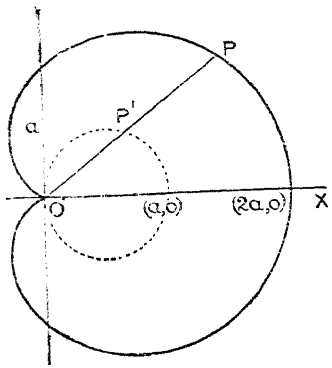


Fig. 107.

From the equation of the curve, and the equation of a circle obtained on p. 23, it is easy to see a simple geometrical construction for the curve. $r = a + a\cos\theta$, and $a\cos\theta$ is the radius vector of a point on a circle of diameter a; therefore, if from a point O on a circle chords OP' are drawn and points P are taken on these chords produced at a distance a from the circumference, the locus of the points P is a cardioid.

If the equation be given in the form $r = a(1 - \cos\theta)$, the graph is the reflexion in the axis of y of the curve shown in the figure.

Examples LXV.

Draw roughly the curves in Examples 1–12:

1. $r = 2 + \cos\theta$. 2. $r = a\theta$. 3. $r = a\cos 2\theta$. 4. $r = 1 + 2\cos\theta$.

5. $r = a\sin 3\theta$. 6. $r = e^{2\theta}$. 7. $r\theta = a$. 8. $r = a\cos 3\theta$.

9. $r = a\sin 2\theta$. 10. $r = 2a\sin\theta\tan\theta$. 11. $r^2 = a^2\cos\theta$.

12. $r = a\sec\theta + b$: (i) when $a > b$, (ii) when $a = b$, (iii) when $a < b$.

13. Transform $y^2(2a - x) = x^3$ to polars.

14. Find the polar equation of a rectangular hyperbola.

15. Find the polar equation of a parabola.

163. **Angle between tangent and radius vector.**

Let (r, θ) be the polar coordinates of a point P (Fig. 108) on the curve, and $(r+\delta r,\ \theta+\delta\theta)$ the coordinates of a neighbouring point Q; therefore the angle $POQ = \delta\theta$.

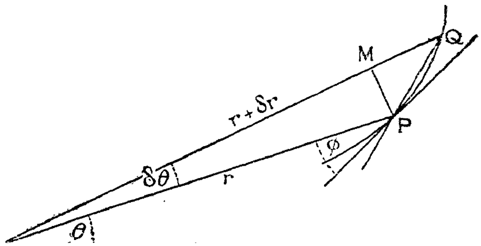


Fig. 108.

Draw PM perpendicular to OQ.

Then $\sin OQP = \dfrac{MP}{PQ} = \dfrac{r\sin\delta\theta}{PQ} = r\dfrac{\sin\delta\theta}{\delta\theta} \times \dfrac{\delta\theta}{\delta s} \times \dfrac{\delta s}{PQ}.$

When Q moves along the curve and approaches indefinitely near to P, the limiting position of PQ is the tangent at P, and the angle OQP becomes the angle between the tangent at P and the radius vector OP. This angle is usually denoted by ϕ.

Now $\quad$ Lt $(\sin\delta\theta)/\delta\theta = 1,\quad$ Lt $\delta s/PQ = 1,\quad$ Lt $\delta\theta/\delta s = d\theta/ds$;

$\therefore$ ultimately
$$\sin\phi = r\frac{d\theta}{ds}.$$

Similarly,
$$\cos OQP = \frac{MQ}{PQ} = \frac{OQ-OM}{PQ}$$
$$= \frac{r+\delta r - r\cos\delta\theta}{\delta s} \times \frac{\delta s}{PQ}$$
$$= \left[\frac{r(1-\cos\delta\theta)}{\delta s} + \frac{\delta r}{\delta s}\right]\frac{\delta s}{PQ}.$$

$\therefore$ ultimately
$$\cos\phi = \left[0 + \frac{dr}{ds}\right] \times 1 = \frac{dr}{ds},$$

since it follows from Art. 13 (10) that

$$\mathrm{Lt}\,\frac{r(1-\cos\delta\theta)}{\delta s} = \mathrm{Lt}\, r\cdot\frac{1-\cos\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta s} = r\times 0\times\frac{d\theta}{ds} = 0;$$

Similarly it may be shown that $\tan\phi = r\dfrac{d\theta}{dr}.$

The last result can also be deduced as follows:

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = r\frac{d\theta}{ds} \times \frac{ds}{dr} = r\frac{d\theta}{dr}. \quad \text{(Art. 34.)}$$

Again, since $\sec^2\phi = 1+\tan^2\phi$, and $\operatorname{cosec}^2\phi = 1+\cot^2\phi$,

it follows that
$$\left(\frac{ds}{dr}\right)^2 = 1+r^2\left(\frac{d\theta}{dr}\right)^2,$$

and
$$\frac{1}{r^2}\left(\frac{ds}{d\theta}\right)^2 = 1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2,$$

i. e.
$$\left(\frac{ds}{d\theta}\right)^2 = r^2+\left(\frac{dr}{d\theta}\right)^2.$$

Examples:

(i) *Prove that in the cardioid* $r = a(1-\cos\theta)$ *the angle between the tangent and the radius vector is half the vectorial angle.*

We have $dr/d\theta = a\sin\theta$;

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta,$$

whence
$$\phi = \tfrac{1}{2}\theta.$$

(ii) *Find the polar equation of the curve in which the inclination of the tangent to the radius vector is constant.*

Let the tangent be inclined at an angle α to the radius vector;

then
$$\tan\alpha = r\,d\theta/dr, \quad \therefore \quad d\theta/dr = (\tan\alpha)/r,$$

whence
$$\theta = \log r.\tan\alpha + C.$$

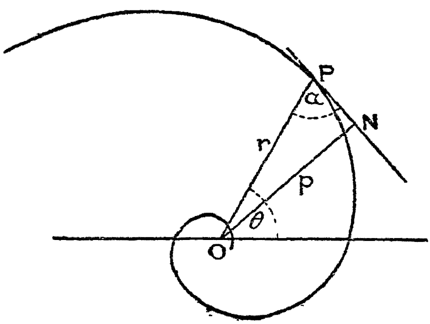


Fig. 109.

Let the curve cut the initial line from which θ is measured at distance a from the origin, i. e. let $r = a$ when $\theta = 0$.

Then $\quad 0 = \log a.\tan\alpha + C$, and $C = -\log a.\tan\alpha$;

$$\therefore \quad \theta = \tan\alpha(\log r - \log a)$$

i. e. $\quad \theta\cot\alpha = \log(r/a)$, $\quad$ whence $r = ae^{\theta\cot\alpha}$.

This curve is called an *equiangular spiral* (Fig. 109).

164. Perpendicular from origin to tangent.

If p be the perpendicular from the origin to the tangent and u the reciprocal of the radius vector, to prove that

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

Since (Fig. 110) $p = r\sin\phi$, we have

$$\frac{1}{p^2} = \frac{\operatorname{cosec}^2\phi}{r^2} = \frac{1+\cot^2\phi}{r^2}$$

$$= \frac{1}{r^2}\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

Since $u = \frac{1}{r}$, $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$;

$$\therefore \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

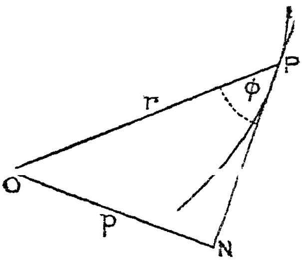


Fig. 110.

If a perpendicular to OP through O meet the tangent and normal at P in T and G respectively, OT and OG are sometimes called the *polar subtangent* and *polar subnormal.*

Evidently the polar subtangent $= r\tan\phi = r^2\, d\theta/dr$,
and the polar subnormal $= r\cot\phi = dr/d\theta$.

165. Tangential-polar or p and r equation.

If r be the radius vector of a point P on a curve, and p the perpendicular from the origin to the tangent at P, the equation which gives the relation between p and r is called the *tangential-polar or* p-r *equation* of the curve. In many curves this relation takes a very simple form.

The tangential-polar equation can easily be deduced from the ordinary polar equation. It was shown, in the preceding article, that

$$p^2 = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

By eliminating θ between this equation and the polar equation of the curve, the tangential-polar equation is obtained.

In a few cases it can be obtained quite easily geometrically.

It is obvious at once that the equation of a circle is $p = r$, if the centre be taken as origin; the equation of a straight line is $p =$ constant; that of an equiangular spiral (Art. 163, Ex. (ii)) is $p = r\sin\alpha$.

Again, if P (Fig. 111) be any point on a circle, ON the perpendicular from

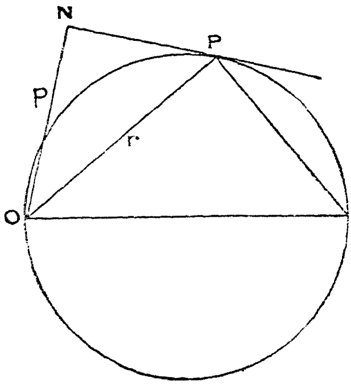


Fig. 111.

a fixed point O on the circumference to the tangent at P, and OA the diameter through O, the triangles ONP, OPA are similar.

$$\therefore \quad ON/OP = OP/OA, \quad \text{i.e. } p/r = r/2a \text{ or } r^2 = 2ap.$$

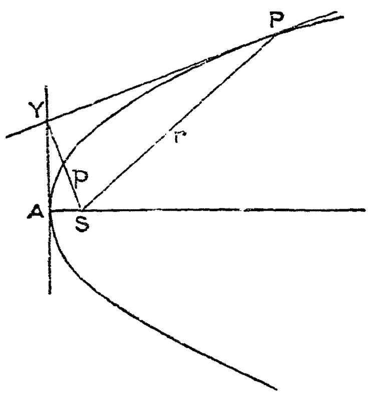


Fig. 112.

In the parabola, it is easily proved that the perpendicular from the focus to a tangent meets it on the tangent at the vertex.

The triangles ASY, YSP (Fig. 112) are similar;

$$\therefore \quad AS/SY = SY/SP; \quad \text{i.e. } a/p = p/r \text{ or } p^2 = ar.$$

In the case of the ellipse, it is a well-known theorem that the rectangle contained by the perpendiculars SY, $S'Y'$ (Fig. 113) from the foci to any tangent is equal to b^2.

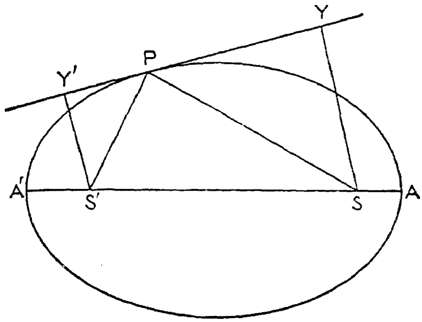


Fig. 113.

The triangles SPY, $S'PY'$ are similar; hence, taking the focus S as origin,

$$\frac{b^2}{p^2} = \frac{SY \cdot S'Y'}{SY^2} = \frac{S'Y'}{SY} = \frac{S'P}{SP} = \frac{AA'-SP}{SP} = \frac{2a-r}{r} = \frac{2a}{r} - 1.$$

Similarly, the corresponding equation for the hyperbola is $\frac{b^2}{p^2} = \pm\frac{2a}{r} + 1$.

As examples of the way in which the tangential-polar equation can be deduced from the polar equation, we will take the lemniscate and the cardioid.

In the lemniscate, $r^2 = a^2 \cos 2\theta$ (Art. 162). Differentiating with respect to θ,

$$2r\,dr/d\theta = -2a^2 \sin 2\theta;$$

$$\therefore \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{a^4 \sin^2 2\theta}{r^2} = \frac{a^4(1-\cos^2 2\theta)}{r^2} = \frac{a^4 - r^4}{r^2}.$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4} \cdot \frac{a^4 - r^4}{r^2} = \frac{a^4}{r^6}; \quad \therefore \quad r^3 = a^2 p.$$

In the cardioid, $r = a(1+\cos\theta)$, $dr/d\theta = -a\sin\theta$,

$$\therefore \quad (dr/d\theta)^2 = a^2 \sin^2\theta = a^2 - a^2\cos^2\theta = a^2 - (r-a)^2 = 2ar - r^2;$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(2ar - r^2) = \frac{2a}{r^3}; \quad \therefore \quad r^3 = 2ap^2.$$

Examples LXVI.

1. Prove that $\phi = \frac{1}{2}(\pi - \theta)$ in the parabola $r(1+\cos\theta) = 2a$.
2. Prove that, in the curve $r = ae^{b\theta}$, the tangent is inclined at a constant angle to the radius vector.
3. Find the angle between the tangent and the radius vector at the point $(\frac{3}{2}a, \frac{1}{3}\pi)$ on the cardioid $r = a(1+\cos\theta)$.

4. Find in terms of r the value of $ds/d\theta$ in the cardioid $r = a(1+\cos\theta)$.

5. Prove that in the curve $r^2 = a^2 \sin 2\theta$ the angle between the tangent and the radius vector is double the vectorial angle.

6. Prove that in the curve $r\theta = a$ (the *reciprocal* or *hyperbolic spiral*) the polar subtangent is constant.

7. Show that in the curve $r = a\sin^3 \frac{1}{3}\theta$ the inclination of the tangent at any point to the initial line is four times the angle between the tangent and the radius vector.

8. Find the angle between the radius vector and the tangent at the point on the curve $r\theta = a$ where $\theta = \pi$.

9. Prove that in the curve $r^n = a^n \sin n\theta$, $\phi = n\theta$.

10. Show that in the curve $r = ae^\theta$ the polar subtangent and subnormal are equal.

11. Prove that, in the curve $r(1-\cos\theta) = 2a$, $\phi + \frac{1}{2}\theta = \pi$.

12. If ON be the perpendicular from the origin to the tangent at P, prove that $PN = r\,dr/ds$.

13. Prove that, in the curve $r^n = a^n \cos n\theta$, $ds/d\theta = a\sec^{(n-1)/n} n\theta$.

14. Show that $r^2 \cos 2\theta = a^2$ represents a rectangular hyperbola.

15. Show that, in the curve $r^2\cos 2\theta = a^2$, $pr = a^2$.

16. Prove that, in the curve $r = a\theta$, $p^2 = r^4/(a^2+r^2)$.

    This curve is called the *spiral of Archimedes*. It is the path of a point which moves along a straight line with constant velocity, while at the same time the line rotates about a fixed point in itself with constant angular velocity.

17. Prove that, in the curve $r = a/\theta$, $p^2 = a^2 r^2/(a^2+r^2)$.

18. Show that, if $r^n = a^n \cos n\theta$, $r^{n+1} = a^n p$.

19. Show that in any curve $ds/dr = r/\sqrt{(r^2-p^2)}$.

20. Prove also that $dr/d\theta = r\sqrt{(r^2-p^2)}/p$.

21. Deduce from the preceding result the equation of the curve in which $r^3 = 2ap^2$.

22. Prove that all chords of the cardioid $r = a(1+\cos\theta)$ through the origin are equal in length.

23. Find the maximum double ordinate of the cardioid.

24. Find the distance from the origin of the tangent (perpendicular to the axis) which touches the cardioid at two points.

25. Find the maximum ordinate of the lemniscate $r^2 = a^2\cos 2\theta$.

26. If u and v be the components of the velocity of a moving point P along and perpendicular to the radius vector OP, prove that $u = \dot{r}$, $v = r\dot{\theta}$.

27. Show that, in the rectangular hyperbola $r^2\cos 2\theta = a^2$, $p^2 = a^2\cos 2\theta$.

28. The curve $r = 2+4\cos\theta$ consists of two loops through the origin, one within the other; find the directions of the tangents to the curve at the origin.

29. Find the maximum double ordinate of the curve $r = a+b\cos\theta$ (which is called a *limaçon*).

30. Find the 'p and r' equation of a hyperbola, taking a focus as origin.

31. ,, ,, ,, ,, ,, an ellipse, taking the centre as origin.

32. Find the distance from the origin of the tangent which touches the curve $r = a+b\cos\theta$ at two points. Compare this result with that of Ex. 24.

33. Prove that in the equiangular spiral the polar subtangent varies as the radius vector.

34. Prove that in the curve $r = a\theta$ the polar subnormal is constant.

35. Prove that in the curve $r = a\sin\theta$ the tangent and the initial line are equally inclined to the radius vector.

36. In the curve $r^2\cos 2\theta = a^2$, find the inclination of the tangent to the radius vector when $\theta = \frac{3}{4}\pi$. Explain the result geometrically.

166. Areas in polar coordinates.

Let OA, OB (Fig. 114) be two fixed radii of a curve making angles α and β respectively with the initial line. Let (r, θ) be the polar coordinates of any point P on the arc AB, and let z be the area between the curve and the radii OA, OP; let Q be the point $(r+\delta r, \theta+\delta\theta)$. The increase $\delta\theta$ in the angle θ produces the increase POQ in the area z.

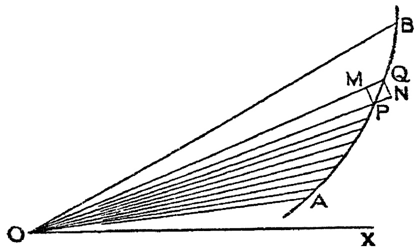


Fig. 114.

If circles with O as centre and OP, OQ as radii cut OQ and OP respectively in M and N, then the area OPQ is intermediate in value between the sectors OPM and OQN,

i.e. $$\delta z > \tfrac{1}{2}r^2\,\delta\theta \text{ and } < \tfrac{1}{2}(r+\delta r)^2\,\delta\theta,$$

$$\therefore\ \delta z/\delta\theta \text{ is between } \tfrac{1}{2}r^2 \text{ and } \tfrac{1}{2}(r+\delta r)^2.$$

In the limit, when $\delta\theta \longrightarrow 0$, $r+\delta r \longrightarrow r$ and $\delta z/\delta\theta \longrightarrow dz/d\theta$;

$$\therefore\ \frac{dz}{d\theta} = \tfrac{1}{2}r^2, \text{ and } z = \int_\alpha^\beta \tfrac{1}{2}r^2\,d\theta.$$

As in the case of rectangular coordinates, the same result is obtained by taking the area AOB as the limiting value of $\Sigma\,(\Delta\,OPM)$,

i.e. $$\mathrm{Lt}\sum_\alpha^\beta \tfrac{1}{2}r^2\,\delta\theta \text{ as } \delta\theta \longrightarrow 0, \text{ i.e. } \int_\alpha^\beta \tfrac{1}{2}r^2\,d\theta.$$

Example. Find the area of one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

Since $r = 0$ when $\theta = \pm\frac{1}{4}\pi$, and the curve is symmetrical about $\theta = 0$,

$$\text{the area } = 2\int_0^{\frac{1}{4}\pi} \tfrac{1}{2}r^2\,d\theta = a^2\int_0^{\frac{1}{4}\pi} \cos 2\theta\,d\theta = a^2\left[\tfrac{1}{2}\sin 2\theta\right]_0^{\frac{1}{4}\pi}$$

Hence the total area of both loops $= a^2$, i.e. the area of a square of side a.

167. Lengths of arcs in polar coordinates.

It was shown in Art. 163 that $\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$;

therefore, measuring s so that it increases with θ,

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}; \qquad \therefore\ s = \int_\alpha^\beta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta,$$

where α, β are the values of θ at the extremities of the arc.

As in the case of rectangular coordinates, the same expression is obtained by taking the length of the arc as the limit of the perimeter of an inscribed polygon.

Example. Find the total length of the cardioid.

In the cardioid $r = a(1 + \cos\theta)$,

$$(ds/d\theta)^2 = r^2 + (dr/d\theta)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta = a^2(2 + 2\cos\theta)$$
$$= 4a^2\cos^2\tfrac{1}{2}\theta;$$

$$\therefore \text{ total length of arc} = 2\int_0^\pi 2a\cos\tfrac{1}{2}\theta\, d\theta = 4a\left[2\sin\tfrac{1}{2}\theta\right]_0^\pi = 8a.$$

168. Volumes and areas in polar coordinates.

There are no simple general formulae for the volumes and superficial areas of solids of revolution in polar coordinates. The following example will show the method of dealing with such cases.

Example. Find the volume and the area of the surface of the solid formed by the rotation of the cardioid $r = a(1 + \cos\theta)$ *about its line of symmetry.*

Starting with the Cartesian formula, we have

$$\text{the area} = \int 2\pi y\, ds = \int_0^\pi 2\pi r\sin\theta \frac{ds}{d\theta}\, d\theta$$

(the limits are 0 and π since the rotation of the upper half gives the solid),

$$= \int_0^\pi 2\pi a(1+\cos\theta)\sin\theta \,.\, 2a\cos\tfrac{1}{2}\theta\, d\theta$$

(it was shown in Art. 167 that $ds/d\theta = 2a\cos\frac{1}{2}\theta$)

$$= 4\pi a^2\int_0^\pi 2\cos^2\tfrac{1}{2}\theta \,.\, 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta \,.\, \cos\tfrac{1}{2}\theta\, d\theta$$

$$= 16\pi a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta\, d\theta$$

Let $\frac{1}{2}\theta = \phi$; then the limits for ϕ are 0 and $\frac{1}{2}\pi$, and the integral

$$= 32\pi a^2\int_0^{\frac{1}{2}\pi} \cos^4\phi\sin\phi\, d\phi$$

$$= 32\pi a^2 \cdot \frac{3 \,.\, 1}{5 \,.\, 3 \,.\, 1}$$

Similarly, the volume $= \int_0^{2a} \pi y^2\, dx = \int_\pi^0 \pi r^2 \sin^2\theta \frac{dx}{d\theta}\, d\theta$,

since $\theta = 0$ when $x = 2a$, and $\theta = \pi$ when $x = 0$.

Now $$x = r\cos\theta = a(\cos\theta + \cos^2\theta),$$

$$\therefore\ dx/d\theta = a(-\sin\theta - 2\cos\theta\sin\theta) = -a\sin\theta(1+2\cos\theta);$$

$$\therefore\ \text{the volume} = \int_\pi^0 \pi a^2(1+\cos\theta)^2\sin^2\theta \times -a\sin\theta(1+2\cos\theta)\, d\theta$$

$$= \pi a^3 \int_0^\pi (1+\cos\theta)^2(1+2\cos\theta)\sin^3\theta\, d\theta$$

$$\pi a^3 \int_0^\pi [1 + 4\cos\theta + 5\cos^2\theta + 2\cos^3\theta]\sin^3\theta\, d\theta.$$

Of the four integrals contained in this expression, the second and fourth are, from Theorem V, Art. 146, equal to 0, and in the other two, the integrals from 0 to π are double the integrals from 0 to $\frac{1}{2}\pi$.

$\therefore$ the volume

$$= 2\pi a^3 \int_0^{\frac{1}{2}\pi} (\sin^3\theta + 5\cos^2\theta\sin^3\theta)\, d\theta$$

$$= 2\pi a^3 \left[\frac{2}{3} + 5 \cdot \frac{2}{5 \cdot 3}\right] \quad \text{(Art. 149)}$$

$$= \tfrac{8}{3}\pi a^3.$$

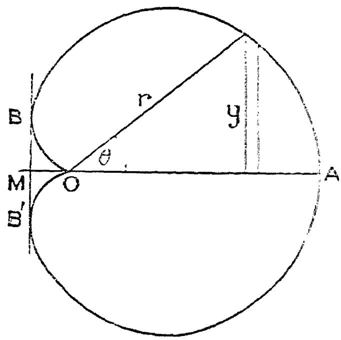


Fig. 115.

It will be noticed that, if BMB' (Fig. 115) be the double tangent, $dx/d\theta$ is $-$ from A to B, and $+$ from B to O; therefore the integral from 0 to π gives the volumes formed by the rotation of ABM and MBO with different signs, i. e. it gives the volume whose section is $ABOB'A$.

Examples LXVII.

1. Find the area of the cardioid $r = a(1+\cos\theta)$.
2. Find the area between the curve $r = 2e^{3\theta}$ and the two radii whose lengths are 2 and 4.
3. Show that, in the curve $r\theta = a$, the area described by the radius starting from some fixed position is proportional to the increase in the length of the radius.
4. Find the area of the curve $r = 2 + \cos\theta$.
5. The curve $r = 2 + 4\cos\theta$ consists of two loops through the origin, one within the other; find the area of each loop. (See Ex. LXVI. 28.)
6. Find the area of the circle $r = 2a\cos\theta$.
7. Trace the curve $r^2 = a^2\cos\theta$, and find its area.
8. Find the area of one loop of the curve $r = a\cos 3\theta$.
9. Find the area of one loop of the curve $r = a\sin 4\theta$.

10. Find the area of the segment of the circle $r = 2a\cos\theta$ cut off by the straight line $\theta = \frac{1}{8}\pi$.

11. Find the areas of the several portions into which the cardioid $r = a(1+\cos\theta)$ is divided by the axis of y.

12. The polar equation of a parabola referred to its focus as origin is $r(1+\cos\theta) = 2a$; find the area cut off by the latus rectum.

13. Find the length of the curve $r = ae^{\theta}$ between two radii of lengths r_1 and r_2.

14. Find the length of the spiral $r = a\theta$ from $\theta = 0$ to $\theta = 2{\cdot}4$.

15. Find the length of the curve $r = a\cos^3\frac{1}{3}\theta$.

16. Find the length of the arc of a parabola (see Question 12) cut off by the latus rectum.

17. Express the length of one loop of the lemniscate $r^2 = a^2\cos 2\theta$ as a definite integral.

18. If A be the area of a curve whose tangential-polar equation is given, prove that $dA/dr = \frac{1}{2}r/\sqrt{(r^2-p^2)}$.
Deduce from this result the area of the cardioid.

19. Deduce from the result of Ex. LXVI. 19, the length of the cardioid.

20. Prove that $2\,dA/d\theta = p\,ds/d\theta = r^2$; and verify geometrically.

21. Find the volume of the solid formed by rotating the curve $r^2 = a^2\cos\theta$ about its line of symmetry.
This solid is called the *solid of greatest attraction.*

22. The curve $r = 4+2\cos\theta$ rotates about its axis; find the area of the surface described.

23. Find the volume of the solid described in the previous example.

24. The curve $r = a\cos\theta$ rotates about the line which bisects it; find the superficial area of the solid thereby formed.

25. Find the volume of the solid in the preceding example.

26. The area mentioned in Question 12 rotates about its axis; find the area of the surface of the solid formed.

27. Find also the volume of the solid in the preceding question.

28. The curve $r = e^{\theta}$ between $\theta = 0$ and $\theta = \pi$ rotates about the line from which θ is measured; find the superficial area of the solid formed.

169. Epicycloids and hypocycloids.

If a circle rolls (without sliding) on the outside of the circumference of another circle, the locus of a fixed point on its circumference is called an *epicycloid*; if it rolls on the inside, the locus is called a *hypocycloid.*

The equations of these curves are easily obtained in terms of a third variable, as in the case of the cycloid (Art. 50).

Let P (Fig. 116) be the position of the tracing point when the point of contact of the circles has moved from A to H; P was originally at A. Let a and b be the radii of the fixed and rolling circles, and θ, ϕ the angles turned through by OH and CH respectively; then the arc $AH = a\theta$, and the arc $PH = b\phi$.

Since these arcs are equal, $a\theta = b\phi$, i.e. $\phi = a\theta/b$.

Let (x, y) be the coordinates of P referred to O as origin and OA as axis of x. Then

$$x = OK - PM = OC\cos\theta - PC\cos CPM$$
$$= (a+b)\cos\theta - b\cos(\theta+\phi), \text{ [since } CPM = CLK = \theta+\phi]$$
$$= (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta, \quad \left(\text{since } \phi = \frac{a}{b}\theta\right).$$
$$y = KC - MC = OC\sin\theta - PC\sin CPM = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta.$$

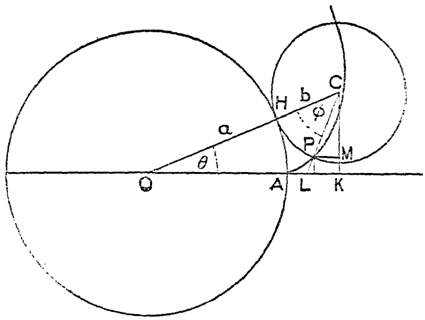


Fig. 116.

If the rolling circle be *inside* the fixed circle, it will be seen at once, by drawing a figure, that the coordinates of the tracing point are obtained by changing the sign of b.

Hence, in this case,

$$x = (a-b)\cos\theta + b\cos\frac{a-b}{b}\theta$$
$$y = (a-b)\sin\theta - b\sin\frac{a-b}{b}$$

If the rolling circle surround the fixed circle, $b > a$; but the latter equations still give the coordinates of the tracing point. The locus in this case is sometimes called a *pericycloid*.

All these curves are special cases of a class of curves known as *roulettes*.

It can be shown exactly as in the case of the cycloid (Art. 50) that, if P be joined to H and also to H', the other extremity of the diameter HC, then PH' and PH are respectively the tangent and the normal to the curve at P.

Particular cases. (i) In the case of the epicycloid, if $b = a$, the equations become $\quad x = 2a\cos\theta - a\cos 2\theta, \quad y = 2a\sin\theta - a\sin 2\theta.$

In this case the curve is a cardioid; for, if r be the distance of P from A, we have $r^2 = (x-a)^2 + y^2$, which reduces to $4a^2(1-\cos\theta)^2$.

Hence $$r = 2a(1-\cos\theta);$$
and it is obvious geometrically that in this case AP is parallel to OC, and the angle PAK is equal to θ, so that the locus of P is a cardioid with A as pole.

(ii) In the case of the hypocycloid, if $a = 2b$, the equations become
$$x = b\cos\theta + b\cos\theta = 2b\cos\theta, \quad y = b\sin\theta - b\sin\theta = 0.$$
Hence the tracing point moves along the axis of x and describes a diameter of the fixed circle.

(iii) If $a = 4b$, the equations become
$$x = 3b\cos\theta + b\cos 3\theta = b[3\cos\theta + 4\cos^3\theta - 3\cos\theta] = 4b\cos^3\theta,$$
$$y = 3b\sin\theta - b\sin 3\theta = b[3\sin\theta - 3\sin\theta + 4\sin^3\theta] = 4b\sin^3\theta;$$
$$\therefore \quad x^{2/3} + y^{2/3} = (4b)^{2/3}(\cos^2\theta + \sin^2\theta) = a^{2/3}.$$
In this case, the curve is the astroid [Art. 49, Ex. (i)].

Examples LXVIII.

1. Give the coordinates of any point on an epicycloid and a hypocycloid when $a = 3b$. Sketch the curves.
2. Find the value of dy/dx in an epicycloid; deduce that, if HCH' be a diameter of the rolling circle (Fig. 116), $H'P$ is the tangent at P.
3. Find $ds/d\theta$ in an epicycloid, and deduce the length of the curve traced out in one revolution of the rolling circle.
4. Find $ds/d\theta$ and the length of the curve in the case of the hypocycloid.
5. Find the equation of the tangent to the epicycloid in which $a = 2b$, at the point where $\theta = \frac{1}{4}\pi$.
6. Find the equation of the tangent to the hypocycloid in which $a = 3b$, at the point where $\theta = \frac{1}{3}\pi$.
7. Find the area between the epicycloid and the fixed circle when $a = 2b$.
8. Prove (geometrically) that the tangential-polar equation of the epicycloid is $r^2 = a^2 + \dfrac{4(a+b)b}{(a+2b)^2}p^2$.
9. Find the tangential-polar equation of the hypocycloid when $a = 3b$.
10. Obtain the coordinates of a point on an epicycloid when b becomes infinite, so that the rolling circle becomes a straight line.
    The epicycloid in this case is called an *involute* of the fixed circle.

CHAPTER XVIII

PHYSICAL APPLICATIONS

CENTRES OF GRAVITY

170. Centre of gravity. Centre of mass or inertia.

It is proved in text-books on Mechanics that the resultant of any number of parallel forces $P_1, P_2, \ldots$, acting at fixed points $A_1, A_2, \ldots$, is their algebraical sum $\Sigma(P)$, and that it acts at a point whose position relative to $A_1, A_2, \ldots$ is fixed. This point is called the *centre* of the system of parallel forces.

If $(x_1, y_1), (x_2, y_2), \ldots$ be the coordinates of $A_1, A_2, \ldots$, referred to rectangular axes OX, OY, it follows, by supposing the forces to be parallel to each axis in turn and taking moments about O, that the coordinates $(\bar{x}, \bar{y})$ of the centre are given by the equations

$$\bar{x} \,.\, \Sigma(P) = P_1 x_1 + P_2 x_2 + \ldots = \Sigma(Px),$$
$$\bar{y} \,.\, \Sigma(P) = P_1 y_1 + P_2 y_2 + \ldots = \Sigma(Py).$$

Each particle of a body is acted upon by a force, viz. its weight, along the line joining it to the centre of the earth (regarded as a sphere). In the case of all ordinary bodies, the distance of the centre of the earth is so great compared with the dimensions of the body that the weights of the different particles of the body may be regarded as a system of parallel forces. This system possesses a 'centre' which is fixed relative to the positions of the particles, i.e. fixed with respect to the body. The resultant of this system of parallel forces is the weight of the body, and its centre is called the *centre of gravity* (frequently denoted by the letters C. G.) of the body.

If $m_1, m_2, \ldots$ denote the masses of a system of particles whose coordinates are $(x_1, y_1), (x_2, y_2), \ldots$, the equations above, which determine the position of the centre of gravity of the system, become

$$\bar{x}\,\Sigma(mg) = \Sigma(mgx); \qquad \bar{y}\,\Sigma(mg) = \Sigma(mgy);$$

i.e. dividing by g,

$$M\bar{x} = \Sigma(mx); \qquad M\bar{y} = \Sigma(my),$$

if M be the total mass of the system.

$\Sigma(mx)$ and $\Sigma(my)$ are sometimes referred to as the *first moments* of the system about the axes of y and x respectively.

We here confine ourselves to the case in which the body is symmetrical about a plane; the centre of gravity lies in this plane, and the preceding equations determine its position relative to fixed axes in this plane.

In the case of a continuous distribution of mass, the summations above become definite integrals. The centre of gravity, as given by the preceding equations, coincides with the point (defined in various ways independently of the *weight* of the body) known as the *centroid* or *centre of mass* or *centre of inertia* of the body.

If the preceding equations be differentiated with respect to the time, we have, using the notation of Art. 62,

$$M\dot{\bar{x}} = \Sigma(m\dot{x}); \qquad M\dot{\bar{y}} = \Sigma(m\dot{y});$$

and, differentiating a second time,

$$M\ddot{\bar{x}} = \Sigma(m\ddot{x}); \qquad M\ddot{\bar{y}} = \Sigma(m\ddot{y}).$$

Hence the velocities and accelerations of the C. G. of a system of particles are obtained from the velocities and accelerations of the several particles by the same rule which gives the coordinates of the C. G. in terms of the coordinates of the particles.

171. Centre of mass of a lamina and of a solid of revolution.

(1) To find the centre of mass of a uniform thin lamina bounded by a curve $y = f(x)$, the axis of x, and two ordinates $x = a$, $x = b$, let the area be divided into elements by ordinates as in Fig. 117; let the coordinates of P and Q be (x, y) and $(x + \delta x, y + \delta y)$.

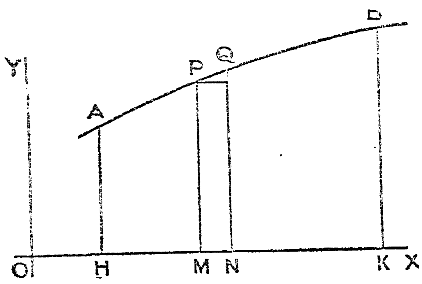


Fig. 117.

Consider the rectangle PN. Its area is $y\delta x$, and its mass $my\delta x$, if m be the mass per unit area of the lamina. The coordinates of the centre of mass of PN are $(x + \frac{1}{2}\delta x, \frac{1}{2}y)$; therefore, if $(\bar{x}, \bar{y})$ denote the coordinates of the centre of mass of $AHKB$, and M the total mass,

$$M\bar{x} = \mathrm{Lt}\sum_{x=a}^{x=b} my\delta x\,(x + \tfrac{1}{2}\delta x) = \int_a^b myx\,dx,$$

since $x + \frac{1}{2}\delta x \to x$ as $\delta x \to 0$; and

$$M\bar{y} = \mathrm{Lt}\sum_{x=a}^{x=b} my\delta x \,.\, \tfrac{1}{2}y = \int_a^b m \,.\, \tfrac{1}{2}y^2\,dx.$$

If the area be symmetrical about one of the axes, the centre of mass will be on the axis of symmetry, and only one coordinate has to be determined.

(2) If the area $AHKB$ makes a complete revolution about the axis of x, the centre of mass of the solid of revolution so formed will be on this axis. If m be the mass per unit volume, i.e. the density, then we have, taking moments about the origin,

$$M\bar{x} = \mathrm{L}_t \sum_{x=a}^{x=b} m\pi y^2\, \delta x \times (x + \tfrac{1}{2}\delta x) = \int_a^b m\pi y^2 x\, dx,$$

which gives the position of the centre of mass of the solid.

Examples:

(i) *Find the centre of mass of the area between the parabola* $y^2 = 4ax$, *the axis of* x, *and the ordinate* $x = b$.

We have

$$M\bar{x} = \int_0^b x \,.\, my\, dx = m\int_0^b x \,.\, 2\, a^{1/2} x^{1/2}\, dx = 2\, ma^{1/2}\int_0^b x^{3/2}\, dx = 2ma^{1/2} \,.\, \tfrac{2}{5} b^{5/2},$$

and $M = m \times \text{area} = m \,.\, \tfrac{2}{3} b \,.\, 2\, a^{1/2} b^{1/2}$ [Art. 79, Ex. (i)] $= \tfrac{4}{3} ma^{1/2} b^{3/2}$.

$$\therefore \quad \text{by division,} \quad \bar{x} = \tfrac{3}{5} b.$$

Similarly $\quad M\bar{y} = \int_0^b \tfrac{1}{2} y \,.\, my\, dy = m\int_0^b 2\, ax\, dx = mab^2;$

$$\therefore \quad \bar{y} = mab^2/M = mab^2/\tfrac{4}{3} ma^{1/2} b^{3/2} = \tfrac{3}{4}\sqrt{(ab)} = \tfrac{3}{8} \,.\, 2\sqrt{(ab)} = \tfrac{3}{8} BK.$$

(ii) *Find the centre of mass of the volume formed by the rotation of the same figure about the axis of* x.

In this case

$$M\bar{x} = \int_0^b m\,\pi\, y^2\, dx \times x = m\,\pi\int_0^b 4\, ax^2\, dx = \tfrac{4}{3} m\,\pi\, ab^3.$$

Also $\quad M = \int_0^b m\pi y^2\, dx = m\pi\int_0^b 4\, ax\, dx = 2\, m\pi ab^2.$

Therefore $\quad \bar{x} = \tfrac{2}{3} b.$

(iii) *Find the C. G. of a quadrilateral with two parallel sides.*

Let a and b be the lengths of the parallel sides AB and CD (Fig. 118), and c the distance between them; the C. G. obviously lies on the line MN which joins the middle points of AB and CD.

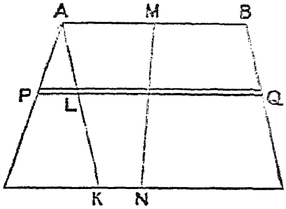


Fig. 118.

Let PQ be a strip of length x at distance y from AB. If m be the mass per unit area, the whole mass

$$= m \times \text{area} = \tfrac{1}{2}(a+b)\, cm.$$

$\therefore$ taking moments,

$$\tfrac{1}{2}(a+b)\, cm\, \bar{y} = \int_0^c mx\, dy \times y.$$

Let ALK, parallel to BC, meet PQ at L; then, by similar triangles,

$$\frac{y}{c}=\frac{AP}{AD}=\frac{PL}{DK}=\frac{x-a}{b-a}; \quad \text{whence} \quad x=a+\frac{b-a}{c}y,$$

and
$$\tfrac{1}{2}(a+b)cm\,\bar{y}=m\int_0^c\left(ay+\frac{b-a}{c}y^2\right)dy=m\left[\frac{ac^2}{2}+\frac{b-a}{c}\cdot\frac{c^3}{3}\right]$$
$$=mc^2\left[\tfrac{1}{2}a+\tfrac{1}{3}(b-a)\right]=\tfrac{1}{6}mc^2(a+2b);$$
$$\therefore\ \bar{y}=c\cdot\frac{a+2b}{3(a+b)}.$$

It follows that the C. G. divides MN in the ratio $a+2b:2a+b$.

From this result, the following simple geometrical construction for the C. G. easily follows:

Produce AB to E and CD to F so that $BE=CD$ and $DF=AB$. Let EF meet MN in G.

Then $\quad MG/GN=ME/NF=(\tfrac{1}{2}a+b)/(a+\tfrac{1}{2}b)=(a+2b)/(2a+b)$;

hence G is the C.G. of the figure.

We will now find the C. G. of a solid of revolution when the axis of rotation is not one of the axes of coordinates.

(iv) *The part of the parabola* $y^2=4ax$ *between the axis of* x *and the latus rectum rotates about the latus rectum; find the C. G. of the solid formed.*

The centre of gravity is obviously on the latus rectum SL (Fig. 119). Let $AS=a$, therefore $SL=2a$ (Ex. II. 20). Imagine the solid divided by planes perpendicular to SL into thin circular plates.

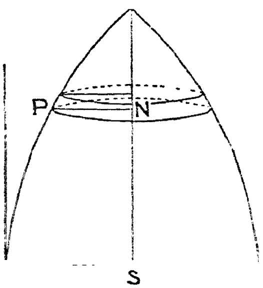


Fig. 119.

The mass of an element

$$=m\pi PN^2\,\delta y=m\pi(a-x)^2\,\delta y,$$

and its C. G. is at the height $y+\tfrac{1}{2}\delta y$, which $\to y$ as $\delta y\to 0$.

$\therefore$ the whole mass

$$=\int_0^{2a}m\pi(a-x)^2dy=m\pi\int_0^{2a}\left(a-\frac{y^2}{4a}\right)^2dy$$
$$=m\pi\int_0^{2a}(a^2-\tfrac{1}{2}y^2+\tfrac{1}{16}y^4/a^2)\,dy$$
$$=m\pi\left[a^2.2a-\tfrac{1}{6}(2a)^3+\tfrac{1}{80}(2a)^5/a^2\right]=\tfrac{16}{15}m\pi a^3.$$

Therefore, taking moments,

$$\tfrac{16}{15}m\pi a^3.\bar{y}=\int_0^{2a}m\pi(a-x)^2y\,dy=m\pi\int_0^{2a}\left[a^2y-\tfrac{1}{2}y^3+\tfrac{1}{16}y^5/a^2\right]dy$$
$$=m\pi\left[a^2.\tfrac{1}{2}(2a)^2-\tfrac{1}{8}(2a)^4+\tfrac{1}{96}(2a)^6/a^2\right]=\tfrac{2}{3}m\pi a^4,$$

whence $\quad \bar{y}=\tfrac{5}{8}a=\tfrac{5}{16}SL.$

172. Centres of gravity connected with the circle and sphere.

We will now solve some examples connected with the circle and the sphere.

To find the C. G. or centre of mass of:

(i) *A uniform circular arc.* Take the line which bisects the arc, upon which the C. G. obviously lies, as axis of x, and let the arc subtend an angle 2α at the centre.

Let m be the mass per unit length. Let s be the length of the arc measured from X to P, and let the angle $XOP = \theta$ (Fig. 120).

Then the mass of an element of arc PQ of length $\delta s = m\,\delta s = mr\,\delta\theta$, and the whole mass $= m\,.\,2r\alpha$.

Therefore, taking moments about O,

$$m.2r\alpha.\bar{x} = \int^{\alpha} mr\,d\theta\,.\,x = 2\int_0 mr\,d\theta\,.\,r\cos\theta$$
$$= 2mr^2\sin\alpha,$$

whence $\quad (r\sin\alpha)/\alpha.$

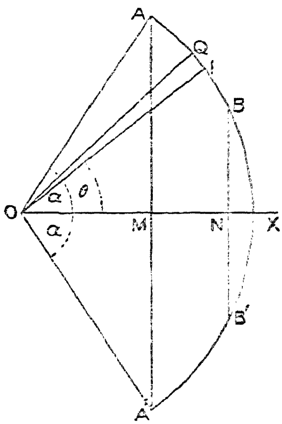


Fig. 120.

(ii) *A sector of a circle.* To find the C. G. of the sector $OAXA'$, we may regard it as the limit of $\Sigma(\Delta POQ)$. The area of this triangle $= \frac{1}{2}r^2\sin\delta\theta = \frac{1}{2}r^2\,\delta\theta \times (\sin\delta\theta)/\delta\theta$, and in the limit, the last factor is 1. Its C. G. is on the median from O to PQ, and therefore is ultimately at distance $\frac{2}{3}r$ from O. The area of the whole sector $= \frac{1}{2}r^2 \times 2\alpha$; hence, if m be the mass per unit area, and therefore $r^2\alpha m$ the whole mass, we have

$$r^2\alpha m \times \bar{x} = \int^{\alpha} m\,.\,\tfrac{1}{2}r^2\,d\theta\,.\,\tfrac{2}{3}r\cos\theta = \tfrac{2}{3}mr^3\sin$$

whence $$\bar{x} = \tfrac{2}{3}r(\sin\alpha)/\alpha.$$

(iii) *The area of the surface of a sphere intercepted by two parallel planes.*

Consider the surface formed by the rotation of the arc AB (Fig. 120) about the axis of x, and take, as in Art. 161, an element of surface $2\pi y\,\delta s$. Its C. G. is at a distance from O which tends to the limit x as $\delta s \to 0$.

$\therefore\ M\bar{x} = \int m\,2\pi y\,ds\,.\,x \quad$ (between suitable limits)

$$= 2m\pi\int_{\beta}^{\alpha} r\sin\theta\,.\,r\cos\theta\,.\,r\,d\theta \quad (\text{if } \alpha, \beta \text{ be the values of } \theta \text{ at } A \text{ and } B)$$

$$= m\pi r^3\int_{\beta}^{\alpha} 2\sin\theta\cos\theta\,d\theta = m\pi r^3\Big[-\cos^2\theta\Big]_{\beta}^{\alpha}$$

$$= m\pi r^3(\cos^2\beta - \cos^2\alpha).$$

Also, $M = m\int 2\pi y\,ds = m\int_{\beta}^{\alpha} 2\pi r\sin\theta\,.\,r\,d\theta = 2m\pi r^2(\cos\beta - \cos\alpha).$

$\therefore$ by division, $\quad \bar{x} = \frac{1}{2}r(\cos\beta + \cos\alpha) = \frac{1}{2}(OM + ON).$

Hence the C. G. is half-way between the bounding planes.

(iv) *The volume of the portion of a sphere cut off by a plane.*

Consider the volume formed by the rotation of AMX (Fig. 120) about the axis of x, and take an element of volume $\pi y^2\,\delta x$. Its C. G. is at a distance from O which tends to the limit x as $\delta x \to 0$. Therefore, denoting OM by h,

$$M\bar{x} = \int_h^r m\pi y^2\,dx \times x.$$

In this case, it is more convenient to integrate with respect to x.

$$M\bar{x} = \int_h^r m\pi x(r^2 - x^2)\,dx = m\pi\left[\tfrac{1}{2}r^2x^2 - \tfrac{1}{4}x^4\right]_h^r$$

$$= m\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{4}r^4 - (\tfrac{1}{2}r^2h^2 - \tfrac{1}{4}h^4)\right]$$

$$= \tfrac{1}{4}m\pi(r^4 - 2r^2h^2 + h^4) = \tfrac{1}{4}m\pi(r^2 - h^2)^2.$$

$$\text{Also, } M = \int_h^r m\pi(r^2 - x^2)\,dx = m\pi\left[r^2x - \tfrac{1}{3}x^3\right]_h^r = \tfrac{1}{3}m\pi(2r^3 - 3r^2h + h^3).$$

$$\therefore \text{ by division, } \bar{x} = \frac{3}{4}\frac{(r^2-h^2)^2}{2r^3 - 3r^2h + h^3} = \frac{3}{4}\frac{(r+h)^2}{2r+h},$$

after removing the common factor $(r-h)^2$.

Particular Cases.

If in (i) we take $\alpha = \frac{1}{2}\pi$, we have the C. G. of a semicircular arc at a distance $2r/\pi$ from the centre along the middle radius.

If in (ii) we take $\alpha = \frac{1}{2}\pi$, we have the C. G. of a semicircular area at a distance $4r/3\pi$ from the centre along the middle radius.

If in (iii) we take $\alpha = \frac{1}{2}\pi$, $\beta = 0$, we have the C. G. of the surface of a hemisphere or of an indefinitely thin hemispherical shell at a distance $\frac{1}{2}r$ from the centre along the middle radius.

If in (iv) we take $h = 0$, we have the C. G. of a solid hemisphere at a distance $\frac{3}{8}r$ from the centre along the middle radius.

173. Application of Simpson's Rule to centres of gravity.

If the equation of the bounding curve (in the case of an area) or the generating curve (in the case of a solid of revolution) be not known, or if the expressions obtained by the method of Art. 171 cannot be integrated, the position of the C. G. can be found approximately by Simpson's Rule (Art. 156), as shown in the following example:

A curve is drawn through the points (1, 2), (1·5, 2·4), (2, 2·7), (2·5, 2·8), (3, 3), (3·5, 2·6), (4, 2·1); *find the C. G. of the area between this curve, the axis of* x, *and the ordinates* x = 1 and x = 4.

We have $\bar{x} = \int_1^4 xy\,dx \div \int_1^4 y\,dx$; $\quad \bar{y} = \int_1^4 \frac{1}{2}y^2\,dx \div \int_1^4 y\,dx$.

The value of $\int_1^4 y\,dx$ has been found in Art. 156, Ex. (i), to be 7·8 nearly.

To find $\int_1 xy\,dx$, we first write down the successive values of xy at each point; they are 2, 3·6, 5·4, 7, 9, 9·1, and 8·4.

The sum of the first and last values $= 10·4$, twice the other odd values $= 2(5·4+9) = 28·8$, four times the even values $= 4(3·6+7+9·1) = 78·8$.

∴ the approximate value of the integral $= \frac{1}{3} \times ·5 (10·4 + 28·8 + 78·8) = 19·67$.

Similarly, the successive values of y^2 are 4, 5·76, 7·29, 7·84, 9, 6·76, and 4·41. Hence the approximate value of

$$\frac{1}{2}\int_1^4 y^2\,dx = \frac{1}{2} \times \frac{·5}{3}[4 + 4·41 + 2(7·29 + 9) + 4(5·76 + 7·84 + 6·76)]$$

$$= \tfrac{1}{12}(122·43) = 10·2.$$

Therefore the coordinates of the C. G. of the given area are approximately 19·67/7·8 and 10·2/7·8, i.e. (2·52, 1·31).

174. Pappus' theorems.

These are two useful theorems first given by Pappus of Alexandria about 300 A.D.

(1) If an arc of a plane curve rotate about an axis in its own plane which does not divide it into two parts, the area of the surface thereby formed is equal to the length of the arc multiplied by the length of the path of the centre of gravity of the arc.

Let the axis about which the curve rotates be taken as the axis of x.

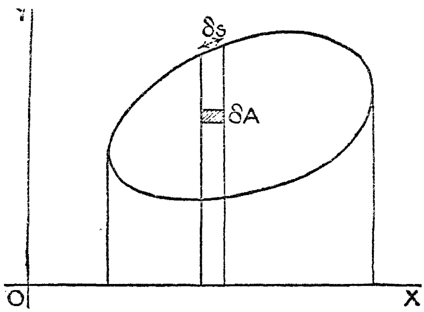


Fig. 121.

If l be the total length of the arc, and $\bar{y}$ the ordinate of its centre of gravity,

$$l\bar{y} = \int y\,ds \text{ between suitable limits.}$$

Hence the area of the surface generated

$$= \int 2\pi y\,ds = 2\pi l\bar{y} = l \times \text{length of path of C.G. of arc,}$$

which gives the theorem stated.

(2) If a plane area rotate about an axis in its own plane which does not divide it into two parts, the volume of the solid thereby formed is equal to the area multiplied by the length of the path of the centre of gravity of the area.

If δA be an element of area, y' the ordinate of its centre of gravity, and $\bar{y}'$ the ordinate of the centre of gravity of the area,

$$A\bar{y}' = \int y' dA.$$

Hence the volume generated $= \int 2\pi y' dA = 2\pi \bar{y}' A$

$= A \times$ length of path of C. G. of A.

These results are evidently true if the arc or the area does not make a complete revolution; in this case, the factor 2π in the preceding proofs is replaced by the factor α, where α is the circular measure of the angle turned through.

Examples:

(i) A circle of radius r rotates about an axis in its own plane at distance $c\,(>r)$ from its centre; find the volume and superficial area of the solid formed (which is called a *tore* or *anchor-ring*).

The centre of the circle is the centre of gravity of both arc and area.

Hence the superficial area $= 2\pi r \times 2\pi c = 4\pi^2 rc$,

and the volume $= \pi r^2 \times 2\pi c = 2\pi^2 r^2 c$.

(ii) These theorems can also be used to find the centre of gravity of a semicircular arc or area, for the rotation of semicircular area gives a sphere.

The volume of the sphere $\frac{4}{3}\pi r^3 =$ area of semicircle $\times$ length of path of its C. G. $= \frac{1}{2}\pi r^2 \times 2\pi \bar{y}$, whence $\bar{y} = 4r/3\pi$ for a semicircular area.

Similarly, the area of the surface of the sphere, i. e. $4\pi r^2 = \pi r \times 2\pi \bar{y}$, whence $\bar{y} = 2r/\pi$ for a semicircular arc.

Examples LXIX.

Find the C. G. of the following, 1–25:

1. (i) A quadrant of a circle. (ii) A quadrant of an ellipse.
2. A solid cone.
3. The area between the curve $xy = a^2$, the axis of x, and the ordinates $x = b$, $x = c$.
4. The figure bounded by one semi-undulation of the sine curve $y = b \sin(x/a)$ and the axis of x.
5. The part of a solid sphere of radius 10 inches intercepted between two parallel planes at distances 3 and 8 inches from the centre.
6. The area between $ay = x^2$, the axis of x, and $x = a$.
7. The area between $y = x^3$, the axis of y, and $y = 1$.
8. The solid formed when the portion of a parabola cut off by the latus rectum rotates about the axis.
9. Half a prolate spheroid bounded by a plane perpendicular to the major axis.

10. Half an oblate spheroid bounded by a plane perpendicular to the minor axis.
11. A segment of a circle cut off by a chord which subtends 60° at the centre.
12. A cardioid. [Proceed as in Art. 172 (ii).]
13. One of the four areas between the axes and the curve $x^{2/3} + y^{2/3} = a^{2/3}$.
14. The surface generated by the rotation of a quadrant of a circle about the tangent at one extremity.
15. The smaller of the two portions into which a solid sphere is divided by a plane which bisects a radius at right angles.
16. A frustum of a solid right circular cone, the radii of its ends being 8 inches and 6 inches, and its length 12 inches.
17. The solid formed by the rotation of the figure bounded by a quadrant of a circle and the tangents at its extremities about one of the tangents.
18. The area between the curve $y = (x-2)(5-x)$ and the axis of x.
19. The portion of an elliptical lamina between the minor axis and the latus rectum.
20. The area between the parabola $y = x^2 - 7x + 12$ and the axes of x and y.
21. The portion of the solid obtained by rotation of $y = x^2 - 4x + 6$ about the axis of x, between the sections $x = 1$ and $x = 4$.
22. The solid formed when the portion of the parabola $y = x^2 - 3x$ cut off by the axis of x rotates about the axis of x.
23. The arc of one arch of a cycloid.
24. The area between one arch of a cycloid and the axis of x.
25. The area between the catenary $y = c \cosh (x/c)$, the axis of x, and $x = \pm a$.
26. Find (by Pappus' Theorems) the surface and volume of the solid formed by the rotation of an equilateral triangle about its base.
27. Also of the solid formed by the rotation of a square about an axis in its plane through one corner perpendicular to the diagonal which passes through the corner.
28. A circle rotates about a tangent; find the superficial area and volume generated.
29. An ellipse rotates about its directrix; find the volume of the solid ring thereby formed.
30. A semicircular bend of iron pipe has a mean radius of 10 inches; the internal diameter of the pipe is 5 inches, and the thickness of the iron $\frac{1}{2}$ inch. Find the weight, supposing 1 cubic inch of iron weighs ·28 lb.
31. A square of side 6 inches with an isosceles triangle of height 6 inches standing on one side rotates about the opposite side; find the area of the surface and the volume of the solid which is formed.
32. Deduce from Pappus' Theorems the volume and area of surface of a cone and a cylinder.
33. An iron ring is in the form of the solid generated by the rotation of an ellipse whose semi-axes are 3 and 2 inches about an axis in its plane parallel to its major axis and distant 8 inches from it; find the weight of the ring if a cubic inch of iron weighs ·28 lb.
34. A curve is drawn through the points (2, 1·4), (3, 2), (4, 2·3), (5, 1·8), (6, 1·2); find the C. G. of the area between this curve, the extreme ordinates, and the axis of x.
35. Find the C. G. of the solid formed by rotating the curve in the preceding question about the axis of x.

CENTRES OF PRESSURE

175. Centre of pressure.

It is proved in text-books on Hydrostatics that the intensity of pressure at any point of an area immersed in a liquid varies as the depth of the point below the surface of the liquid and is equal to wh, where w is the 'specific weight', i.e. the weight per unit volume, of the liquid. The point of an immersed area at which the resultant pressure on the area acts is called the *centre of pressure* of the area. Its position can easily be determined by the Integral Calculus, as follows:

If δA be an element of the area at depth y below the surface, the pressure on $\delta A = wy\,\delta A$, and the total pressure on the area $= \int wy\,dA$, taken all over the area.

If $\bar{y}$ be the depth of the centre of gravity of the area, $A\bar{y} = \int y\,dA$; $\therefore$ the total pressure $= wA\bar{y} =$ the area $\times$ the pressure at its C.G.

If z be the depth of the centre of pressure below the surface, we have, by taking moments,

$$\text{the total pressure } \times z = \int y \times wy\,dA$$

i.e.

$$wA\bar{y}\,z = w\int y^2\,dA,$$

and

$$z = \frac{\int y^2\,dA}{A\bar{y}},$$

the integral being taken over the whole of the immersed area.

In evaluating the definite integral, the area is usually divided into strips parallel to the surface of the liquid.

Examples:

(i) *Find the centre of pressure of a triangle immersed with its base in the surface.*

Let b be the length of the base and h the height of the triangle. The resultant pressure on the triangle $= \frac{1}{2}bh \times$ pressure at C. G. $= \frac{1}{2}bh \times w.\frac{1}{3}h$ $= \frac{1}{6}wbh^2$.

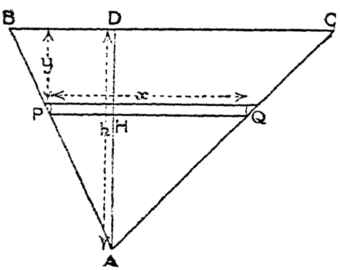


Fig. 122.

Dividing the triangle up by lines parallel to the surface, the pressure on a strip PQ (Fig. 122), whose upper edge is at depth y,

$$= PQ\,.\,\delta y \times w\,(y + \tfrac{1}{2}\delta y) = w\,.PQ\,.y\,\delta y,$$

neglecting small quantities of the second order.

$\therefore$ taking moments about the surface,

$$z \times \tfrac{1}{6}wbh^2 = \int_0^h w\,.PQ\,.\,y\,dy \times y.$$

By similar triangles, $\dfrac{PQ}{b} = \dfrac{AH}{AD} = \dfrac{h-y}{h}$, i. e. $PQ = \dfrac{b}{h}(h-y)$.

$$\therefore \quad z \times \tfrac{1}{6} wbh^2 = \int_0^h \frac{wb}{h}(hy^2 - y^3)\,dy = \frac{wb}{h}\Big[\tfrac{1}{3}hy^3 - \tfrac{1}{4}y^4\Big]_0^h = \frac{wb}{h}\cdot\tfrac{1}{12}h^4$$
$$= \tfrac{1}{12} wbh^3.$$

$\therefore$ the depth of the centre of pressure $= \frac{1}{2}h$.

Since the centre of pressure is obviously on the median through A, its position is determined.

(ii) *Find the centre of pressure of a rectangle, sides* a *and* b, *immersed vertically in a liquid with the sides* a *parallel to the surface, and its centre of gravity at a depth* h *below the surface.*

Dividing the rectangle into strips by lines parallel to the surface (Fig. 123), the pressure on a strip at depth y is $a\,\delta y \times wy$.

The resultant pressure on the rectangle $= ab \times wh$.

$\therefore$ taking moments about the surface,

$$ab\,.\,whz = \int_{h-\frac{1}{2}b}^{h+\frac{1}{2}b} a\,wy\,dy \times y = aw\Big[\tfrac{1}{3}y^3\Big]_{h-\frac{1}{2}b}^{h+\frac{1}{2}b}$$
$$= \tfrac{1}{3}aw\left[(h+\tfrac{1}{2}b)^3 - (h-\tfrac{1}{2}b)^3\right]$$
$$= \tfrac{1}{3}aw\,.\,2\left[3h^2\,.\,\tfrac{1}{2}b + \tfrac{1}{8}b^3\right] = \tfrac{1}{3}awb\left[3h^2 + \tfrac{1}{4}b^2\right],$$

whence $$z = h + \tfrac{1}{12}b^2/h.$$

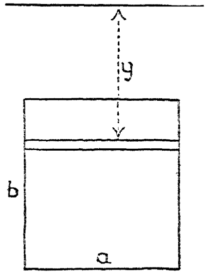


Fig. 123.

(iii) *Find the centre of pressure of a circle of radius* r *immersed with its plane vertical and its centre at depth* h ($>$ r) *below the surface.*

The resultant pressure on the circle $= \pi r^2 \times wh$.

The pressure on a strip PQ (Fig. 124) parallel to the surface and at depth y below it is $PQ\,\delta y \times wy$.

If PQ subtends an angle 2θ at the centre of the circle, $PQ = 2r\sin\theta$, $y = h - r\cos\theta$, $dy/d\theta = r\sin\theta$. Hence

$\pi r^2 hw \times$ depth of centre of pressure

$$= \int_{h-r}^{h+r} wy\,.\,PQ\,dy \times y$$
$$= \int_0^\pi w\,.\,2r\sin\theta\,(h - r\cos\theta)^2\,r\sin\theta\,d\theta$$
$$= 2r^2 w\int_0^\pi \left[h^2\sin^2\theta - 2hr\sin^2\theta\cos\theta + r^2\sin^2\theta\cos^2\theta\right]d\theta$$

(The second integral is 0 by Art. 146)

$$= 4r^2 w\int_0^{\frac{1}{2}\pi}(h^2\sin^2\theta + r^2\sin^2\theta\cos^2\theta)\,d\theta$$
$$= 4r^2 w\left[h^2\,.\,\tfrac{1}{2}\,.\,\tfrac{1}{2}\pi + r^2\,.\,\frac{1}{4\,.\,2}\,.\,\tfrac{1}{2}\pi\right] \text{ (Art. 149)}$$
$$= \pi r^2 w\left[h^2 + \tfrac{1}{4}r^2\right].$$

$\therefore$ depth of centre of pressure $= h + \frac{1}{4}r^2/h$.

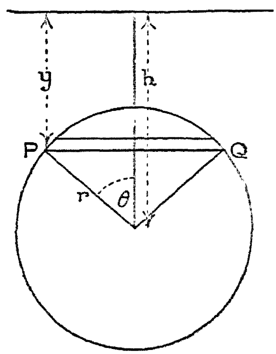


Fig. 124.

Examples LXX.

Find the C.P. of the following, 1–7.

1. A triangle immersed with its vertex in the surface and its base parallel to the surface.
2. A triangle immersed with its vertex upwards and at depth h below the surface and its base parallel to the surface.
3. A rectangle 3 ft. by 4 ft. immersed vertically with its shorter sides horizontal and the upper one 2 ft. below the surface.
4. A semicircle immersed with its bounding diameter in the surface.
5. An ellipse immersed with its major axis vertical and one vertex in the surface.
6. A trapezium immersed with one of its parallel sides in the surface.
7. The area cut off from a parabola by its latus rectum, immersed with the latus rectum in the surface.
8. A triangle is immersed in water with its base in the surface; show that the pressures on the two parts into which it is divided by a horizontal line through its centre of pressure are equal.
9. Find the displacement of the centre of pressure caused by increasing the depth of an immersed area by a given amount h.
10. Prove that the limiting position of the C.P., as h is increased indefinitely, coincides with the C.G.

MOMENTS OF INERTIA

176. Moments of inertia.

If particles of masses $m_1, m_2, \ldots$ be situated at points whose perpendicular distances from a given straight line are $r_1, r_2, \ldots$, then $\Sigma(mr^2)$, i.e. $m_1 r_1^2 + m_2 r_2^2 + \ldots$ is called the *moment of inertia* of the system about the given line.

It is sometimes called the *second moment* of the system about the given line, $\Sigma(mr)$ being called the first moment [cf. Art. 170].

In the case of a continuous distribution of mass, the summation becomes a definite integral. If δm be an element of mass of a body at distance r from a fixed line, $\mathrm{Lt}\,\Sigma r^2 \delta m$, i.e. $\int r^2\,dm$ taken throughout the body, is the moment of inertia of the body about the given line.

The moment of inertia of a body is of very great importance in Dynamics in dealing with rotation (see Art. 196); it plays a part in the rotation of a body similar to that played by the mass in a motion of translation. For example, if the given system of masses have a common velocity v, parallel to a given straight line, the kinetic energy of the system $= \frac{1}{2}$ (mass of system) v^2. If the system of particles above mentioned be rigidly connected by a framework of negligible mass, and rotate about the fixed straight line with angular

velocity ω, the linear velocities of the particles $m_1, m_2, \ldots$ will be $r_1\omega, r_2\omega, \ldots$ respectively, and the kinetic energy of the system will be

$$\tfrac{1}{2}m_1(r_1\omega)^2 + \tfrac{1}{2}m_2(r_2\omega)^2 + \ldots, \quad \text{i.e. } \tfrac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \ldots)\,\omega^2,$$

i.e. $\frac{1}{2}$(moment of inertia of the system) $\times\, \omega^2$.

If the moment of inertia of a body of mass M about a line be written in the form Mk^2, k is called the *radius of gyration* of the body about the line. In the case of a uniform wire of negligible thickness bent into a circle of radius r, every point of the wire is at distance r from an axis through its centre perpendicular to its plane; hence its moment of inertia about this axis is mr^2, and the radius of gyration is equal to the radius of the circle.

The moment of inertia and the kinetic energy of a body rotating about a fixed axis are the same as if the whole mass were collected at a distance k from the axis.

The letters M. I. are generally used as an abbreviation for the term 'moment of inertia'. Methods of evaluating moments of inertia are shown in the following examples:

Examples:

(i) *Find the M.I. of a uniform straight rod about an axis perpendicular to its length through a point at a distance* b *from its centre.*

Let $2a$ be the length of the rod and m the mass per unit length; therefore the whole mass M is $2am$. Taking the axis of x along the rod and the axis about which the M.I. is required as axis of y, the mass of an element PQ (Fig. 125) is $m\,\delta x$, and its M.I. about OY is (to the first order of small quantities) $m\,\delta x \times x^2$, if $OP = x$.

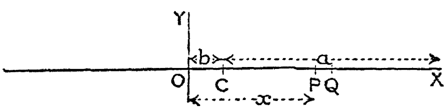


Fig. 125.

Hence the M. I. of the rod

$$= \int_{-a+b}^{a+b} mx^2\,dx = m\Big[\tfrac{1}{3}x^3\Big]_{-a+b}^{a+b} = \tfrac{1}{3}m\,[(a+b)^3 - (-a+b)^3]$$

$$= \tfrac{1}{3}m\,(2a^3 + 6ab^2) = M(\tfrac{1}{3}a^2 + b^2).$$

If the axis pass through the centre of the rod, $b = 0$, and the M. I. $= \frac{1}{3}Ma^2$. Therefore the radius of gyration $= a/\sqrt{3}$.

If the axis pass through one end of the rod, $b = a$, and the M. I. $= \frac{4}{3}Ma^2$. Therefore the radius of gyration $= 2a/\sqrt{3}$.

(ii) *Find the M. I. of a rectangle about an axis parallel to a side.*

If $2b$ (Fig. 126) be the length of the sides parallel to the axis, and $2a$ the length of the other sides, the M. I. of a strip parallel to the axis

$$= m \,.\, 2b\,\delta x \times x^2,$$

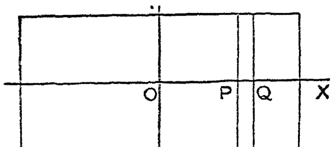


Fig. 126.

and the whole mass M is $4mab$; the working is the same as in the preceding example with the addition of the factor $2b$ until the M is introduced, and the results are exactly the same. In fact the rectangle may be regarded as made up of rods perpendicular to the axis, and the preceding result, being true for each one, is true for the sum of them.

(iii) *Find the M. I. of a circular disc about an axis through its centre perpendicular to its plane.*

Let a be the radius of the disc and m its mass per unit area; therefore its total mass M is $\pi a^2 m$. Divide the disc into elements by means of concentric circles (Fig. 127); the mass of the element between two circles of radii r and $r+\delta r$ is ultimately $m \,.\, 2\pi r\,\delta r$, and its radius of gyration is r.

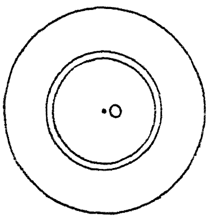


Fig. 127.

$$\text{M. I. of disc} = \int_0^a m \,.\, 2\pi r\, dr \times r^2$$

$$= 2m\pi \left[\tfrac{1}{4} r^4\right]_0^a = m\pi \,.\, \tfrac{1}{2} a^4 = \tfrac{1}{2} Ma^2.$$

Hence the radius of gyration $= a/\sqrt{2}$.

This result may now be used to obtain the M. I. of a solid of revolution about the axis of revolution, as shown in the following example.

(iv) *Find the M. I. of a sphere about a diameter.*

Let r be the radius and m the mass per unit volume; therefore the whole mass M is $\frac{4}{3} m\pi r^3$. Divide the sphere into thin slices by planes perpendicular to the diameter about which the M. I. is required (Fig. 128).

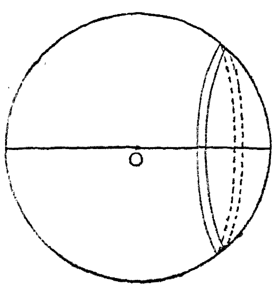


Fig. 128.

The mass of an element $= m\pi y^2 \delta x$, and, by the preceding result, its M. I.

$$= (\text{its mass}) \times \tfrac{1}{2} y^2 = m\pi y^2\,\delta x \,.\, \tfrac{1}{2} y^2.$$

$\therefore$ M. I. of whole sphere

$$\tfrac{1}{2} m\pi y^4\, dx = \tfrac{1}{2} m\pi \int_{-r}^{r} (r^2 - x^2)^2\, dx$$

$$\tfrac{1}{2} m\pi \times 2 \int_0^r (r^4 - 2r^2x^2 + x^4)\, dx$$

$$= m\pi \left[r^4 x - \tfrac{2}{3} r^2 x^3 + \tfrac{1}{5} x^5\right]_0^r = m\pi\left[r^5 - \tfrac{2}{3} r^5 + \tfrac{1}{5} r^5\right] = \tfrac{8}{15} m\pi r^5$$

$$= \tfrac{4}{3} m\pi r^3 \times \tfrac{2}{5} r^2 = \tfrac{2}{5} Mr^2.$$

Therefore the radius of gyration $= r\sqrt{\tfrac{2}{5}}$.

(v) *Find the M. I. of an elliptic lamina about its major axis.*

Divide the ellipse into indefinitely thin strips by lines perpendicular to the major axis (Fig. 129). The mass of an element is $m \,.\, 2y\,\delta x$, and its M. I. about the major axis is, by Ex. (i),

$$m\,2y\,\delta x \times \tfrac{1}{3} y^2.$$

$\therefore$ M.I. of ellipse

$$= \int_{-a}^{a} \tfrac{2}{3} m y^3\,dx = \tfrac{2}{3} m \,.\, 2 \int_0^a y^3\,dx$$

[let $x = a\cos\theta$, $y = b\sin\theta$, (Art. 50)]

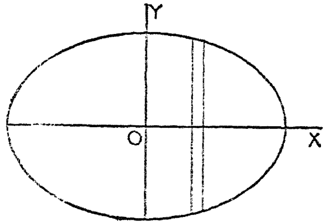


Fig. 129.

$$= \tfrac{4}{3} m \int_{\frac{1}{2}\pi}^{0} b^3 \sin^3\theta \times -a\sin\theta\,d\theta$$

$$= \tfrac{4}{3} m b^3 a \int_0^{\frac{1}{2}\pi} \sin^4\theta\,d\theta = \tfrac{4}{3} m b^3 a \times \frac{3\,.\,1}{4\,.\,2} \cdot \tfrac{1}{2}\pi = \tfrac{1}{4}\pi\, m a b^3$$

$$= \tfrac{1}{4} M b^2, \text{ since } M = m \times \text{area} = m\pi ab.$$

The radius of gyration $= \frac{1}{2} b$.

This is given as an example of the way in which an area may be divided up into strips perpendicular to the axis about which the M. I. is required. The result might have been obtained by dividing the area into strips parallel to the major axis. The mass of such a strip is $m \,.\, 2x\,\delta y$, and its radius of gyration is y; hence the M. I. $= 2\int_0^b m \,.\, 2xy^2\,dy$, which may be evaluated in a similar manner, and gives the same result.

As in the case of areas, volumes, and C. G., if the equation of the bounding curve is not known, or if the general formula gives an expression which cannot be integrated, an approximate value of the M. I. can be found by the use of Simpson's Rule. For instance,

(vi) *To find the radius of gyration about the axis of revolution of the solid described in Ex.* (iii), p. 303.

If A be the area of a section of radius y, perpendicular to the axis of x, we have

$$Mk^2 = \int mA \,.\, \tfrac{1}{2} y^2\,dx = m \,.\, \tfrac{1}{2}\pi \int y^4\,dx, \text{ since } A = \pi y^2.$$

The values of y are $18/\pi, 21/\pi, 23/\pi, 25/\pi, 26/\pi$, whence, by logarithms, the values of y^4 are found to be 1077, 1995, 2870, 4006, 4690, approximately, and $M = m \times \text{volume} = 4025m$, using the result obtained in Art. 159.

$\therefore$ by Simpson's Rule,

$$4025\,k^2 = \tfrac{1}{2}\pi \times \tfrac{6}{3}[1077 + 4690 + 2(2870) + 4(1995 + 4006)] = \pi \,.\, 35511,$$

whence k is found to be 5·265 approximately.

Examples LXXI.

Find the M. I. of

1. A square about a side.
2. A rectangle, sides a and b, about a line parallel to the sides a and distant $\frac{1}{3}b$, $\frac{2}{3}b$ from them respectively.
3. A flat circular ring, whose outer and inner radii are r and $2r$, about an axis through its centre perpendicular to its plane.
4. A circle about a diameter.
5. An isosceles triangle about an axis through its vertex parallel to its base.
6. The same triangle about its base.
7. The same triangle about its axis.
8. The same triangle about a line through its C. G. parallel to its base.
9. A right circular cylinder about its axis.
10. A right circular cone about its axis.
11. A spheroid about its axis of revolution.
12. An elliptic lamina about a latus rectum.
13. The portion of a paraboloid of revolution bounded by the section $x = b$, about its axis.
14. A thin uniform circular wire about a diameter.
15. An indefinitely thin spherical shell about a diameter.
16. The area between the parabola $y^2 = 4ax$ and the double ordinate $x = b$, about the tangent at the vertex.
17. The same area about its axis.
18. The same area about the ordinate $x = b$.
19. The area described in Ex. LXIX. 34, about the axis of x.
20. The volume described in Ex. LXIX. 35, about the axis of x.
21. A uniform arc of a circle about its chord.
22. The area between one arch of a cycloid and its base, about the base.
23. The solid formed by the rotation of a cycloid about its base, about the axis of revolution.
24. The area enclosed by the curve $x^{2/3} + y^{2/3} = a^{2/3}$, about one of the axes.

177. General theorems on moments of inertia.

The evaluation of moments of inertia is facilitated by several simple general theorems which establish relations between moments of inertia about different axes.

I. *The M. I. of a lamina about an axis perpendicular to its plane through a point O in its plane is equal to the sum of the M. I. about any two rectangular axes through O in the plane.*

If r (Fig. 130) be the distance of an element δm from the origin O, and (x, y) its coordinates referred to two rectangular axes through O, the M. I. about a line through O perpendicular to the plane XOY

$$= \int r^2\, dm = \int (x^2 + y^2)\, dm$$

$$= \int x^2\, dm + \int y^2\, dm, \text{ taken all over the area,}$$

$$= \text{M. I. about } OY + \text{M. I. about } OX.$$

Examples. This result may be used to deduce the M.I. of a circular disc about a diameter from the M. I. about an axis through its centre perpendicular to its plane. For we have

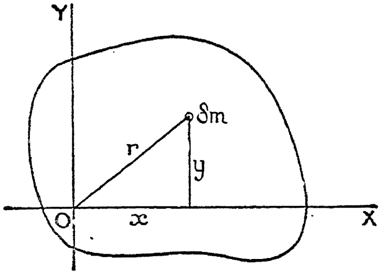


Fig. 130.

$\frac{1}{2}Ma^2$ = M. I. about a perpendicular axis through the centre [Art. 176 (iii)]
= sum of M. I. about two rectangular axes in its plane
= 2 × M. I. about a diameter, from symmetry.

∴ M. I. about a diameter $= \frac{1}{4}Ma^2$.

Again, the M. I. of a rectangle, sides $2a$, $2b$, about an axis through one corner perpendicular to its plane = sum of M. I. about the two sides through that corner $= M \cdot \frac{4}{3}a^2 + M \cdot \frac{4}{3}b^2 = M \cdot \frac{4}{3}(a^2+b^2)$.

Hence also the M. I. of a square lamina of side a about an axis through one corner perpendicular to its plane = sum of M. I. about two sides $= \frac{2}{3}Ma^2$; and since a cube may be regarded as made up of square laminae, for each one of which the preceding result is true, it follows that the M. I. of a cube of side a about an edge $= \frac{2}{3}Ma^2$.

II. *The M. I. of a body about any axis exceeds the M. I. about a parallel axis through the centre of gravity by the product of the mass into the square of the distance between the parallel axes* (i.e. by the M. I. of the whole mass collected at the centre of gravity about the original axis).

From this theorem it follows that the M. I. about an axis through the C. G. is less than the M. I. about any parallel axis.

Let G (Fig. 131) be the centre of gravity of the body. Let the given axis meet the plane through G perpendicular to it in A, at distance a from G, and let a linear element δm, parallel to the given axis, cut this plane at P; let AG be taken as the axis of x, and let (x, y) be the coordinates of P.

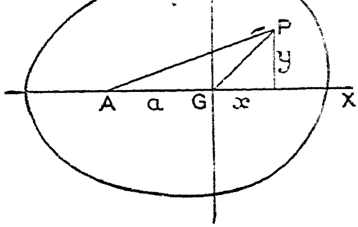


Fig. 131.

The M.I. about the line through A perpendicular to the plane XGY

$$= \int AP^2\,dm = \int[(a+x)^2+y^2]\,dm = \int (a^2+2ax+x^2+y^2)\,dm$$
$$= a^2\int dm + 2a\int x\,dm + \int GP^2\,dm$$
$$= a^2 M + 0 + \text{M.I. about the line through } G \text{ perpendicular to the plane } XGY,$$

since $\int x\,dm = M\bar{x}$, where $\bar{x}$ is the abscissa of the C.G. [Art. 170]
$= 0$, since the C.G. is the origin.

Hence the M.I. about the axis through A exceeds the M.I. about the parallel axis through the centre of gravity by Ma^2.

Examples. The M.I. of a rod or rectangle of length $2a$ about an axis through its centre perpendicular to its length is $\frac{1}{3}Ma^2$; hence the M.I. about a parallel axis through one extremity $= \frac{1}{3}Ma^2 + Ma^2 = \frac{4}{3}Ma^2$.

The M.I. of a circular disc of radius r about a line through a point on its edge perpendicular to its plane

$= $ M.I. about axis through centre perpendicular to its plane $+Mr^2$
$= \frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2$.

The M.I. of the disc about a tangent line $= \frac{1}{4}Mr^2 + Mr^2 = \frac{5}{4}Mr^2$.

It must be carefully noticed that the theorem does not connect the M.I. about any two parallel axes; one of them must go through the centre of gravity.

E.g. the M.I. of an isosceles triangle (Fig. 132), of height h and vertical angle 2α, about a line through its vertex parallel to its base

$$= \int_0^h m\,.\,2y\,dx\,.\,x^2 = 2m\int_0^h x^3\tan\alpha\,dx = 2m\tan\alpha\,.\,\tfrac{1}{4}h^4 = \tfrac{1}{2}Mh^2,$$

since $M = mh \times \frac{1}{2}\text{base} = mh^2\tan\alpha$.

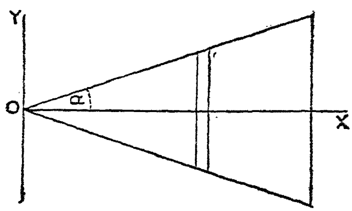


Fig. 132.

To deduce the M.I. about the base, we must first find the M.I. about a parallel axis through the C.G. of the triangle. The distance between the C.G. and the vertex is $\frac{2}{3}h$;

$\therefore\ \frac{1}{2}Mh^2 = $ M.I. about a parallel axis through C.G. $+ \frac{4}{9}Mh^2$,

$\therefore$ M.I. about axis through C.G. parallel to base $= \frac{1}{18}Mh^2$.

The distance from the C.G. to the base $= \frac{1}{3}h$;

$\therefore$ M.I. about base $= \frac{1}{18}Mh^2 + \frac{1}{9}Mh^2 = \frac{1}{6}Mh^2$.

III. *To find the M. I. of a lamina about a line through the origin inclined to the axes.*

Let the straight line OA (Fig. 133) be inclined to OX at an angle α. Let δm be an element of mass situated at the point P whose coordinates are (x, y). Draw PM, PN perpendicular to OA, OX and NL, NH perpendicular to PM, OA respectively.

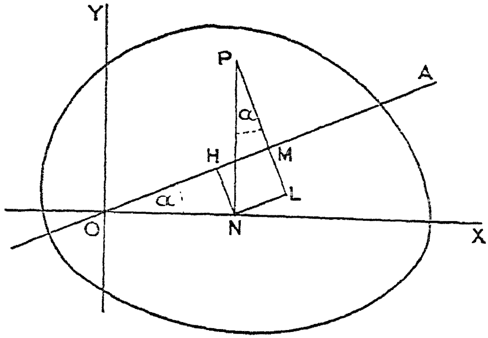


Fig. 133.

The M. I. of the lamina about OA

$$= \int MP^2\, dm = \int (LP - NH)^2\, dm = \int (y \cos\alpha - x \sin\alpha)^2\, dm$$

$$= \cos^2\alpha \int y^2\, dm - 2 \sin\alpha \cos\alpha \int xy\, dm + \sin^2\alpha \int x^2\, dm,$$

the integrals being taken all over the lamina.

$\int xy\, dm$ is called the 'product of inertia' about the axes OX, OY. If the body be symmetrical about either of the coordinate axes, it is evident that this integral $\int xy\, dm$ is zero; for, if symmetrical about the axis of x, then, to any value of $xy\delta m$ for a positive value of y, there is a value for the corresponding negative value of y which will be equal in magnitude and opposite in sign; hence, as in Art. 146, the terms of the sum whose limit is the definite integral cancel in pairs, and the integral is zero. Similarly if the lamina is symmetrical about the axis of y.

In this case, the M. I. of the lamina about OA

$$= \cos^2\alpha \int y^2\, dm + \sin^2\alpha \int x^2\, dm,$$

i.e. *if the lamina is symmetrical about one (or both) of the axes OX, OY, the M. I. about a line inclined at angle α to OX is equal to*

$$(\textit{M.I. about } OX) \cos^2\alpha + (\textit{M. I. about } OY) \sin^2\alpha.$$

In this case the M. I. about OX, OY are called the 'principal moments of inertia relative to O', and the axes are called the 'principal axes at O'. For further information as to principal axes and moments of inertia, the student is referred to works on rigid dynamics, it is there shown that at every point of a lamina there is a pair of axes for which the product of inertia is zero.

It is easily seen that this theorem is also true for a solid body which is symmetrical about the plane XOY. For in this case, if $2z$ be the length of the element through P perpendicular to the plane XOY, of which P is the middle point, the M.I. of the element about $OA = \delta m\,(\frac{1}{3}z^2 + MP^2)$; and its M.I. about OX and OY are respectively $\delta m\,(\frac{1}{3}z^2 + y^2)$ and $\delta m\,(\frac{1}{3}z^2 + x^2)$.

$\therefore$ the M.I. of the body about OA, as before,

$$= \int \tfrac{1}{3} z^2\, dm + \cos^2\alpha \int y^2\, dm + \sin^2\alpha \int x^2\, dm$$

$$= \cos^2\alpha \int (\tfrac{1}{3}z^2 + y^2)\, dm + \sin^2\alpha \int (\tfrac{1}{3}z^2 + x^2)\, dm, \text{ [since } \sin^2\alpha + \cos^2\alpha = 1]$$

$$= \cos^2\alpha \times \text{M. I. about } OX + \sin^2\alpha \times \text{M. I. about } OY.$$

Examples:

(i) *Find the M.I. of a rectangle, sides* $2a$ *and* $2b$, *about a diagonal.*

The M.I. about lines through the centre parallel to the edges are $\frac{1}{3}Ma^2$ and $\frac{1}{3}Mb^2$, and the rectangle is symmetrical about these lines.

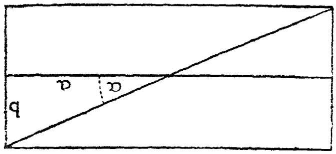


Fig. 134.

If α (Fig. 134) be the angle between the diagonal and a side whose length is $2a$

$$\cos^2\alpha = a^2/(a^2+b^2), \text{ and } \sin^2\alpha = b^2/(a^2+b^2).$$

$$\therefore \text{ M. I. about diagonal } = \tfrac{1}{3}Mb^2\cos^2\alpha + \tfrac{1}{3}Ma^2\sin^2\alpha$$
$$= \tfrac{1}{3}Mb^2 \,.\, a^2/(a^2+b^2) + \tfrac{1}{3}Ma^2 \,.\, b^2/(a^2+b^2)$$
$$= \tfrac{2}{3}Ma^2b^2/(a^2+b^2).$$

(ii) *Find the M. I. of a solid right circular cone about an axis through its vertex parallel to its base.*

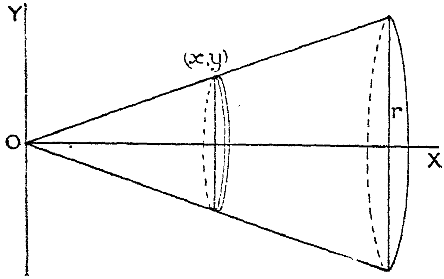


Fig. 135.

Divide the cone (Fig. 135) into thin circular slices by planes perpendicular to its axis.

The mass of an element is $m\pi y^2\,\delta x$, and its M. I. about one of its diameters is $m\pi y^2\,\delta x\,.\,\frac{1}{4}y^2$; $\therefore$ its M. I. about the given axis, which is at distance x from a parallel axis through the C. G., is $m\pi y^2\,\delta x\,(\frac{1}{4}y^2+x^2)$.

$$\begin{aligned}
\therefore\ \text{M. I. of cone} &= \int_0^h m\pi y^2\left(\tfrac{1}{4}y^2+x^2\right)dx \\
&= m\pi\int_0^h\left(\frac{1}{4}\frac{r^4}{h^4}x^4+\frac{r^2}{h^2}x^4\right)dx,\ \text{ since }\ \frac{y}{x}=\frac{r}{h}, \\
&= \left(\frac{r^4}{4h^4}+\frac{r^2}{h^2}\right)\int_0^h x^4\,dx \\
&= \frac{m\pi r^2}{4h^4}\left(r^2+4h^2\right).\frac{h^5}{5} \\
&= \tfrac{1}{20}m\pi r^2h\left(r^2+4h^2\right) \\
&= \tfrac{3}{20}M\left(r^2+4h^2\right),\ \text{ since the whole mass } M=\tfrac{1}{3}m\pi r^2h.
\end{aligned}$$

From this result the M. I. about a parallel axis through the C. G., and then the M. I. about a diameter of the base can be deduced.

The M. I. about a generating line can also be deduced by Theorem III of this article; for the M. I. about the axis is easily found by direct integration to be $\frac{3}{10}Mr^2$, and the cone is symmetrical both about the axis, and about any plane through the axis.

Hence, if α be the semi-vertical angle of the cone, the M. I. about a generating line

$$\begin{aligned}
&= \tfrac{3}{10}Mr^2\cos^2\alpha+\tfrac{3}{20}M\left(r^2+4h^2\right)\sin^2\alpha \\
&= \tfrac{3}{10}Mr^2.\frac{h^2}{r^2+h^2}+\tfrac{3}{20}M\left(r^2+4h^2\right)\frac{r^2}{r^2+h^2} \\
&= \tfrac{3}{20}M\frac{r^2}{r^2+h^2}\left(2h^2+r^2+4h^2\right)=\tfrac{3}{20}M\frac{r^2\left(r^2+6h^2\right)}{r^2+h^2}.
\end{aligned}$$

Examples LXXII.

Find the M. I. of

1. A flat circular ring, radii r and r', about a diameter.
2. A square about an axis through one corner perpendicular to its plane.
3. An ellipse about an axis through its centre perpendicular to its plane.
4. A square lamina of side a about an axis through its centre perpendicular to its plane.
5. An equilateral triangular lamina about an axis through the middle point of its base perpendicular to its plane.
6. An ellipse about (i) the tangent at one end of the major axis, (ii) a latus rectum, (iii) a directrix.
7. An equilateral triangle about an axis through its C. G. perpendicular to its plane.
8. A cylinder about a generating line.
9. A sphere about a tangent line.

10. A straight rod of length $2a$ about an axis perpendicular to its length at distance b from one end.

11. A square of side a about any line through its centre in its plane. (Deduce both from Example 4 above, and from Theorem III).

12. A square about any line in its plane at distance b from its centre.

13. An ellipse about the line joining the extremities of the axes.

14. An isosceles triangle, of height h and base $2b$, about a line joining the middle point of the base to the middle point of one of the equal sides.

15. A solid cone about (i) an axis through the C. G. parallel to the base, (ii) a diameter of the base.

16. A solid cylinder about a diameter of one end.

17. A solid cylinder about (i) a line through the C. G. perpendicular to the axis, (ii) a tangent to one of the circular ends.

18. The solid formed by the rotation of a rectangle, sides a and b, about a line in its plane distant c ($> \frac{1}{2}b$) from its centre and parallel to the sides a, about the axis of rotation.

19. A solid anchor-ring about the axis of rotation.

20. An arc of a circle about an axis through its middle point perpendicular to its plane.

21. A rod in which the line-density varies as the distance from one end, about an axis through that end perpendicular to the rod.

22. A circular disc in which the surface-density varies as the distance from the centre, about an axis through the centre perpendicular to the disc.

23. A right-angled triangle about a line through the right angle perpendicular to its plane.

24. A paraboloid of revolution bounded by the section $x = b$, about a tangent line at the vertex.

25. A spheroid about a tangent at an extremity of the axis of rotation.

POTENTIAL

178. Potential.

If $m_1, m_2, \ldots$ be the masses of a system of particles situated at distances $r_1, r_2, \ldots$ respectively from a point P, then

$$\sum \frac{m}{r} \equiv \frac{m_1}{r_1} + \frac{m_2}{r_2} + \ldots$$

is called the *potential* of the system at the point P. This function is of great importance in the theory of attractions and in electricity. In the case of a continuous distribution of mass, the summation becomes a definite integral.

Examples of the calculation of the function in several important cases are here given.

Examples:

(i) *Find the potential of a circular disc at a point on its axis.*

If the disc be divided into elements by concentric circles, the potential of

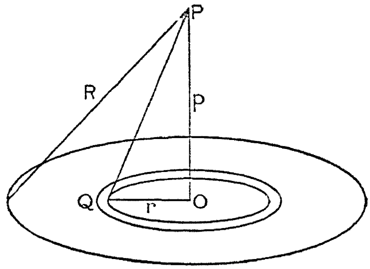


Fig. 136.

an element at a point P (Fig. 136) on the axis is $m\,.\,2\pi r\,\delta r/PQ$ (to the first order of small quantities), m being the mass per unit area.

Let p be the distance of P from the disc and a the radius of the disc.

Then the potential of the disc at $P = \int_0^a \frac{2m\pi r}{PQ}\,dr$

$$= 2m\pi \int_0^a \frac{r}{\sqrt{(r^2+p^2)}}\,dr = 2m\pi\left[\sqrt{(r^2+p^2)}\right]_0^a$$
$$= 2m\pi\left[\sqrt{(p^2+a^2)}-p\right] = 2m\pi(R-p),$$

if R be the distance of P from a point on the edge of the disc.

(ii) *Find the potential of a thin spherical shell at any point.*

Let c be the distance of P (Fig. 137) from the centre of the spherical shell. Divide the shell into elements by planes perpendicular to OP.

The potential of an element at $P = \frac{m\,.\,2\pi y\,\delta s}{PQ} = \frac{m\,.\,2\pi r\sin\theta\,.\,r\delta\theta}{PQ}$

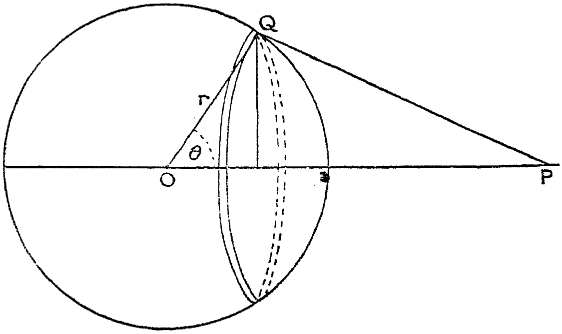


Fig. 137.

$\therefore$ the potential of the whole shell $= \int \frac{m\,.\,2\pi r^2\sin\theta\,d\theta}{PQ}$

$$= 2m\pi r^2 \int \frac{\sin\theta}{\sqrt{(c^2+r^2-2cr\cos\theta)}}\,d\theta.$$

This can be integrated by putting $c^2+r^2-2cr\cos\theta = z^2$;

$$\therefore \quad 2cr\sin\theta = 2z\,dz/d\theta, \quad \text{i.e.} \quad cr\sin\theta/z = dz/d\theta,$$

and the limits for z (which is PQ) are $c-r$ and $c+r$, if P be outside the shell.

$$\text{the potential} = \frac{2m\pi r^2}{}\int\frac{dz}{d\theta}\,d\theta = \frac{2m\pi r}{c}\int_{c-r}^{c+r} dz$$

$$= [c+r-(c-r)] = \frac{4m\pi r^2}{} = \frac{M}{c}.$$

If the point P be inside the shell, the expression to be integrated is the same, but the limits for z or PQ are $r-c$ and $r+c$.

$$\therefore \quad \text{in this case, the potential} = \frac{2m\pi r}{c}[r+c-(r-c)] = 4m\pi r = M/r.$$

If the point P be on the shell, the limits for z are 0 and $2r$, and $c=r$; hence in this case also the potential $= M/r$.

Hence the potential of a thin spherical shell at an *external* point $= M/c$, i.e. it is the same as if the whole mass were concentrated at the centre; at an *internal* point, the potential $= M/r$, i.e. it is constant, and therefore is the same as if the point were the centre.

(iii) *Find the potential of a solid sphere at any point.*

Let the sphere be divided by concentric spheres into thin spherical shells.

If the point P be outside the sphere, the potential of each shell and hence, by addition, of the whole sphere is the same as if the whole mass were collected at the centre, and therefore is equal to M/c, where c is the distance of the point P from the centre.

If P be inside the sphere, the potentials of the spherical shells which do not contain P are the same as if their whole mass were collected at the centre, and hence their sum $= \frac{4}{3}\pi c^3 m/c = \frac{4}{3}\pi mc^2$; the potentials of the shells which do contain P are the same as if P were at the centre, and therefore their sum

$$= \int_c^r \frac{m\,.\,4\pi r^2\,dr}{r} = 4\pi m\int_c^r r\,dr = 4\pi m\left(\tfrac{1}{2}r^2-\tfrac{1}{2}c^2\right) = 2\pi m\left(r^2-c^2\right).$$

Hence the total potential of the sphere at P

$$= \tfrac{4}{3}\pi mc^2 + 2\pi m\left(r^2-c^2\right) = 2\pi m\left(r^2-\tfrac{1}{3}c^2\right).$$

Examples LXXIII.

Find the potential of the following, 1–11:

1. A circular arc at its centre.
2. A thin cylindrical shell (with open ends) at the centre of one end.
3. A solid cylinder at the centre of one end.
4. A hollow cone at its vertex.
5. A solid cone at its vertex.
6. A thick shell bounded by two concentric spheres of radii r and r'.

7. A thin hemispherical shell at its centre.
8. A shell bounded by two non-intersecting and non-concentric spheres. (Take it as the difference of potentials of two solid spheres.)
9. A flat circular ring at a point on its axis.
10. A sector of a circle at the centre of the circle.
11. Prove that the potential of a thin uniform rod AB of length $2l$ at a point P on its perpendicular bisector is $m\log[(r+l)/(r-l)]$, where $PA = r$. Show that this may be put in the form $2m \log\cot\frac{1}{2}\alpha$, where α is the angle PAB.
12. If V be the potential of a solid sphere of radius r at a point distant x from the centre, prove that V and dV/dx are continuous functions of x, but that d^2V/dx^2 is discontinuous when $x = r$.

ATTRACTIONS

179. Attraction.

The law of gravitation, as enunciated by Newton, states that two particles of masses m, m', at distance r apart, attract each other with a force which varies directly as the product of the masses and inversely as the square of the distance between them, i.e. the attraction is equal to kmm'/r^2. It is usual to choose the units so that the constant k may be unity; they are then called *astronomical units.* In terms of these units, the attraction of a particle of mass m on unit mass at distance r from it is equal to m/r^2. 'The attraction at P' is the phrase used to denote the attraction on a particle of unit mass situated at P.

The force between two electrified particles obeys the same law, being attractive if the product of the charges be negative, and repulsive if the product be positive.

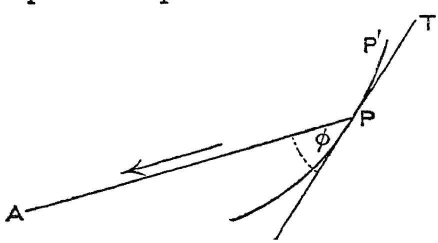


Fig. 138.

Let V be the potential of mass m situated at A at a point P distant r from it, so that $V = m/r$. Let s be the distance of P (Fig. 138) measured along its path from some fixed point in the path; then V is a function of s. Let ϕ be the angle between the radius vector AP and the tangent at P.

We have $$\frac{dV}{ds}=\frac{dV}{dr}\cdot\frac{dr}{ds}=-\frac{m}{r^2}\cdot\frac{dr}{ds}=-\frac{m}{r^2}\cos\phi \qquad \text{(Art. 163)}$$

$$=\frac{m}{r^2}\cos APT = \text{attraction of } m \text{ at } P\times\cos APT$$

$=$ resolved part of attraction of m in direction PT (in which s increases).

This result will be true for each particle of an attracting system, and therefore will be true for the whole system. Hence, if V be the potential of an attracting system at an external point P, the attraction of the system at P, in the direction in which s is measured, is equal to dV/ds.

Examples:

(i) *Find the attraction of a uniform circular disc at a point on its axis.*

From Ex. (i) of the preceding article, if r be the radius of the disc,

$$V=2\pi m(R-p)=2\pi m[\sqrt{(r^2+p^2)}-p],$$

where p is the distance of the point P (Fig. 139) from the disc. The attraction of the disc at P is obviously along the axis, from symmetry, and

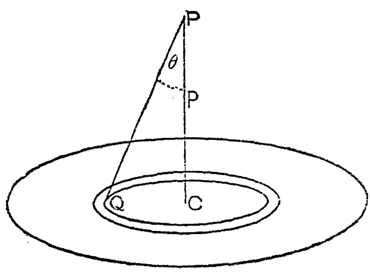


Fig. 139.

$$\frac{dV^*}{dp}=-2\pi m\left(\frac{p}{\sqrt{(r^2+p^2)}}-1\right)=2\pi m\left(1-\frac{p}{R}\right)=2\pi m(1-\cos\alpha),$$

if α be the angle subtended at P by the radius of the disc.

The same result may be obtained directly, by resolving the attraction of an element of the disc along OP and integrating the result.

Taking $\alpha=\frac{1}{2}\pi$, we see that the attraction of an infinite disc at a point at finite distance from it, or of a finite disc at a point whose distance from it is indefinitely small, has the constant value $2m\pi$.

* $+dV/dp$ is the attraction in the direction in which p increases, i.e. upwards.

(ii) *Find the attraction of a straight uniform rod at a point on its perpendicular bisector.*

If p be the distance of the point P (Fig. 140) from the rod, and θ the inclination of PQ to the perpendicular PN from P to the rod, the attraction of an element $m\,\delta x$ situated at Q is $m\,\delta x/PQ^2$. From symmetry, the resultant attraction of the rod is along PN; hence, resolving along PN and integrating, the total attraction

$$=\int \frac{m\,dx}{PQ^2}\cos\theta = 2\int_0^{\alpha}\frac{mp\sec^2\theta\,d\theta\,.\cos\theta}{p^2\sec^2\theta},$$

since $x=p\tan\theta$, $PQ = p\sec\theta$, 2α being the angle subtended by the rod at P,

$$=\frac{2m}{p}\int_0^{\alpha}\cos\theta\,d\theta=\frac{2m}{p}\sin\alpha.$$

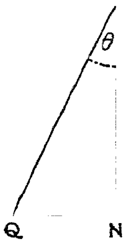


Fig. 140.

(iii) *Find the attraction of a spherical shell at a given point.*

If the point be inside the shell, the potential M/r is constant, and therefore its differential coefficient is zero; hence the attraction of a spherical shell at an internal point is zero.

If the point P be outside the shell at distance c from the centre, the attraction of the shell at P is, from symmetry, along the line joining P to the centre. The potential at P is M/c (from Ex. (ii) of Art. 178); therefore the attraction of the shell, which is towards the centre, i.e. in the direction in which c decreases, $=-dV/dc=M/c^2$; hence the attraction of a spherical shell at an external point is the same as if the whole mass were concentrated at the centre.

(iv) *Find the attraction of a solid sphere at a given point.*

If the point P be outside the sphere at distance c from the centre, the potential $V=M/c$, and the attraction towards the centre $=-dV/dc=M/c^2$, i.e. the same as if the whole mass were concentrated at the centre.

If P be inside the sphere, the potential $V=2\pi m(r^2-\frac{1}{3}c^2)$ (from Ex. (iii) of the last article), and therefore the attraction towards the centre

$$=-dV/dc=-2\pi m\left(-\tfrac{2}{3}c\right)=\tfrac{4}{3}\pi mc.$$

Hence the important results that, in the case of a solid sphere attracting according to the law of gravitation, the resultant attraction at an external point varies inversely as the square of the distance from the centre, and at an internal point, varies directly as the distance from the centre. It follows that the value of g, the acceleration of a particle due to the earth's attraction, varies in the same manner, if the earth be regarded as a sphere of uniform density.

It should be noticed that, although the expressions for the potential and the attraction of a solid sphere at a point distant c from its centre take

different forms according as the point is inside or outside the sphere, yet both are continuous functions of c; in both cases the two expressions tend to the same value when $c \to r$. Both expressions for the potential become M/r, and both expressions for the attraction become M/r^2, i.e. $\frac{4}{3} m \pi r^3/r^2$ or $\frac{4}{3} m \pi r$. Hence V and dV/dc are both continuous when $c = r$.

The second differential coefficient d^2V/dc^2 is however discontinuous when $c = r$; for at an internal point

$$d^2V/dc^2 = \text{d. c. of } -\tfrac{4}{3}\pi mc = -\tfrac{4}{3}\pi m,$$

and at an external point,

$$d^2V/dc^2 = \text{d.c. of } -M/c^2 = 2M/c^3 = \tfrac{8}{3}\pi ma^3/c^3,$$

which, as $c \to r$, approaches the value $\frac{8}{3}\pi m$.

Hence there is an abrupt change from $-\frac{4}{3}\pi m$ to $\frac{8}{3}\pi m$, i.e. an abrupt increase of $4\pi m$ in the value of d^2V/dc^2, as c increases through the value r.

Examples LXXIV.

Find the attraction of the following, 1–11:

1. A thin uniform rod at a point on its perpendicular bisector, by differentiating the expression for the potential obtained in Ex. LXXIII. 11.
2. A circular disc at a point on its axis, by direct integration.
3. A thin uniform rod at a point on the perpendicular to the rod from one end of it.
4. A thin uniform rod at any point. [See Ex. 14, below.]
5. A thin cylindrical shell (open at the ends) at an external point on its axis.
6. A solid cylinder at an external point on its axis.
7. A solid right circular cone at its vertex.
8. A thick spherical shell, radii r and r' ($r > r'$), at a point distance x from its centre (i) when $x < r'$, (ii) when $r' < x < r$, (iii) when $x > r$.
9. A shell bounded by two non-intersecting and non-concentric spheres (i) at an internal point, (ii) at an external point.
10. A rod AB at a point in AB produced.
11. A flat circular ring at a point on its axis.
12. Taking the value of g as 32·18 at the earth's surface, and the radius of the earth as 4000 miles, find the value of g (i) at a point 100 miles within the surface, (ii) at a point 100 miles outside the surface.
13. Find the work done in raising 100 lb. from the surface of the earth to a height of 100 miles. (Take the radius of the earth as 3960 miles.)
14. A circle is drawn with any point P as centre to touch a straight line AB; if CD be the arc of this circle intercepted by PA, PB, prove that the attraction of the straight rod AB is the same in magnitude and direction as that of the circular rod CD.
15. If V be the potential of a solid sphere at a point distant x from its centre, draw the graphs of (i) V, (ii) dV/dx, (iii) d^2V/dx^2.

COMPOUND INTEREST LAW

180. The compound interest law.

In many cases in nature, the rate of change of a quantity which is a function of some variable is, for any value of the variable, proportional to its actual magnitude for that value; i.e. if t denote the variable * of which y is a function,

$$\frac{dy}{dt} = ky, \text{ which can be written } \frac{1}{y}\frac{dy}{dt} = k.$$

The left-hand side is the d. c. of $\log y$ with respect to t. Therefore, integrating with respect to t, $\log y = kt + C$,

i.e. $$y = e^{kt+C} = e^{kt} \times e^{C} = ae^{kt}$$

writing a instead of the constant factor e^C.

This law of change, viz.: that the rate of increase of a variable is proportional to the value of the variable, is called the *compound interest law* for the following reason:

Let a sum of money £P be invested at compound interest at the rate of r per cent. per annum, and let the interest be payable n times per annum at equal intervals of time.

After the first payment of interest, the amount

$$= P + \frac{P}{100} \cdot \frac{r}{n} = P\left(1 + \frac{r}{100n}\right),$$

and, similarly, the amount at the end of each interval is equal to the amount at the beginning of the interval multiplied by the factor $1 + r/100n$. Therefore after t years, i.e. after nt payments of interest, the amount will be

$$P(1 + r/100n)^{nt}.$$

This is the manner in which money increases in actual practice, not continuously as a mathematical function increases, but by a succession of disconnected finite increments (as in the graph of Fig. 31); n may be 1 (C. I. paid yearly), 4 (C. I. paid quarterly), 12 (C. I. paid monthly), and so on.

If $r/100n$ be denoted by $1/m$, and therefore $n = rm/100$, this amount may be written

$$P(1 + 1/m)^{rmt/100}.$$

Now let n (and therefore also m) increase and ultimately become indefinitely great, so that the interest is added more and more

* The independent variable is frequently the time.

frequently, and ultimately continuously; the amount at the end of t years will then be

$$P \times \underset{m \to \infty}{\mathrm{Lt}} (1+1/m)^{rmt/100} = P \times \left[\underset{m \to \infty}{\mathrm{Lt}} (1+1/m)^m\right]^{rt/100} = Pe^{rt/100} \text{ (Art. 87).}$$

Hence, when compound interest is added continuously to the principal P, the amount A at the end of t years $= Pe^{rt/100}$, and therefore obeys the above law.

The rate of increase of $A = \dfrac{dA}{dt} = P \cdot \dfrac{r}{100} e^{rt/100} = \dfrac{r}{100} A.$

The preceding result can also be obtained directly by integration, for the amount A at any instant is increasing at the rate of r per cent. per annum, i.e.

$$dA/dt = Ar/100.$$

$$\therefore \quad \frac{1}{A}\frac{dA}{dt} = \frac{r}{100}, \quad \text{whence } \log A = \frac{r}{100} t + C.$$

When $t = 0$, A is equal to the sum P originally invested, $\therefore \log P = C$.

i.e. $$\log A = rt/100 + \log P,$$
$$\therefore \quad A = Pe^{rt/100}, \quad \text{as before.}$$

Extension of compound interest law.

Cases in which the rate of increase of the function is partly constant and partly varies directly as the value of the function may be included in the above law, for if $\quad dy/dt = b + ky,$
we may write $\quad dy/dt = k(y + b/k),$
from which it follows that the rate of increase of the function is proportional to the excess of the value of the function over the constant $-b/k$.

The equation may be written $\dfrac{1}{y+b/k}\dfrac{dy}{dt} = k.$

Therefore, integrating, and taking the constant of integration in the form $\log C$, which is more convenient than C, we have

$$\log(y + b/k) = kt + \log C,$$

whence $$y + b/k = Ce^{kt},$$

and $$y = -b/k + Ce^{kt}.$$

(It should be noticed that the preceding equation takes exactly the same form as in the case of the compound interest law if we replace $y + b/k$ by z, and therefore dy/dt by dz/dt.)

181. Particular cases of the compound interest law.

Among the natural processes which follow the compound interest law are the following:

1. *The cooling of a body which is at a higher temperature than its surroundings, according to Newton's Law of Cooling.*

This states that the rate of cooling is proportional to the excess of the

temperature of the body over the temperature of its surroundings; i.e. if θ denote this excess of temperature,

$$d\theta/dt = -k\theta,$$

the $-$ sign being taken because the temperature decreases as time goes on. Hence, from the result at the beginning of the last article, $\theta = Ce^{-kt}$.

If θ_0 be the original excess of temperature, i.e. the value of θ when $t=0$, we have $\theta_0 = C$; $\quad \therefore\ \theta = \theta_0 e^{-kt}$.

Taking a numerical case, suppose that a body cools from 80° C. to 70° C. in 5 minutes; what will its temperature be after a quarter of an hour, and how long will it take to cool to 40° C., the surrounding temperature remaining at 20° C. all the time?

Here $\theta_0 = 80-20 = 60$; therefore $\theta = 60e^{-kt}$.

It is given that θ (the *excess* of temperature) $= 50$ when $t = 5$; therefore

$$50 = 60e^{-5k}, \text{ whence } e^{-5k} = \tfrac{5}{6}, \quad -5k = \log\tfrac{5}{6}, \text{ and } k = \tfrac{1}{5}\log 1{\cdot}2.$$

After a quarter of an hour, $\theta = 60e^{-15k} = 60(e^{-5k})^3 = 60\times(\tfrac{5}{6})^3 = 34{\cdot}7$° C. nearly. Therefore the temperature will be 54·7° C.

The time to cool to 40° C. is given by $20 = 60e^{-kt}$; whence

$$-kt = \log\tfrac{1}{3} = -\log 3, \text{ and } t = (\log 3)/k = 5\log 3/\log 1{\cdot}2 = 30{\cdot}1 \text{ minutes.}$$

The temperature of 40° C. is reached after a little more than half an hour.

2. *The change in the atmospheric pressure due to an alteration in height above sea-level.*

Let p be the pressure at height h above sea-level (or any other fixed level), and $p+\delta p$ the pressure at height $h+\delta h$.

Taking a vertical cylindrical column of air of height δh and section A, the pressure on the lower end exceeds the pressure on the upper end by the weight of the column, i.e. by $g\rho A\delta h$.

Hence $\quad pA-(p+\delta p)A = g\rho A\delta h$, i.e. $A\delta p = -g\rho A\delta h$.

Therefore, when $\delta h \to 0$, $\quad dp/dh = -g\rho = -gp/k$,

since, as is proved in text-books on Hydrostatics, $p = k\rho$, provided the temperature be supposed to remain constant.

Therefore $p = Ce^{-gh/k} = p_0 e^{-gh/k}$, if p_0 be the pressure at the given level.

Hence, if p_1, p_2 be the atmospheric pressures at heights h_1, h_2, we have

$$p_1/p_2 = p_0 e^{-gh_1/k} \div p_0 e^{-gh_2/k} = e^{g(h_2-h_1)/k}.$$

3. *The motion of a particle against a force which is proportional to the velocity.*

(For small velocities, the resistance of the air is roughly proportional to the velocity). Such a force will produce a retardation which varies as the velocity; hence the equation of motion of the particle is $dv/dt = -kv$,

$v = ue^{-kt}$, where u is the initial velocity.

4. *The tension of a rope or belt round a rough pulley or cylinder.*

Let T be the tension at a point P (Fig. 141) whose angular distance from the point A, where the rope leaves the pulley on the slack side, is θ; and let $T+\delta T$ be the tension at Q, distant $\theta+\delta\theta$ from A.

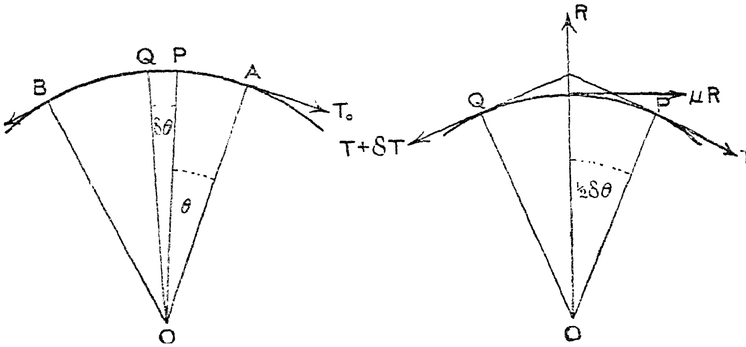


Fig. 141.

Let R be the normal reaction at the middle point of PQ, and μ the coefficient of friction; therefore μR is the friction at that point when the rope is on the point of slipping.

Resolving along the normal and tangent at the middle point of PQ for the equilibrium of the indefinitely small element PQ, we have

$$R = T\sin\tfrac{1}{2}\delta\theta + (T+\delta T)\sin\tfrac{1}{2}\delta\theta,$$

and $$(T+\delta T)\cos\tfrac{1}{2}\delta\theta = \mu R + T\cos\tfrac{1}{2}\delta\theta,$$

whence $$\delta T\cos\tfrac{1}{2}\delta\theta = \mu R = \mu(2T+\delta T)\sin\tfrac{1}{2}\delta\theta,$$

from the preceding equation.

$$\therefore\quad \delta T/\delta\theta\,.\cos\tfrac{1}{2}\delta\theta = \mu(2T+\delta T)\,(\sin\tfrac{1}{2}\delta\theta)/\delta\theta$$

When $\delta\theta\to 0$, $\dfrac{\delta T}{\delta\theta}\to\dfrac{dT}{d\theta}$, $\cos\tfrac{1}{2}\delta\theta\to 1$, $2T+\delta T\to 2T$, $\dfrac{\sin\frac{1}{2}\delta\theta}{\delta\theta}$.

$$\therefore\quad \text{ultimately,}\quad dT/d\theta = \mu\,.\,2T\times\tfrac{1}{2} = \mu T,$$

whence, as before, $T = T_0 e^{\mu\theta}$, where T_0 is the tension at A.

From this it is easily seen how it is that a small force at one end of a rope which takes a turn or two round a rough post can support a very considerable tension at the other end, for if the coefficient of friction be $\frac{1}{2}$ and the rope makes $1\frac{1}{2}$ complete turns, i.e. if $\theta = 3\pi$, we have $T = T_0 e^{\frac{3}{2}\pi} = T_0\times 111{\cdot}2$, so that a given tension at the slack end will support a tension 111 times as great at the other end.

5. *The discharge of a condenser through a large resistance.*

It is shown in works on Electricity that, if C be the capacity of the condenser, and R the resistance through which it is discharging, then $dq/dt = -q/CR$, where q is the amount of the charge at time t.

Hence $q = Ae^{-t/CR}$, where A is constant,
$= q_0 e^{-t/CR}$, if q_0 be the original charge when $t = 0$.

Therefore $\log \frac{q}{q_0} = -\frac{t}{CR}$, and $R = \frac{t}{C} \Big/ \log \frac{q_0}{q} = \frac{t}{C} \Big/ \log \frac{v_0}{v}$,

if v_0, v be the potentials originally and at time t. This gives R in terms of v and t.

6. *The true expansion of a length, area, or volume when the coefficient of expansion is constant.*

Taking the case of a volume (Art. 38), $\frac{1}{V}\frac{dV}{d\theta} = \alpha$, if V be the volume at temperature θ. Therefore, as in the preceding cases, $V = V_0 e^{\alpha\theta}$, if V_0 be the volume at temperature 0°, or $V = V' e^{\alpha(\theta-\theta')}$, if V' be the volume at temperature θ'. Taking the former case, we have, on expanding $e^{\alpha\theta}$ and neglecting squares of α, $V = V_0(1 + \alpha\theta)$ approximately.

7. *The current flowing in an electric circuit.*

It is shown in works on Electricity that if an electric current of strength i be flowing in a circuit of self-induction L and resistance R, and if E be the external E. M. F. on the circuit, then $L\, di/dt + Ri = E$.

(i) If the circuit be left to itself, so that there is no external E. M. F., $E = 0$,

$$\therefore \quad di/dt = -Ri/L.$$

Hence, as before, $i = i_0 e^{-Rt/L}$, where i_0 is the original current.

(ii) If a constant E. M. F. be supplied to the circuit, we have a case of the extension of the compound interest law mentioned above, for then

$$di/dt = E/L - Ri/L, \text{ where } E \text{ is constant.}$$

Therefore, using the result at the end of Art. 80,

$$i = -\frac{E}{L} \Big/ \left(-\frac{R}{L}\right) + Ae^{-Rt/L} = \frac{E}{R} + Ae^{-Rt/L}, \text{ where } A \text{ is a constant.}$$

If the time be measured from the instant the circuit is completed, $i = 0$ when $t = 0$.

$$\therefore \quad 0 = E/R + A, \text{ and } A = -E/R;$$

so that
$$i = E(1 - e^{-Rt/L})/R.$$

Since the last term in the brackets tends to zero as t increases, the current approaches the constant value E/R.

(For another case of this problem, when the circuit is under the influence of a variable E. M. F., see the next article.)

8. *The velocity of certain chemical reactions.*

(*a*) Many chemical reactions follow the law (known as Wilhelmy's Law) which states that the velocity of the reaction is proportional to the concentration of the reacting substance, i. e. if a be the initial concentration of the reagent and x the amount transformed at time t, $dx/dt = k(a - x)$.

This is the extension of the compound interest law, and integrating as in the preceding article, we get

$$\frac{1}{a-x}\frac{dx}{dt}=k; \qquad -\log(a-x)=kt+C.$$

To find C, we have $x=0$ when $t=0$; $\therefore$ $-\log a=C$.

$\therefore$ changing signs,

$$\log(a-x)=-kt+\log a, \quad \text{i.e.} \quad a-x=ae^{-kt}, \quad \text{or} \quad x=a(1-e^{-kt}).$$

The equation may also be written in the form

$$kt=\log a-\log(a-x)=\log\frac{a}{a-x}; \quad \therefore \quad k=\frac{1}{t}\log\frac{a}{a-x}.$$

This gives the value of the constant k when a and a pair of simultaneous values of t and x are known.

(*b*) There are other chemical reactions which follow the more complicated law

$$dx/dt=k(a-x)(b-x).$$

This may be written $\dfrac{1}{(a-x)(b-x)}\dfrac{dx}{dt}=k$;

$\therefore$ integrating with respect to t, $kt=\displaystyle\int\frac{dx}{(a-x)(b-x)}$.

By the method of partial fractions (Art. 123) we find (if $a>b$)

$$\frac{1}{(a-x)(b-x)}=\frac{1}{a-b}\left[\frac{1}{b-x}-\frac{1}{a-x}\right].$$

$$\int\frac{dx}{(a-x)(b-x)}=\frac{1}{a-b}[-\log(b-x)+\log(a-x)]=\frac{1}{a-b}\log\frac{a-x}{b-x}.$$

$$\therefore \quad (a-b)kt=\log\frac{a-x}{b-x}+C.$$

$x=0$ when $t=0$; $\quad \therefore \quad 0=\log(a/b)+C$;

$$\therefore \quad (a-b)kt=\log\frac{a-x}{b-x}-\log\frac{a}{b}=\log\frac{b(a-x)}{a(b-x)};$$

or

$$e^{(a-b)kt}=\frac{b}{a}\cdot\frac{a-x}{b-x}.$$

Solving this equation for x, we obtain

$$x=\frac{ab\,[e^{(a-b)kt}-1]}{ae^{(a-b)kt}-b},$$

which gives the value of x at time t.

If a pair of simultaneous values of x and t are known, the value of k is obtained from the equation above,

$$\kappa=\frac{1}{(a-b)t}\log\frac{b(a-x)}{a(b-x)}.$$

182. Another example from Electricity.

We have, in the preceding article, solved the equation

$$L\frac{di}{dt}+Ri=E,$$

for the particular cases $E=0$ and $E=$ constant.

Let us now take the case when the external E. M. F. is a periodic function of the time; let $E = E_0 \sin pt$, where E_0 is constant.

$$\therefore \quad \frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin pt.$$

In integrating the equation $\frac{di}{dt} + \frac{R}{L} i = 0$ in the preceding article, we obtained $\quad i = Ae^{-Rt/L}$, i. e. $ie^{Rt/L} = A$, a constant.

If we verify this result by differentiation, we get

$$i \,.\, \frac{R}{L} e^{Rt/L} + \frac{di}{dt} e^{Rt/L} = 0, \quad \text{i. e. } e^{Rt/L}\left(\frac{di}{dt} + \frac{R}{L} i\right) = 0.$$

This shows that the left-hand side of our equation above is made an exact differential coefficient (of $ie^{Rt/L}$), and therefore the equation can be integrated, by multiplying it by the factor $e^{Rt/L}$; it then becomes

$$e^{Rt/L} \frac{di}{dt} + i \frac{R}{L} e^{Rt/L} = \frac{E_0}{L} e^{Rt/L} \sin pt.$$

The left-hand side being the d. c. with respect to t of $ie^{Rt/L}$, we have by integration

$$ie^{Rt/L} = \frac{E_0}{L} \int e^{Rt/L} \sin pt \, dt + C.$$

An integral of the same type as that on the right-hand side has already been evaluated in Art. 139; substituting R/L, p, and t for a, b, and x respectively in the result of that article, we get

$$ie^{Rt/L} = \frac{E_0}{L} \cdot \frac{e^{Rt/L}(R/L \,.\, \sin pt - p \cos pt)}{R^2/L^2 + p^2} + C,$$

whence $$i = E_0 \frac{(R \sin pt - pL \cos pt)}{R^2 + p^2 L^2} + Ce^{-Rt/L}.$$

Measuring the time from the instant when the circuit is completed, we have $i = 0$ when $t = 0$,

$$\therefore \; 0 = \frac{E_0}{R^2 + p^2 L^2}(-pL) + C.$$

$$i = \frac{E_0}{R^2 + p^2 L^2}(R \sin pt - pL \cos pt) + \frac{E_0 pL}{R^2 + p^2 L^2} e^{-Rt/L}.$$

The first term can be put in a more convenient form by the following artifice, which is one of frequent use.

Let $R = k \cos \alpha$, $pL = k \sin \alpha$; therefore $\tan \alpha = pL/R$, and, squaring and adding, $R^2 + p^2 L^2 = k^2$.

Then $$R \sin pt - pL \cos pt = k(\cos \alpha \sin pt - \sin \alpha \cos pt)$$
$$= k \sin(pt - \alpha) = \sqrt{(R^2 + p^2 L^2)} \sin(pt - \alpha).$$

Hence $$i = \frac{E_0}{\sqrt{(R^2 + p^2 L^2)}} \sin(pt - \alpha) + \frac{E_0 pL}{R^2 + p^2 L^2} e^{-Rt/L}.$$

The last term becomes very small as t increases, since, R/L being +, $e^{-Rt/L}$ decreases rapidly as t increases, and therefore the current soon approaches the steady oscillation given by

$$\frac{E_0}{\sqrt{(R^2 + p^2 L^2)}} \sin(pt - \alpha), \quad \text{where } \alpha = \tan^{-1} \frac{pL}{R}.$$

Examples LXXV.

In doing these examples, the differential equation should be formed and actually solved in each case. Do not substitute numerical values in the results of Art. 181.

1. Find the compound interest on £200 invested for 3 years at 5 per cent. per annum, when interest is payable (i) monthly, (ii) daily, (iii) continuously.

2. The temperature of a body is 30° above that of the surrounding atmosphere; and its rate of cooling per minute is $\cdot 01\,\theta$, where θ is the excess of its temperature above that of the atmosphere (which is supposed to remain constant); find (i) its temperature after 3 minutes, (ii) when its excess of temperature will have fallen to 20°.

3. The temperature of a liquid in a room of constant temperature 20° is observed to be 70°; after 5 minutes, it is observed to be 60°; what will its temperature be after another half-hour, and when will it be 40°?

4. A rope which is in contact with a circular post is on the point of slipping; if the portion of rope in contact with the post subtends an angle of 120° at the centre, and the coefficient of friction is $\frac{1}{2}$, compare the tensions at opposite ends.

5. A rope is wound just twice round a post and held by a force of 20 lb. wt. at one end; if the coefficient of friction be $\cdot 4$, what force must be applied at the other end to make it slip?

6. The height of the barometer is 30 inches at sea-level; what would it be at the top of a mountain 10,000 feet high, if the temperature were constant? [Take the specific gravity of air at sea-level as $\cdot 0013$, that of mercury as $13\cdot 6$, and determine k from this.]

7. The height of the barometer is 30 inches at the bottom of a mountain and 24 inches at the top; find the height of the mountain. [Take $k = 842000$.]

8. A light string hangs over a fixed rough horizontal cylinder, and is on the point of slipping when masses of 8 lb. and 2 lb. are suspended from its extremities; find the coefficient of friction.

9. A fly-wheel of mass 1 ton and radius of gyration 2 feet is running against a frictional resistance which is proportional to the velocity; its angular velocity was initially 80 radians per second, and after 20 seconds it is 50 radians per second; what will it be after a minute? [If I be the M.I. of the wheel, and ω its angular velocity at time t, $Id\omega/dt = -k\,\omega$.]

10. A point moves so that its acceleration is always numerically $\frac{1}{5}$ of its velocity; if it starts with velocity of 5 ft.-secs., find its velocity after 10 seconds, and when its velocity will be 100 ft.-secs.

11. A particle falls vertically under the action of its weight, and against a resistance which produces a retardation proportional to the velocity; find its velocity after 10 seconds, supposing that it starts from rest, and that its velocity tends to the value 80 ft.-secs. as t increases indefinitely.

12. A chemical reaction takes place according to the law mentioned in Art. 181. 8 (a). If $a = 9\cdot 5$, and $x = 3\cdot 2$ after two minutes, find (i) the value of k, (ii) the value of x after 5 minutes.

13. A chemical reaction takes place according to the law mentioned in Art. 181. 8 (b). If $a = 35\cdot 4$, $b = 12\cdot 5$, and $x = 2\cdot 3$ after one minute, find (i) the value of k, (ii) the value of x after 3 minutes.

14. An electric condenser of capacity 18×10^{-14} is discharging through a resistance 3×10^{12}; if the initial charge be $\cdot 001$, find (i) the charge after $\cdot 01$ second, (ii) when the charge is reduced to 10 per cent. of its original value.

15. A condenser of capacity 5×10^{-15} is discharging through a resistance, and in 2 seconds the voltage falls to one-tenth of its original value; find the resistance.

16. A current is flowing in a circuit of resistance 10 ohms and self-induction $\cdot 02$ henry; if its value was originally 40 amps., find (i) its value after $\cdot 01$ second, (ii) when it is 10 amps., the circuit being left to itself.

17. Find the current after $\cdot 01$ second if the same circuit is under the influence of a constant E. M. F. of 50 volts, and $i = 0$ when $t = 0$.

18. Find the current after $\cdot 01$ second if there is an E. M. F. of $50 \sin 500t$, and $i = 0$ when $t = 0$.

19. Find the current after half a second in a circuit of resistance 10 and self-induction 5 under an E. M. F. of $40 \sin 200t$.

20. The rate at which liquid is flowing out of a vessel at any instant is proportional to the amount left in at that instant; if the vessel is half emptied in 1 minute, how much will flow out in 2 minutes, and when will it be four-fifths empty?

21. A pane of glass absorbs 4 per cent. of the light passing through it; how much of the light will get through 20 such panes of the same kind of glass? How many panes will absorb 40 per cent. of the light?
[If I be the intensity after passing through a thickness l, $dI/dl = -kI$.]

22. An electric current, left to itself, drops to $\frac{1}{2}$ of its original value in $\frac{1}{10}$ second; how long will it take to drop to one-millionth of its original value?

23. An electric current left to itself drops 20 per cent. in 2 minutes; when will it be imperceptible to a galvanometer which can just detect one thousand-millionth part of its original value?

24. The population of a country is at any instant increasing at a rate which is proportional to its value at that instant; if it be doubled in 20 years, when will it have increased 5-fold?

CHAPTER XIX

APPLICATIONS TO MECHANICS

WORK

183. Work and energy.

It was shown in Art. 65, that, if W be the work done in moving a particle from some standard position to a point P, and E the kinetic energy at P, then $F = dW/dx$ and also $= dE/dx$, x being the distance of P from some fixed point in the line of motion.

Therefore $dW/dx = dE/dx$, and hence (Art. 76) W and E differ by a constant only, i.e. $W = E + C$.

If E_0 be the kinetic energy of the particle in the standard position, we have $E = E_0$ when $W = 0$; hence $0 = E_0 + C$, and $C = -E_0$,

$$\therefore \quad W = E - E_0.$$

Therefore the work done in moving the particle from one point to the other is equal to the change in the kinetic energy of the particle.

Also, since $dW/dx = F$, it follows that $W = \int F dx$. Therefore, if F be known in terms of x, the work done in moving the particle from one point to another can be calculated.

As an example of this, we will calculate the work done in stretching an elastic string. Let a be the natural length of the string, and suppose we want to find the amount of work done in stretching it from length $a+b$ to length $a+c$. The tension of such a string is given by Hooke's Law, which states that the tension is proportional to the extension. When the stretched length is $a+x$, the extension is x, and the most convenient way of expressing this law is: $T = \lambda x/a$, where λ is a constant. [If $x = a$, this gives $T = \lambda$, so that the constant λ is the weight which, suspended at the end, would cause the string to hang in equilibrium stretched to twice its natural length, supposing this law continues to hold good.]

Hence we have $$dW/dx = T = \lambda x/a.$$

Therefore the work done in increasing x from b to c

$$= \int_b^c \frac{\lambda x}{a}\, dx = \frac{\lambda}{a}\left[\tfrac{1}{2}x^2\right]_b^c = \tfrac{1}{2}\lambda\,\frac{c^2-b^2}{a}.$$

If T_b, T_c denote the tensions at lengths $a+b$, $a+c$ respectively, $T_b=\lambda b/a$ and $T_c=\lambda c/a$; therefore the work required

$$=\tfrac{1}{2}\lambda(c+b)(c-b)/a=\tfrac{1}{2}(T_c+T_b)(c-b)$$

$=$ the extension $\times$ the mean of the initial and final tensions.

184. Graphical method.

If x is the distance of a moving body from a fixed point O in its line of motion, and if the values of the force acting on the body for different values of x are known, either by calculation from a formula or as the result of observations, then, by plotting these values, we may obtain a curve whose ordinate at any point (x, y) represents the force F acting upon the body when at distance x from O.

The work done in moving the body from x_1 to $x_2 = \int_{x_1}^{x_2} F\,dx$, and since F is represented by the ordinate of the curve, this is represented by the area between the curve, the axis of x and the ordinates $x=x_1$ and $x=x_2$. This area may be calculated by Simpson's Rule or measured by a planimeter, and thus the amount of work done is approximately obtained.

This is the principle of the 'indicator diagram' of an engine, which registers mechanically the pressure in the cylinder at different parts of the stroke; the area of the diagram which is drawn gives the amount of work done during the stroke.

A similar method can be used to estimate the distance travelled in a given interval of time, if the velocities at different instants be known, and to estimate the change of velocity in a given interval, if the accelerations at different instants be known, for

$$\frac{ds}{dt}=v,\quad \therefore\quad s=\int_{t_1}^{t_2} v\,dt;\quad \text{and}\quad \frac{dv}{dt}=f,\quad \therefore\quad v=\int_{t_1}^{t_2} f\,dt.$$

185. Work done by an expanding gas.

Imagine the gas contained within a right circular cylinder of cross-section A sq. ft. and length h feet, in which a piston just fits and slides freely, and suppose that a slight expansion of the gas from volume v to volume $v+\delta v$ moves the piston a distance δh. The pressure on the end of the piston is pA, if p be the intensity of pressure of the gas. Hence, if δW be the work done by the gas in the expansion, $\delta W=pA\,\delta h=p\,\delta v$, and $\delta W/\delta v=p$. Therefore if $\delta v\to 0$, $dW/dv=p$, and the work done in a finite expansion is obtained by integration. It can be shown that this relation is true whatever be the shape of the vessel which contains the gas. If v_1 be the original volume and v_2 the final volume, the total work done by the gas in the expansion $=\int_{v_1}^{v_2} p\,dv$. If the gas is compressed,

this is the amount of work which must be done to reduce the volume from v_2 to v_1. If the expansion be supposed to take place isothermally, i.e. without alteration of temperature, the relation between p and v is given by Boyle's Law, $pv =$ constant; if the expansion is adiabatic, i.e. if no heat is taken from or supplied to the gas, the relation between p and v is given by the law $pv^\gamma =$ constant. [See Art. 236.] The pressure p, volume v, and absolute temperature T of a 'perfect gas' are connected by the relation $pv = kT$.

We will take an example of each case.

Examples:

(i) *In an air-compressor, air is drawn in at atmospheric pressure* 14·7 *lb. wt. per sq. inch, and is compressed until the pressure is* 50 *lb. wt. per sq. inch. Find the work done per minute and the horse-power, if the machine makes* 100 *strokes per minute and draws in* 2 *cubic feet at each stroke, supposing the compression isothermal.*

The total work done *against* the gas in reducing the volume from v_1 to v_2

$$= \int_{v_2}^{v_1} p\,dv = \int_{v_2}^{v_1} \frac{k}{v}\,dv = k\left[\log v\right]_{v_2}^{v_1}$$

$$= k(\log v_1 - \log v_2) = p_1 v_1 \log(v_1/v_2) = p_1 v_1 \log(p_2/p_1).$$

$v_1 =$ the initial volume of the air compressed $= 100 \times 2 = 200$ cu. ft.;

$p_1 =$ the initial pressure $= 14·7 \times 144$ lb. wt. per sq. ft.;

$p_2/p_1 = 50/14·7 = 3·401$.

$\therefore$ the work done $= 14·7 \times 144 \times 200 \times \log 3·401$

$= 331{,}500$ ft.-lb. per minute,

and the necessary H. P. is a little more than 10.

(ii) *A quantity of dry air at temperature* 40° *F. is compressed adiabatically until its volume is one-third of its original volume; find the amount of work done and the change of temperature.*

Taking the general case, the work which must be done

$$\int_{v_2}^{v_1} p\,dv = \int_{v_2}^{v_1} \frac{k}{v^\gamma}\,dv = \left[\frac{kv^{-\gamma+1}}{-\gamma+1}\right]_{v_2}^{v_1} = \frac{1}{-(\gamma-1)}\left[\frac{k}{v_1^{\gamma-1}} - \frac{k}{v_2^{\gamma-1}}\right]$$

$$-\frac{1}{\gamma-1}\left[\frac{p_1 v_1^\gamma}{v_1^{\gamma-1}} - \frac{p_2 v_2^\gamma}{v_2^{\gamma-1}}\right] = \frac{1}{\gamma-1}(p_2 v_2 - p_1 v_1).$$

This result may be expressed in the form

$$\frac{p_1 v_1}{\gamma-1}\left(\frac{p_2 v_2}{p_1 v_1} - 1\right), \text{ and } \frac{p_2}{p_1} = \left(\frac{v_1}{v_2}\right)^\gamma;$$

$$\therefore \text{ the work required} = \frac{p_1 v_1}{\gamma-1}\left[\left(\frac{v_1}{v_2}\right)^{\gamma-1} - 1\right].$$

In the given example, $v_1/v_2 = 3$; $\therefore$ the work $= \dfrac{p_1 v_1}{\gamma-1}(3^{\gamma-1} - 1)$.

If there were originally 60 cubic feet of air at atmospheric pressure, $p_1 = 14{\cdot}7 \times 144$, $v_1 = 60$, and γ is, for air, $1{\cdot}404$.

Therefore the amount of work required

$$= \frac{14{\cdot}7 \times 144 \times 60}{{\cdot}404}(3^{\cdot 404} - 1) = \frac{14{\cdot}7 \times 8640}{{\cdot}404} \times {\cdot}559 = 175700 \text{ ft.-lb.}$$

To find the change of temperature, we have $p_1 v_1 = kT_1$, $p_2 v_2 = kT_2$.

$$\therefore \quad \frac{T_2}{T_1} = \frac{p_2 v_2}{p_1 v_1} = \left(\frac{v_1}{v_2}\right)^\gamma \times \frac{v_2}{v_1} = \left(\frac{v_1}{v_2}\right)^{\gamma - 1}.$$

Taking the absolute zero of temperature as 461° F., $T_1 = 461° + 40° = 501°$;

$$\therefore \quad T_2 = 501° \times 3^{\gamma - 1} = 501° \times 3^{\cdot 404} = 781°.$$

Hence the temperature rises on compression to 320° F.

VIRTUAL WORK

186. Virtual work.

It is proved in text-books on Statics that, if a body or system of bodies be in equilibrium, the work done by the external forces in any small displacement (consistent with the geometrical conditions) which the system may be imagined to take (i.e. in any *virtual displacement*) is zero. More strictly speaking, if the displacement be an infinitesimal of the first order, the work done in any such displacement from a position of equilibrium will be an infinitesimal of the second order. In many cases we can, by the principles of the differential calculus, write down at once the work done by the forces in a small displacement, and then, by equating it to zero, find the position of equilibrium, or obtain relations between the forces in the position of equilibrium. The following examples illustrate the method.

Examples:

(i) *A uniform rod of weight* W (Fig. 142) *rests between the ground* AC *a vertical wall* BC, *both smooth, and is kept from slipping by a horizontal s attached to the lower end of the rod and supporting a weight* P *hanging freely; find the position of equilibrium.*

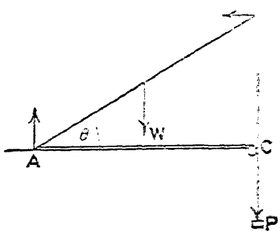


Fig. 142.

Let $2l$ be the length of the rod, θ its inclination to the ground when in equilibrium, and b the length of the string. Imagine the weight P to descend a little, so that θ is increased by a small amount $\delta\theta$. The reactions at A, B, and C do no work, since the displacements at A, B, and C are

perpendicular to them. If z be the height of the middle point of the rod above AC, and x the depth of P below C, the work done by W is $-W\delta z$, and the work done by the weight P is $P\delta x$.

Hence, by the principle of virtual work,

$$-W\delta z + P\delta x = 0.$$

Now $z = l\sin\theta$, $x = b - 2l\cos\theta$; $\therefore$ (Art. 24) $\delta z = l\cos\theta\,\delta\theta$, $\delta x = 2l\sin\theta\,\delta\theta$.

Hence, substituting and dividing by $\delta\theta$, $\quad Wl\cos\theta = P2l\sin\theta$,

$$\text{or} \qquad \tan\theta = W/2P.$$

This gives the position of equilibrium.

(ii) *A frame* ABC (Fig. 143) *consists of* 3 *light rods, of which* AB, AC *are of length* a *and* BC *of length* $\frac{3}{2}$a, *freely jointed together; it rests with* BC *horizontal,* A *below* BC, *and the rods* AB, AC *over two smooth fixed pegs* E *and* F *in the same horizontal line, distance* 2b *apart. A weight* W *is suspended from* A; *find the thrust on the rod* BC.

Denote the angle BAH by θ. Imagine A to descend a little, and that the rod BC is slightly shortened. The only forces which do work are the

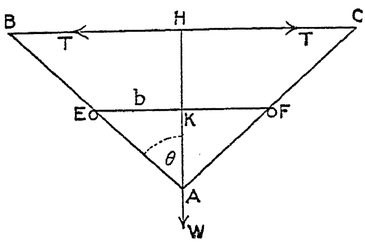


Fig. 143.

weight W and the thrust T. The work done by W is $W\delta(KA)$, and the work done by T is $T\delta(BC)$.

Hence $$W\delta(KA) + T\delta(BC) = 0.$$

Since we are supposing BC to alter its length a little, we must find its length in terms of the variable θ.

$KA = b\cot\theta$, and $\quad BC = 2a\sin\theta$; $\quad \therefore \quad \delta(KA) = -b\operatorname{cosec}^2\theta\,\delta\theta$, and $\delta(BC) = 2a\cos\theta\,\delta\theta$.*

Hence $$-Wb\operatorname{cosec}^2\theta\,\delta\theta + T\,.\,2a\cos\theta\,\delta\theta = 0,$$

$$\therefore \quad T = W.\frac{b}{2a}\frac{\operatorname{cosec}^2\theta}{\cos\theta}.$$

* If A descends, θ diminishes, $\therefore$ $\delta\theta$ is negative; this makes $\delta(KA)$ positive, and $\delta(BC)$ negative.

Since $\quad BC = \frac{3}{2}a$, $\sin\theta = BH/BA = \frac{3}{4}$, and $\cos\theta = \frac{1}{4}\sqrt{7}$;

$$\therefore \quad T = W\frac{}{2a}\cdot\frac{64}{9\sqrt{7}} = \frac{}{9\sqrt{7}}\cdot\frac{b}{a}W.$$

If the weights of the various bodies of the system be the only forces which do work in the displacement, and if y be the height of the centre of gravity of the system above a fixed horizontal line, the principle of virtual work tells us that $W\delta y$, and therefore δy (since the weight W of the system is finite), is of the second order of small quantities. If y be expressed in terms of some variable θ, then, to the first order of small quantities,

$$\delta y = \frac{dy}{d\theta}\delta\theta \text{ (Art. 24).}$$

Hence, since δy is of the second order, we have $dy/d\theta = 0$, i.e. y is a maximum or minimum (provided $d^2y/d\theta^2$ is not zero). Hence the system is in a position of equilibrium when the height of its centre of gravity is a maximum or a minimum. The equilibrium will be stable (i. e. if slightly displaced, the system will tend to return to its original position) if the height of the centre of gravity be a minimum, i. e. if $d^2y/d\theta^2$ is positive; and unstable (i. e. if slightly displaced, the system will tend to move still further away from the position of equilibrium) if the height of the centre of gravity be a maximum, i. e. if $d^2y/d\theta^2$ is negative.

Examples:

(i) *A rod rests with one end against a smooth vertical plane* AB (Fig. 144), *and the other on a smooth inclined plane* AC *of angle* α; *find the position of equilibrium.*

Let θ be the inclination of the rod to the horizontal, and l the length of the rod. The height y of the C. G. above A

$$= AN + \tfrac{1}{2}NB = NC\tan\alpha + \tfrac{1}{2}l\sin\theta$$
$$= l\cos\theta\tan\alpha + \tfrac{1}{2}l\sin\theta.$$

To find the maximum and minimum values of this, we have

$$dy/d\theta = -l\sin\theta\tan\alpha + \tfrac{1}{2}l\cos\theta,$$

which is equal to 0 when $\sin\theta\tan\alpha = \frac{1}{2}\cos\theta$, i.e. when $\cot\theta = 2\tan\alpha$. This gives the position of equilibrium.

Since $\quad d^2y/d\theta^2 = -l\cos\theta\tan\alpha - \frac{1}{2}l\sin\theta$, which is negative, θ being acute, y is a maximum, and the equilibrium is unstable.

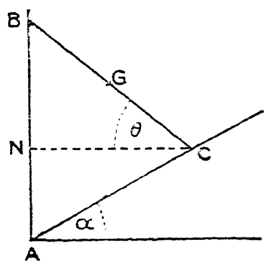


Fig. 144.

(ii) *A uniform rod* AB (Fig. 145) *of length* 2a *is hinged at* A; *a string attached to the middle point* G *of the rod passes over a smooth pulley at* C *at height* a *vertically above* A, *and supports a weight* P *hanging freely; find the positions of equilibrium.*

Let θ be the inclination of the rod to the vertical, therefore $\frac{1}{2}\theta$ is the inclination of the string CG to the vertical, since $AC = AG$. Let l be the length of the string.

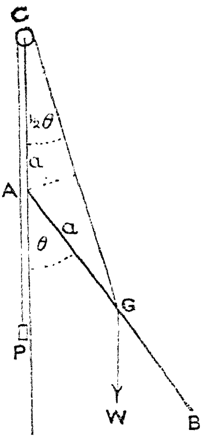


Fig. 145.

The depth of G below $C = a + a\cos\theta$, and the depth of $P = l - CG = l - 2a\cos\frac{1}{2}\theta$.

Hence, if y be the depth of the C. G. of the system,

$$(P+W)\,y = P\,(l - 2a\cos\tfrac{1}{2}\theta) + Wa\,(1+\cos\theta).$$

This is to be a maximum or minimum. Differentiating with respect to θ,

$$(P+W)\,dy/d\theta = P\,(a\sin\tfrac{1}{2}\theta) + Wa\,(-\sin\theta).$$

Hence $dy/d\theta = 0$ when $P\sin\frac{1}{2}\theta = W\sin\theta$

$$= 2W\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta,$$

i.e. when $\sin\frac{1}{2}\theta = 0$, or when $\cos\frac{1}{2}\theta = P/2W$.

Differentiating a second time, we have

$$(P+W)\frac{d^2y}{d\theta^2} = Pa\,.\,\tfrac{1}{2}\cos\tfrac{1}{2}\theta - Wa\cos\theta$$

$$Wa\left[\frac{P}{2W}\cos\tfrac{1}{2}\theta - \cos\theta\right]$$

$$Wa\left[\frac{P}{2W}\cos\tfrac{1}{2}\theta - 2\cos^2\tfrac{1}{2}\theta + 1\right]$$

If $\sin\frac{1}{2}\theta = 0$, $\theta = 0$ and $\cos\frac{1}{2}\theta = 1$; $\therefore\ (P+W)\frac{d^2y}{d\theta^2} = Wa\left[\frac{P}{2W} - 1\right]$,

which is + or − according as $P >$ or $< 2W$.

Hence,

if $P > 2W$, the depth y is a minimum, and the position $\theta = 0$ is unstable;

if $P < 2W$, the depth y is a maximum, and the position $\theta = 0$ is stable.

The second solution, $\cos\frac{1}{2}\theta = P/2W$, is only possible when $P < 2W$, and then

$$(P+W)\frac{d^2y}{d\theta^2} = Wa\left[\frac{P^2}{4W^2} - 2\,.\,\frac{P^2}{4W^2} + 1\right] = Wa\,.\,\frac{4W^2-P^2}{4W}$$

which is +, since $P < 2W$. The depth y is then a minimum, and the position given by $\cos\frac{1}{2}\theta = P/2W$ is unstable.

Examples LXXVI.

1. Find the work done in stretching an elastic string of natural length 6 ft. from length 7 ft. to length 8 ft., λ being 4 lb. weight.

2. Find the work done in stretching a string to three times its natural length, λ being $\frac{1}{2}$ lb. weight.

3. The resultant pressures on the piston of a steam-engine at distances 0, 2, 4, 6, 8, 10, 12, 14, 16 inches from the beginning of the stroke are respectively 18,000, 18,500, 18,400, 18,000, 16,500, 14,200, 11,100, 7200, 1800; find the work done during the stroke of 16 inches.

4. The pressures of a gas at volumes 1, 2, 3, 4, 5, 6, 7 cubic feet are 400, 240, 170, 120, 85, 70, 65 lb. weight per square inch. Find the work done during the expansion from 1 to 7 cubic feet.

5. A gas expands according to the law $pv =$ constant. When its volume is 2 cubic feet, the pressure is 350 lb. weight per square inch. Find the work done as the gas expands to a volume of 5 cubic feet.
Find the work done as the gas expands until its pressure is 55 lb. weight per square inch.

6. A quantity of air expands according to the law $pv^{1\cdot 4} =$ constant. The pressure is 250 lb. weight per square inch when the volume is 3 cubic feet. Find the work done when it expands to a volume of 7 cubic feet. Find how much work is required to compress it to a volume of 2 cubic feet.

7. A body of mass 100 lb. is drawn along a rough horizontal plane ($\mu = \cdot 3$) by means of a rope which passes over a smooth pulley 6 feet above the plane. If it be originally 10 feet distant from the pulley, find the work done in pulling it very slowly a distance of 5 feet along the ground.

8. A chain 500 feet long hangs vertically, and its mass varies uniformly from 10 lb. per foot at its upper end to 7 lb. per foot at its lower end. Find the work done in winding it up.

9. A circular well, 6 feet in diameter and 200 feet deep, is full of water. Find the amount of work done in pumping all the water to the top. At what rate is the level of the water sinking when it is (i) 100 feet, (ii) 150 feet below the ground, supposing the engine works at uniform rate and empties the well in 80 minutes?

10. A quantity of dry air at temperature 50° F. and atmospheric pressure is compressed adiabatically from volume 20 cubic feet to volume 5 cubic feet; find the amount of work done and the change of temperature.

11. Three cubic feet of saturated steam, pressure 150 lb. weight per square inch, expand to volume 8 cubic feet. Find the work done, the law of expansion being $pv^{1\cdot 06} =$ constant.

12. A uniform rod, weight W, rests with the lower end on a smooth horizontal plane AB, and the upper end against a vertical plane BC; it is kept from slipping by a horizontal string attached to a point on the rod distant one-third of its length from the lower end, which passes over a smooth pulley and supports a weight $\frac{1}{2}W$ hanging freely. Find the position of equilibrium.

13. If the lower end of the rod in Question 12 be supported by a string attached to B (and the other string be removed), find the tension of the string when the rod is inclined at 80° to the vertical.

14. Four equal uniform rods, each of weight W, are smoothly jointed together; B and D are kept apart by a rod of negligible weight of such length that $ABCD$ is a square, and the whole is suspended from A. Find the thrust in the rod BD.

15. If, in the preceding question, the rod BD is taken away, and the figure kept in shape by an inextensible string AC, find the tension in the string.

16. If in Question 14 the string be elastic, of natural length equal to the length of the rods, and such that the weight of all the rods would just stretch it to double its natural length, find the inclination of the rods to the vertical when in equilibrium.

17. A string passes over two smooth pegs A, B, 2 feet apart in a horizontal line, and has masses 5 lb. suspended at each end, and 6 lb. at its middle point. Find the position of equilibrium of the 6 lb. mass.

18. A uniform heavy rod, 6 feet long, rests over a smooth peg and against a smooth wall, from which the peg is 1 foot distant. Find the position of equilibrium, and whether it is stable or unstable.

19. A uniform rod rests with its ends on two smooth inclined planes of angles $35°$ and $50°$ which have a common foot. Find the position of equilibrium, and whether it is stable or unstable.

20. A parallelogram $ABCD$ of four uniform rods freely jointed has the side AB fixed horizontally and hangs in a vertical plane. A is attached to C by a light string of length equal to AB, and α is the acute angle of the parallelogram. Find the tension of the string.

21. Four rods of length a and negligible weight are freely jointed; the system rests with AC vertical, and BC, CD against two smooth pegs in the same horizontal line, distant c apart, B and D being kept apart by a light rod of length a. Find the thrust in BD when a weight W is placed on A.

22. A ladder of mass 100 lb. rests with one end on the ground and the other against a smooth vertical wall. It is kept from slipping by a horizontal string attached to its lower end. Find the tension of the string when the ladder is inclined to the horizontal at an angle α.

23. In the preceding question, find the work done in pulling the ladder from inclination $60°$ to inclination $70°$, its length being 40 feet.

24. A cube of wood of side 2 feet and specific gravity $\cdot 6$ floats in water with its base horizontal. Find the work done in pushing it down until its top is level with the surface of the water.

RECTILINEAR MOTION OF A PARTICLE

187. Motion of a particle in a straight line.

We will next consider some applications of the integral calculus to the motion of a particle in a straight line.

Expressions for the velocity and acceleration of a moving point have already been given (Art. 62), together with a few simple examples in which the acceleration is a given function of the time (Arts. 63 and 78).

We will now discuss some cases in which the force acting on the particle is given as a function of the position of the particle.

In the first place, since the force acting on a particle in any direction is equal to the product of its mass and its acceleration in that direction, it follows that the acceleration of the particle will follow the same law as the force which produces it.

(1) *Simple Harmonic Motion.* We commence with the well-known case of simple harmonic motion.

A particle moves in a straight line under the influence of a force which is directed towards a fixed point in the line, and varies as the distance from that point. To find the motion.

Let the particle start from rest at distance a from the fixed point O (Fig. 146). When at P, at distance x from O, its acceleration in the

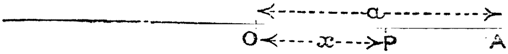


Fig. 146.

direction OP (i.e. *away* from O) is $\ddot{x}$ or $v\,dv/dx$. [Since $v\,dv/dx = \frac{1}{2}$ the d.c. of v^2 with respect to x, it is $+$ when v^2 increases as x increases.] Taking the latter form, since the force and therefore the acceleration varies as x, and is *towards* O, we have

$$v\,dv/dx = -\mu x,$$

where μ is a constant whose value can be found if the mass of the particle and the magnitude of the force acting upon it in any position are given.

Integrating with respect to x, $\frac{1}{2}v^2 = -\mu \cdot \frac{1}{2}x^2 + C$.

Since the particle starts from rest at A, we have $v = 0$ when $x = a$,

$$\therefore \quad 0 = -\tfrac{1}{2}\mu a^2 + C, \quad \text{and} \quad C = {}^1\mu a^2.$$

Substituting this value of C, $v^2 = \mu(a^2 - x^2)$,

$$\therefore \quad v = \pm\sqrt{\{\mu(a^2 - x^2)\}}.$$

This gives the velocity in any position; the double sign indicates that, at distance x from O, the particle is moving sometimes towards O and sometimes away from O; the magnitude of the velocity is the same in either case. This equation may be written

$$dx/dt = v = \pm\sqrt{\mu}\sqrt{(a^2 - x^2)},$$

i.e.

$$\frac{1}{\sqrt{(a^2 - x^2)}}\frac{dx}{dt} = \pm\sqrt{\mu}.$$

Integrating with respect to t, $\sin^{-1}(x/a) = \pm t\sqrt{\mu} + C$.

If t be measured from the instant when the particle starts, we have $x = a$ when $t = 0$.

$$\therefore \ \sin^{-1} 1 = C, \ \text{and} \ C = \tfrac{1}{2}\pi; \qquad \therefore \ \sin^{-1}(x/a) = \tfrac{1}{2}\pi \pm t\sqrt{\mu},$$

or

$$x/a = \sin(\tfrac{1}{2}\pi \pm t\sqrt{\mu}), \quad \text{and} \quad x = a\cos t\sqrt{\mu}.$$

This gives the distance of the particle from O (not the distance travelled from A) after time t.

The velocity at any instant is obtained by differentiating this with respect to t. This gives

$$v = dx/dt = -a\sqrt{\mu}\sin t\sqrt{\mu}.$$

If $t\sqrt{\mu}$ is increased by 2π, both x and v are unchanged in magnitude and sign, i.e. all the circumstances of the motion are repeated

without any alteration when t is increased by $2\pi/\sqrt{\mu}$; hence the motion is oscillatory, and the time of a complete oscillation, or the *period*, $= 2\pi/\sqrt{\mu}$. This is independent of a, the *amplitude*.

(2) *Law of Gravitation.* Next, let us take an example of the law of gravitation.

A particle moves in a straight line under the action of a force towards a fixed point in the line varying inversely as the square of the distance from that point; find the motion, and, as particular cases, find (i) *with what velocity a meteorite would reach the earth after moving from a very great distance under the influence of the earth's attraction, and* (ii) *how long it would take the moon, if suddenly stopped in its orbit, to reach the earth.*

As in the preceding case, the acceleration at distance x from O is $v\,dv/dx$ in the direction OP, $\therefore$ $v\,dv/dx = -\mu/x^2$.

Integrating with respect to x, $\frac{1}{2}v^2 = \mu/x + C$.

If the particle starts from rest at distance a, $v = 0$ when $x = a$.

$$\therefore\ 0 = \mu/a + C, \text{ and } C = -\mu/a;$$

hence $$v^2 = 2\mu(1/x - 1/a).$$

This gives the velocity of the particle in any position.

In the case of the meteorite starting at a very great distance, we may take $v = 0$ when $x = \infty$; $\therefore$ $C = 0$, and $v^2 = 2\mu/x$. At the earth's surface, the distance x of the particle from the centre of the earth * is equal to r, the radius of the earth; hence, neglecting the retarding effect of the earth's atmosphere, the velocity on reaching the earth's surface is given by $v^2 = 2\mu/r$. We can find μ in this case, because we know the acceleration of a particle at the earth's surface due to the attraction of the earth; it is approximately 32 ft.-secs. per sec., therefore $\mu/r^2 = 32$.

Hence $v^2 = 2\mu/r = 64r = 64 \times 4000 \times 5280$, taking r as 4000 miles. This gives the value of v as approximately 7 miles per second.

Conversely, if we suppose the direction of motion reversed, a stone projected vertically upwards from the earth's surface with this (or any greater) velocity would (neglecting the effect of the atmosphere) never return, but would recede to an infinite distance.

The retardation due to the resistance of the earth's atmosphere has here been neglected. As a matter of fact, this is so enormous that few meteorites ever reach the earth's surface. They are usually dissipated by the heat generated by their passage through the air. If in the preceding formula we take $r = 4050$ miles, the result will not be very different, and this would give the velocity at a point 50 miles distant from the earth's surface; at this distance the atmosphere of the earth will be extremely rare, and its retarding force very slight.

* It was shown in Art. 179, Ex. (iv) that the attraction of the earth, regarded as a sphere, at an external point is the same as if its whole mass were concentrated at its centre.

Returning to the general case, we have

$$dx/dt = v = -\sqrt{\{2\mu(1/x - 1/a)\}} = -\sqrt{\{2\mu(a-x)/ax\}},$$

taking the $-$ sign since x decreases as t increases.

i.e. $$\sqrt{\left(\frac{ax}{a-x}\right)}\frac{dx}{dt} = -\sqrt{(2\mu)};$$

$\therefore$ integrating with respect to t, $\int\sqrt{\left(\frac{ax}{a-x}\right)}dx = -\sqrt{(2\mu)}\,t + C.$

The expression on the left-hand side is rationalized by putting $x = a\cos^2\theta$;

then $$C - \sqrt{(2\mu)}\,t = \int\sqrt{\left(\frac{a^2\cos^2\theta}{a\sin^2\theta}\right)} \times -2a\cos\theta\sin\theta\,d\theta$$

$$= -a^{\frac{3}{2}}\int 2\cos^2\theta\,d\theta$$

$$= -a^{\frac{3}{2}}\int(1+\cos 2\theta)\,d\theta$$

$$= -a^{\frac{3}{2}}(\theta + \tfrac{1}{2}\sin 2\theta).$$

When $t = 0$, $x = a$, and therefore $\cos\theta = 1$ and $\theta = 0$; hence $C = 0$.

$$\therefore \quad \sqrt{(2\mu)}\,t = a^{\frac{3}{2}}(\theta + \tfrac{1}{2}\sin 2\theta),$$

and $$t = a^{\frac{3}{2}}(\theta + \sin\theta\cos\theta)/\sqrt{(2\mu)}$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{(2\mu)}}\left[\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right].$$

This gives the time to reach a point distant x from O, after starting from rest at distance a from O. The time to reach the origin is found by putting $x = 0$, which gives $t = \dfrac{a^{\frac{3}{2}}}{\sqrt{(2\mu)}}\cdot\dfrac{\pi}{2}$.

Taking the particular case mentioned above, if the moon be supposed to describe a circle of radius a about the earth with angular velocity ω, its acceleration towards the earth's centre is $\omega^2 a$ (Art. 68); but at distance a, the acceleration due to the earth's attraction is μ/a^2 towards the earth's centre; $\therefore$ $\omega^2 a = \mu/a^2$, and $\omega = \sqrt{\mu}/a^{\frac{3}{2}}$.

Hence the time of a complete revolution of the moon $= 2\pi/\omega = 2\pi a^{\frac{3}{2}}/\sqrt{\mu}$. Therefore

$$\frac{\text{time to reach the earth}}{\text{time of a revolution about the earth}} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8} = \cdot 1768 \ldots \text{ nearly.}$$

The time of a revolution of the moon about the earth is 27 days $7\frac{3}{4}$ hours nearly; hence the time it would take the moon to reach the earth is $\cdot 1768$ of 27 days $7\frac{3}{4}$ hours, i. e. a little less than 4 days 20 hours.

This supposes the moon to reach the centre of the earth, i. e. it neglects their radii in comparison with their initial distance apart.

188. Motion of a particle suspended by an elastic string.

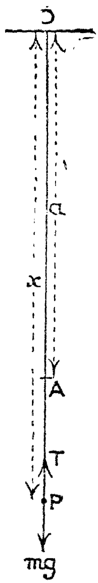


Fig. 147.

Let OA $(= a)$ (Fig. 147) be the natural length of the string (whose mass is neglected), and suppose a mass m to be gently attached to A, and then let go; it will begin to descend. Let x be the length of the string after time t. The forces on the particle are its weight mg vertically downwards, and the tension of the string vertically upwards. The tension of the string is given by Hooke's Law (Art. 183); in this case the tension is $\lambda(x-a)/a$, x being the total length.

Let v be the velocity when the length of the string is x; therefore the acceleration is $v\,dv/dx$ downwards.

The equation of motion is

$$mv\frac{dv}{dx} = \text{the force downwards} = mg - T = mg - \frac{\lambda(x-a)}{a};$$

i.e.
$$v\frac{dv}{dx} + \frac{\lambda}{ma}\left(x - a - \frac{mag}{\lambda}\right) = 0.$$

If $x - a - mag/\lambda$ be denoted by z,* then

$$\frac{dv}{dx} = \frac{dv}{dz}\cdot\frac{dz}{dx} = \frac{dv}{dz};$$

and the equation may be written

$$v\frac{dv}{dz} + \frac{\lambda}{ma}z = 0.$$

This is the same equation as was obtained and solved in Art. 187, (1), with λ/ma instead of μ. The initial conditions in this case are that when $t = 0$, $v = 0$ and $x = a$, therefore $z = -mag/\lambda$. Hence, substituting these values in the result there obtained, we have

$$z = -(mag/\lambda)\cos\{t\sqrt{(\lambda/ma)}\},$$

i.e.
$$x = a + (mag/\lambda)\left[1 - \cos\{t\sqrt{(\lambda/ma)}\}\right].$$

The maximum value of x (when $t\sqrt{(\lambda/ma)} = \pi$) is $a + 2mag/\lambda$, and the minimum value (when $t = 0$) is a; hence the particle descends a distance $2mag/\lambda$, then rises to its original position again, and continues to oscillate between these two positions with simple harmonic motion. The centre of the oscillation is the position of equilibrium of the particle, which is at a depth $a + mag/\lambda$ below O. The time of a complete oscillation is $2\pi\sqrt{(ma/\lambda)}$.

* It will be seen from the result below that z is the depth of the particle below its position of equilibrium.

Examples LXXVII.

1. A particle starts from rest and moves towards a fixed point O under the influence of a force directed towards O, and varying as the distance from O; if the particle was initially 4 feet from O, and the force on it was then equal to twice its weight, find (i) the velocity when 2 feet from O, (ii) the velocity at O, (iii) the distance from O after half a second, (iv) the time of a complete oscillation.

2. If in the preceding question all the circumstances are the same except that, instead of starting from rest, the particle is projected towards O with a velocity of 8 foot-seconds, find the corresponding velocities, distance, and time.

3. Supposing the earth suddenly stopped in its orbit round the sun, how long would it then take to fall into the sun?

4. A particle moves in a straight line under the influence of a force towards a fixed point O in the line, which varies inversely as the square of the distance from that point; it starts from rest at distance 4 feet from O, and the force at starting is four times its weight; find (i) the velocity when 1 foot from O, (ii) the time to reach O, (iii) the time to reach a point 1 foot from O.

5. If a particle could move in a straight line from the surface of the earth to its centre, how long would it take, starting from rest, to reach the centre, and what velocity would it have on arriving there? [See Art. 179, Ex. (iv).] After what time would it return to the starting-point?

6. Find the velocity of a particle which has moved from rest at a distance of 1000 miles under the influence of the earth's attraction, when it arrives at a distance of 100 miles from the earth's surface.

7. If a particle were projected vertically upwards with a velocity of 1 mile per second from a point on the earth's surface, find how far it would go under the influence of the earth's attraction (neglecting the effect of the atmosphere).

8. With what velocity would a stone have to be projected from the surface of the moon in order not to return? The radius of the moon is about 1100 miles, and the value of g at its surface about $5\frac{1}{3}$ ft.-secs. per sec.

9. A particle starts from rest and moves towards a fixed point O with an acceleration which varies as the square of the distance from O, and which is 16 ft.-secs. per sec. when the particle is 4 ft. from O; find the velocity with which it arrives at O, if it was originally 10 feet from O.

10. Suppose that in the preceding question all the circumstances are the same except that the acceleration varies inversely as the distance; find the velocity of the particle when it is 1 foot from O.

11. Find the velocity in any position when a particle moves from an infinitely great distance under the action of a force which varies inversely as the cube of the distance.

12. A particle of mass $\frac{1}{2}$ oz. is attached to the end of an elastic string 2 feet long which hangs vertically from a fixed point, and is then let go; find (i) the greatest depth it reaches, (ii) the time of oscillation, (iii) the velocity at depth of 2·5 feet, (iv) the time to reach depth 2·8 feet, (v) the depth after half a second, supposing that a mass 1 oz. hangs in equilibrium with the string 4 feet long.

13. A particle is repelled from a point O with a force which varies as the distance from O; if it starts with velocity u from O, find its velocity at any distance from O, and the time it takes to reach a given distance from O.

14. Answer the same questions if the particle starts from rest at distance a.

15. Answer the same questions if the repulsive force varies inversely as the square of the distance.

16. A particle moves towards a fixed point O with an acceleration which varies as its distance from O; its velocity when 4 feet from O is 20 foot-seconds, and its acceleration is then $6\frac{2}{3}$ foot-seconds per second; at what distance from O did it start from rest?

17. A particle moves towards a point O with an acceleration which varies inversely as the cube of the distance from O; find the time to reach O, supposing it starts from rest at distance a.

18. A particle attached to the end of an elastic string of natural length 3 feet hangs in equilibrium with the string stretched to a length of 4 feet; if the particle is held with the string at its natural length and then let go, find (i) the time to reach the greatest depth, (ii) the maximum velocity, (iii) the velocity after 1 second, (iv) the depth after 3 seconds.

19. If in the preceding question the particle is held with the string stretched to a length of $4\frac{1}{2}$ feet, and let go, find the values of (ii)–(iv).

20. A particle on a smooth horizontal plane is attached to two horizontal elastic strings, each of natural length 2 feet, which are in the same straight line, and have their other ends attached to fixed points 6 feet apart; the particle is in equilibrium with each string stretched to a length of 3 feet, and the modulus of elasticity λ is twice the weight of the particle for each string. If the particle is displaced a distance of 1 foot, so that the strings are 2 and 4 feet long, and then let go, find the time of a complete oscillation, and the position of the particle at any time.

21. Answer the same question when the strings are stretched vertically between two points 8 feet apart, and the particle is displaced 1 foot upwards from the position of equilibrium.

MOTION IN A RESISTING MEDIUM

189. Resistance proportional to velocity.

We now proceed to discuss several cases of the motion of a particle in a medium whose resistance is a function of the velocity of the particle.

A particle falls from rest in a medium whose resistance varies as the velocity; find the velocity at any subsequent instant and the distance fallen.

Let m be the mass of the particle. It is convenient to take the resistance as kmv; this varies as v, since m is supposed constant during the motion of the particle.

Taking the acceleration in the form dv/dt (since we want to find v in terms of t), the equation of motion is

$$mdv/dt = \text{force vertically downwards} = mg - mkv,$$

i.e. $\dfrac{dv}{dt} = g - kv$, which may be written $\dfrac{1}{g-kv}\cdot\dfrac{dv}{dt} = 1.$

Integrating, $\{\log(g-kv)\}/(-k) = t + C.$

Since the particle starts from rest, $v = 0$ when $t = 0$,

$$\therefore \quad -(\log g)/k = C;$$

hence, substituting this value of C, and multiplying by $-k$,

$$\log(g-kv) = -kt + \log g,$$

$$\therefore \quad \log\frac{g-kv}{g} = -kt, \quad \text{or} \quad 1 - \frac{k}{g}v = e^{-kt},$$

whence $$v = \frac{g}{k}(1 - e^{-kt}).$$

As t increases, $v \longrightarrow$ the limiting value g/k, since $e^{-kt} \longrightarrow 0$.

This is called the *terminal velocity*; its value can be obtained at once from the equation of motion, for it is clear that so long as the weight of the particle is greater than the resistance, the velocity continues to increase, and therefore the resistance continues to increase, and the resultant force on the particle tends to become zero; the acceleration then tends to zero, and the velocity tends to a constant value. The acceleration $\dot{v}$ is zero when $g - kv = 0$, i.e. when $v = g/k$. This then is the terminal velocity, the limit to which the velocity of the particle tends.

The velocity rapidly approaches the terminal velocity (unless k be very small) since the term e^{-kt} diminishes rapidly. For suppose the terminal velocity is 96 foot-seconds, i.e. $g/k = 96$, and therefore $k = \frac{1}{3}$. The velocity after 9 seconds $= (g/k)(1 - e^{-kt}) = 96(1 - e^{-3}) = 96 \times {\cdot}95$ nearly; i.e. after 9 seconds, the velocity is only about 5 per cent. short of the terminal velocity.

The distance fallen in time t is now obtained by writing

$$\frac{ds}{dt} = v = \frac{g}{k}(1 - e^{-kt}).$$

$\therefore$ integrating, $$s = \frac{g}{k}\left(t - \frac{e^{-kt}}{-k}\right) + C.$$

$s = 0$ when $t = 0$, $\therefore \; 0 = \dfrac{g}{k}\left(\dfrac{1}{k}\right) + C,$

i.e. $$C = -g/k^2.$$

Hence $$s = \frac{g}{k}\left(t + \frac{1}{k}e^{-kt}\right) - \frac{g}{k^2} = \frac{gt}{k} - \frac{g}{k^2}(1 - e^{-kt}).$$

190. Resistance proportional to square of velocity.

In this case, if we take a particle falling from rest, the equation of motion becomes

$$dv/dt = g - kv^2, \text{ which may be written } k\,(g/k - v^2),$$

or, putting c^2 instead of g/k for convenience, $dv/dt = k\,(c^2 - v^2)$,

$$\therefore \ \frac{1}{c^2 - v^2}\,\frac{dv}{dt} = k.$$

Integrating with respect to t, $\dfrac{1}{2c}\log\dfrac{c+v}{c-v} = kt + C.$

When $t = 0$, $v = 0$, since the particle starts from rest;

$$\therefore \ 0 = C, \text{ and } \log\frac{c+v}{c-v} = 2ckt;$$

$$\therefore \ \frac{c+v}{c-v} = e^{2ckt}, \text{ whence } v = c\,.\,\frac{e^{2ckt}-1}{e^{2ckt}+1} = c\tanh ckt.$$

As t increases indefinitely, $\tanh ckt \to 1$ (Art. 93) and $v \to c$, i.e. $\sqrt{(g/k)}$. This is the 'terminal velocity', as is likewise evident from the equation of motion.

The distance fallen through in any time t is at once obtained from the preceding result, for

$$dx/dt = v = c\tanh ckt,$$

$$\therefore \ x = c\int\tanh ckt\,.\,dt = c\int\frac{\sinh ckt}{\cosh ckt}\,dt = \frac{c}{ck}\log\cosh ckt + C.$$

When $t = 0$, $x = 0$, and $\cosh ckt = 1$; $\therefore$ $\log\cosh ckt = 0$.

Hence $C = 0$, and $x = \dfrac{1}{k}\log\cosh ckt = \dfrac{1}{k}\log\cosh\sqrt{(gk)}\,t.$

To find the height attained by a particle projected vertically upwards with velocity u, we take the acceleration as $v\,dv/dx$. The equation of motion is then

$$v\,dv/dx = -g - kv^2,$$

i.e.
$$\frac{v}{g+kv^2}\,\frac{dv}{dx} = -1.$$

Integrating, $\quad (1/2k)\log(g+kv^2) = -x + C.$

Initially, $v = u$ and $x = 0$, $\quad \therefore \ (1/2k)\log(g+ku^2) = C.$

$$\therefore \ x = \frac{1}{2k}\log(g+ku^2) - \frac{1}{2k}\log(g+kv^2) = \frac{1}{2k}\log\frac{g+ku^2}{g+kv^2}.$$

At the highest point, $v = 0$, and $x = \dfrac{1}{2k}\log\left(1 + \dfrac{k}{g}u^2\right).$

This gives the greatest height attained.

191. Numerical examples.

(i) *A particle is projected vertically upwards with a velocity of* 80 *foot-seconds in a medium whose resistance varies as the square of the velocity; with what velocity will the particle return to the starting-point, given that the terminal velocity of the particle falling in the same medium is* 80 *foot-seconds? Find also the time which elapses before it returns to the starting-point.*

The acceleration of the particle when falling is $g-kv^2$, and this is zero when v is the terminal velocity, i.e. $32-k\,.\,6400=0$; hence $k=\frac{1}{200}$.

When the particle is ascending, the equation of motion is

$$v\frac{dv}{dx}=-g-kv^2=-32-\frac{v^2}{200}=-\tfrac{1}{200}(6400+v^2).$$

i.e.
$$\frac{2v}{6400+v^2}\frac{dv}{dx}=-\tfrac{1}{100}.$$

Integrating, $\log(6400+v^2)=-\frac{1}{100}x+C.$

$$v=80 \text{ when } x=0; \quad \therefore \log 12800=C,$$

and $\frac{1}{100}x=\log 12800-\log(6400+v^2)=\log\{12800/(6400+v^2)\}.$

At the highest point, $v=0$; $\therefore$ $x=100\log 2.$

We now have to find the velocity of the particle after falling this distance from rest.

When descending, $v\,dv/dx=g-kv^2=\frac{1}{200}(6400-v^2),$

whence, as above, $\log(6400-v^2)=-\frac{1}{100}x+C.$

$$v=0 \text{ when } x=0; \quad \therefore \log 6400=C.$$

$$\therefore \log(6400-v^2)=-\tfrac{1}{100}x+\log 6400.$$

Hence, when $x=100\log 2$, we have

$$\log(6400-v^2)=-\log 2+\log 6400=\log 3200;$$

$$\therefore 6400-v^2=3200,$$

and $v=\sqrt{3200}=40\sqrt{2}=56{\cdot}56$ foot-seconds.

To find the time, we take the acceleration in the form dv/dt.

When ascending, $\dfrac{dv}{dt}=-\frac{1}{200}(6400+v^2)$, $\therefore$ $\dfrac{1}{6400+v^2}\dfrac{dv}{dt}=-\frac{1}{200}.$

Integrating, $\frac{1}{80}\tan^{-1}\frac{1}{80}v=-\frac{1}{200}t+C.$

When $t=0$, $v=80$; $\therefore$ $C=\frac{1}{80}\tan^{-1}1=\frac{1}{320}\pi,$

and $\frac{1}{80}\tan^{-1}\frac{1}{80}v=\frac{1}{320}\pi-\frac{1}{200}t.$

At the highest point, $v=0$; $\therefore$ $t=\frac{5}{8}\pi=1{\cdot}96$ seconds.

When descending, $\dfrac{dv}{dt}=\frac{1}{200}(6400-v^2)$, $\therefore$ $\dfrac{1}{6400-v^2}\dfrac{dv}{dt}=\frac{1}{200}.$

Integrating, $\frac{1}{160}\log\dfrac{80+v}{80-v}=\dfrac{t}{200}+C.$

When $t=0$, $v=0$; $\quad C=0$, and $t=\frac{5}{4}\log\dfrac{80+v}{80-v}.$

If we now use the result obtained above, that $v = 40\sqrt{2}$ on reaching the starting-point again, we have

$$t = \tfrac{5}{4}\log\frac{80+40\sqrt{2}}{80-40\sqrt{2}} = \tfrac{5}{4}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} = \tfrac{5}{4}\log(3+2\sqrt{2}) = \tfrac{5}{4}\log 5{\cdot}83 = 2{\cdot}2.$$

Hence the total time which elapses is 4·16 seconds approximately.

(ii) A *toboggan descends a uniform slope of* 1 *in* 5 *which is a hundred yards in length. The coefficient of friction is* $\frac{1}{20}$, *and the resistance of the air varies as the square of the velocity, and is* 2 *lb. weight per square foot of surface exposed to it when the velocity is* 20 *foot-seconds. If the toboggan when loaded weighs* 200 *lb., and presents a surface of* 4 *square feet to the air-resistance, find its velocity when it reaches the foot of the incline and the time it takes to descend. Show that, however long the incline may be, the velocity can never exceed about* $26\frac{1}{2}$ *miles per hour.*

The resolved part of the weight down the incline

$$= 200\sin\alpha = 200\times\tfrac{1}{5} = 40 \text{ lb. weight} \quad \text{(Fig. 148)}.$$

The friction $\quad = \frac{1}{20}R = \frac{1}{20}\times 200\cos\alpha = 10\times\dfrac{4{\cdot}9}{5} = 9{\cdot}8$ lb. weight.

The air-resistance per square foot $= kv^2$, and is equal to 2 lb. weight when $v = 20$;

$$\therefore\ 2 = k\,.\,400, \text{ and } k = \tfrac{1}{200}.$$

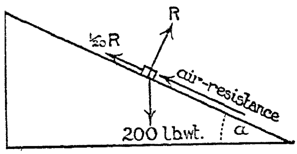


Fig. 148.

Hence, since the surface exposed to it is 4 square feet, the total air-resistance $= \frac{1}{50}v^2$ lb. weight.

Therefore the equation of motion of the toboggan is

$$200\,v\,dv/ds = (40 - 9{\cdot}8 - \tfrac{1}{50}v^2)g.$$

$$\therefore\ v\,dv/ds = \tfrac{32}{200}(30{\cdot}2 - \tfrac{1}{50}v^2) = \tfrac{32}{10000}(1510 - v^2),$$

i.e. $$\frac{v}{1510-v^2}\frac{dv}{ds} = \frac{32}{10000}, \text{ or } \frac{-2v}{1510-v^2}\frac{dv}{ds} = -{\cdot}0064.$$

Integrating, $\qquad \log(1510 - v^2) = -{\cdot}0064s + C.$

Now $v = 0$ at starting, i.e. when $s = 0$; $\quad\therefore\ \log 1510 = C$;

$$\therefore\ \log(1510 - v^2) = \log 1510 - {\cdot}0064s.$$

Hence $$1510 - v^2 = 1510\,e^{-{\cdot}0064s},$$

and $$v^2 = 1510\,[1 - e^{-{\cdot}0064s}].$$

At the foot of the incline, $s = 300$, and $e^{-{\cdot}0064s} = e^{-1{\cdot}92} = {\cdot}1466$,

$$\therefore\ v^2 = 1510\times{\cdot}8534 \text{ and } v = 35{\cdot}9 \text{ foot-seconds.}$$

Hence the toboggan reaches the bottom with a velocity of 35·9 foot-seconds, or $24\frac{1}{2}$ miles an hour very nearly.

As s increases indefinitely, $v^2 \to 1510$ and $v \to 38{\cdot}9$. Therefore, however long the incline may be, the velocity will never exceed 38·9 feet per second, or roughly $26\frac{1}{2}$ miles per hour.

To find the time of descent, we may, in the equation of motion, take the acceleration as dv/dt instead of $v\,dv/ds$, and proceed as in the last example; it will be found that, when $v = 35{\cdot}9$, $t = 13$ nearly, so that the time of descent is approximately 13 seconds.

(iii) *Find the horizontal distance travelled in* 1 *second by a body projected horizontally with velocity* 1000 *foot-seconds, assuming the resistance of the air varies as the cube of the velocity (which is found by experiment to be approximately the case for large velocities).*

The equation of motion is $v\dfrac{dv}{dx} = -kv^3$, i.e. $-\dfrac{1}{v^2}\dfrac{dv}{dx} = k$;

$\therefore$ integrating, $\quad 1/v = kx + C.$

Initially, $v = 1000$, and $x = 0$; $\therefore\ C = \frac{1}{1000}$,

and $\quad 1/v = kx + \frac{1}{1000}$, $\quad$ i.e. $dt/dx = kx + \frac{1}{1000}$.

Integrating again, $\quad t = \frac{1}{2}kx^2 + \frac{1}{1000}x + C.$

Initially both t and x are 0; $\therefore\ C = 0$,

and $$\tfrac{1}{2}kx^2 + \tfrac{1}{1000}x - t = 0.$$

Taking $t = 1$, $$500\,kx^2 + x - 1000 = 0;$$

$$\therefore\ x = [-1 \pm \sqrt{(1 + 4\,.\,500\,k\,.\,1000)}]/1000\,k.$$

The positive root of this equation gives the distance required; it is found by experiment that $k = 4{\cdot}45 \times 10^{-8}$ nearly. Substituting this value, we find $x = 976$ feet approximately.

MOTION IN A CURVE

192. Motion in an ellipse.

We have discussed (Art. 187) the motion of a particle in a straight line when attracted to a fixed point in the line by a force which varies as the distance from the point. Let us now determine the motion of a particle under a similar force, when projected in a different direction. Suppose it is projected from a point A, at a distance a from the fixed point O towards which the force acts, with velocity u in the direction perpendicular to OA (Fig. 149).

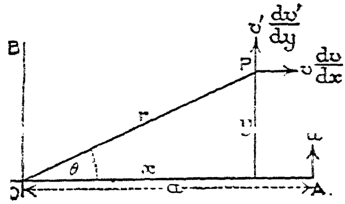


Fig. 149.

Let (x, y) be the coordinates of the position P of the particle at the end of time t, referred to rectangular axes OA, OB, and let (r, θ) be the polar coordinates of P. The force on the particle at P may be written in the form μmr, and the accelerations of P parallel to the axes are $v\,dv/dx$ and $v'\,dv'/dy$, where v and v' are the components of the velocity of P parallel to the axes respectively.

Resolving parallel to the axes, we have the equations

$$mv\,dv/dx = -\mu mr\cos\theta = -\mu mx,$$

$$mv'\,dv'/dy = -\mu mr\sin\theta = -\mu my,$$

i.e. $\quad v\,dv/dx = -\mu x, \quad \text{and} \quad v'\,dv'/dy = -\mu y;$

together with the initial conditions $x = a$, $y = 0$, $v = 0$, $v' = u$.

These are the equations of simple harmonic motion obtained in Art. 187, and the equations can be solved as in that article. Hence the motion of the particle is compounded of two simple harmonic motions in directions at right angles and having the same period, since μ is the same in both.

The equation of simple harmonic motion can also be solved in another way as follows: Taking the above equations, they may be written in the forms $\ddot{x} = -\mu x$, and $\ddot{y} = -\mu y$.

If we ask ourselves what kind of function satisfies an equation of this type, we remember that the second differential coefficients of $\sin mt$ and $\cos mt$ with respect to t are $-m^2 \sin mt$ and $-m^2 \cos mt$; hence, if $x = \sin t\sqrt{\mu}$ or $\cos t\sqrt{\mu}$, it follows that $\ddot{x} = -\mu x$, and therefore the same result is true if $x = A \sin t\sqrt{\mu} + B \cos t\sqrt{\mu}$, where A and B are constants. This therefore is a solution of the equation, and it will be seen later that it is the most general solution.

Hence $x = A \sin t\sqrt{\mu} + B\cos t\sqrt{\mu}$; $y = C\sin t\sqrt{\mu} + D\cos t\sqrt{\mu}$, and it remains to determine the constants A, B, C, D.

Differentiating,

$$dx/dt, \text{ i.e. } v, = A\sqrt{\mu}\cos t\sqrt{\mu} - B\sqrt{\mu}\sin t\sqrt{\mu};$$

$$dy/dt, \text{ i.e. } v', = C\sqrt{\mu}\cos t\sqrt{\mu} - D\sqrt{\mu}\sin t\sqrt{\mu}.$$

Substituting in these four equations for x, y, dx/dt, dy/dt the initial values $x = a$, $v = 0$, $y = 0$, $v' = u$ when $t = 0$, we get

$$a = B; \; 0 = A\sqrt{\mu}, \; \text{and} \; 0 = D, \; u = C\sqrt{\mu}.$$

$$\therefore \; x = a\cos t\sqrt{\mu} \; [\text{as in Art. 187}], \; \text{and} \; y = (u/\sqrt{\mu})\sin t\sqrt{\mu}.$$

Eliminating t, we have as the equation of the path of P

$$\frac{x^2}{a^2} + \frac{y^2}{u^2/\mu} = \cos^2 t\sqrt{\mu} + \sin^2 t\sqrt{\mu} = 1,$$

which is the equation of an ellipse whose centre is the origin, and whose axes lie along the axes of coordinates (p. 19), and are of lengths $2a$ and $2u/\sqrt{\mu}$.

Hence the path of the attracted particle is an ellipse described about the 'centre of force' as centre.

If $u^2 = \mu a^2$, the axes are equal and the path of the particle is a circle. In this case $\dot{x} = -a\sqrt{\mu}\sin t\sqrt{\mu}$ and $\dot{y} = a\sqrt{\mu}\cos t\sqrt{\mu}$. $\therefore\ \dot{x}^2 + \dot{y}^2 = a^2\mu$, and the resultant velocity of the moving point is constant and equal to $a\sqrt{\mu}$, i.e. the moving point describes a circle of radius a with uniform angular velocity $\sqrt{\mu}$. Uniform circular motion may therefore be regarded as the resultant of two simple harmonic motions at right angles of equal periods and amplitudes, one of which is a quarter oscillation ahead of the other. This follows at once geometrically, if we draw perpendiculars PN, PM from a point P on the circle to two diameters at right angles, and consider the motion of N and M.

193. Motion of a particle along a smooth curve in a vertical plane.

Let u and v be the velocities of the particle at A and P respectively, and s the length of the arc AP (Fig. 150).

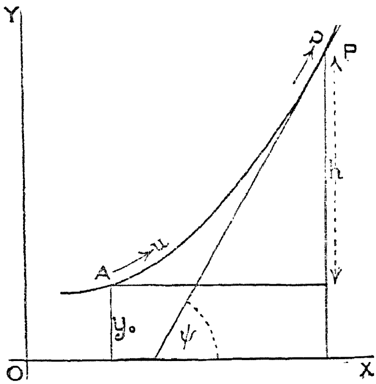


Fig. 150.

The acceleration of the particle along the tangent at P is $v\,dv/ds$; therefore, resolving along the tangent,

$$mv\,dv/ds = -mg\sin\psi = -mg\,dy/ds \quad \text{(Art. 82)}.$$

Integrating with respect to s, $\frac{1}{2}v^2 = -gy + C$.

If y_0 be the ordinate of A, then $v = u$ when $y = y_0$.

$$\therefore\ \tfrac{1}{2}u^2 = -gy_0 + C.$$

Hence, by subtraction, $\frac{1}{2}(v^2 - u^2) = -g(y - y_0)$ (i)

i.e. $$v^2 = u^2 - 2gh,$$

if h be the vertical distance between A and P.

Therefore, if a particle moves along a smooth curve under the action of gravity, the change in its velocity depends only upon the vertical distance it travels.

Multiplying equation (i) by m, it may be written

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2 = mgy - mgy_0,$$

i.e. the decrease in the kinetic energy of the particle is equal to the increase in its potential energy; hence *the sum of the kinetic and potential energies is constant.*

Examples LXXVIII.

1. A particle is projected with velocity u, and moves horizontally in a medium whose resistance varies as the velocity. Find (i) the velocity after travelling a given distance, (ii) the velocity after a given time, (iii) the distance travelled in a given time, (iv) when it comes to rest, (v) where it comes to rest.

2. A particle is projected with velocity 1000 foot-seconds, and moves horizontally in a medium whose resistance to mass m moving with velocity v is $\frac{1}{10}mv^3$. Find (i) the velocity after travelling a distance x, (ii) the velocity after t seconds.

3. A particle is projected vertically upwards with velocity 80 foot-seconds in a medium whose resistance varies as the square of the velocity, and is equal to $\frac{1}{200}mv^2$ poundals in the case of mass m lb. moving with velocity v foot-seconds. Find (i) the time to the highest point, (ii) the greatest height, (iii) the velocity after 2 seconds, (iv) the velocity at height 20 feet.

4. Answer the first three questions of Ex. 3, if all the conditions are the same except that the resistance is equal to $\frac{1}{20}mv$ poundals.

5. A particle falls from rest in a medium whose resistance varies as the velocity. The resistance is $\frac{1}{10}$ of the weight when the velocity is 10 foot-seconds. Find (i) the terminal velocity, (ii) the velocity after 5 seconds, (iii) the distance fallen in 4 seconds.

6. Answer the same questions if all the conditions are the same except that the resistance varies as the square of the velocity. Find also the velocity after falling 40 feet.

7. A particle falls from rest in a medium whose resistance varies as the cube of the velocity. If the terminal velocity be 16 foot-seconds, find the resistance to a mass of 2 lb. moving with velocity 10 foot-seconds.

8. A particle is projected vertically upwards with velocity 40 foot seconds in a medium whose resistance varies as the square of the velocity, and is equal to $\frac{1}{4}$ of the weight of the particle at starting. Find (i) the time to the highest point, (ii) the greatest height, (iii) the velocity on reaching the ground again, (iv) the time taken to fall to the ground again.

9. A particle is projected vertically upwards with velocity 80 foot-seconds; find its velocity after rising 10 feet, if the resistance produces a retardation $\cdot 00005\,v^4$ ft.-secs. per sec., where v is the velocity of the particle.

10. Find the terminal velocity if a particle falls in a medium whose resistance varies as the n^{th} power of the velocity.

11. A particle is projected vertically upwards with velocity u in a medium whose resistance varies as the velocity. Find the time to the highest point and the greatest height.

12. In the preceding question, find the time to the highest point if the resistance varies as the square of the velocity.

13. A particle of mass 4 lb. starts with velocity 100 foot-seconds, and moves horizontally against a resistance $v^4/2^8$ lb. weight. Find (i) its velocity after travelling 20 feet, (ii) the distance travelled in 1 second, (iii) how far it travels before its velocity is reduced to one-half of its original value.

14. A particle of mass m lb. moves horizontally in a medium whose resistance $= m\sqrt{v}/k$ lb. weight. Find the time before the particle comes to rest and the distance travelled, if it starts with velocity u.

15. A man descends from a balloon by means of a parachute. How large should the parachute be in order that, whatever be the height from which he starts, his velocity on reaching the ground may not exceed 20 foot-seconds? The total mass of the man and parachute is 160 lb. and the resistance of the air varies as the square of the velocity, and is equal to 1 lb. weight per square foot of surface exposed to it when the velocity is 20 foot-seconds.

16. Steam is shut off, and the brakes are applied to a train running at 60 miles per hour. If the brakes exert a constant retarding force equal to $\frac{1}{18}$ of the weight of the train, and if the other resistances are proportional to the velocity and equal to $\frac{1}{160}$ of the weight of the train when the velocity is 60 miles per hour, find the time and the distance travelled before the train comes to rest.

17. Two particles move in the same vertical straight line in a medium whose resistance varies as the velocity. One is projected vertically upwards with velocity u, and starting at the same time the other falls from rest at a height h. After what time will they meet?

18. An inclined plane is half a mile long and has a vertical fall of 300 feet. A toboggan of mass 200 lb. slides down it. If the coefficient of friction is ·05, and the air-resistance varies as the square of the velocity and is equal to 5 lb. weight when the velocity is 40 foot-seconds, find the velocity at the bottom of the incline and the time of descent. Show that the velocity will never exceed 64 foot-seconds, however long the incline be.

19. In Ex. 15, find how long and how far the man falls before his velocity is 19·5 foot-seconds.

20. OA, OB are two equal straight lines at right angles; a particle is projected from A in the direction AB with velocity 20 foot-seconds, and is attracted to O by a force which varies as the distance from O. If OA be 5 feet, and if the initial acceleration of the particle be 20 foot-seconds per second, find its path.

21. Determine the path, if, in the preceding question, the direction of projection is inclined to OA at an angle $\sin^{-1}\frac{4}{5}$, the other circumstances of the motion being unaltered.

22. Find the coordinates of the particle at the end of time t, and deduce the equation of the path, if in the theorem of Art. 192, the force is repulsive instead of attractive.

23. A particle moves in a parabola under the action of a force parallel to its axis; prove that the force must be constant.

24. A particle moves under the action of an attractive force which is perpendicular to a given straight line and varies as the distance from it; show that it describes a sine-curve.

MOTION OF A PENDULUM

194. The simple pendulum.

A particle of mass m *is attached by a string of length* l *to a fixed point and makes oscillations in a vertical plane. To find the time of a small oscillation.*

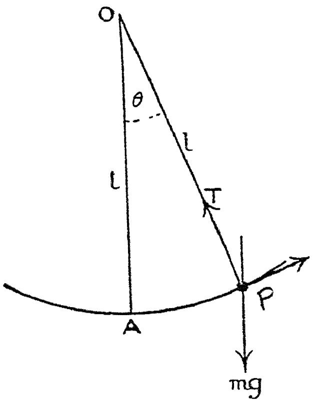


Fig. 151.

If θ be the angle which the string OP makes with the vertical at time t, the acceleration of m along the tangent at P in the direction in which θ increases is $l\,d^2\theta/dt^2$ or $l\,d\omega/dt$ (Art. 68), if ω be the angular velocity. Hence, resolving along the tangent,

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

i.e.
$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

This equation cannot be integrated in finite terms so as to give θ in terms of t. A first integral which gives the relation between the angular velocity ω and the angle θ can however be found. For

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\times\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta};$$

Hence the equation may be written $\omega\,d\omega/d\theta = -(g/l)\sin\theta$.

Integrating with respect to θ, $\qquad \frac{1}{2}\omega^2 = (g/l)\cos\theta + C.$

If the particle be held with the string inclined at an angle α to the vertical and then let go, we have $\omega = 0$ when $\theta = \alpha$,

$$\therefore\quad C = -(g/l)\cos\alpha;\ \text{ and }\ \omega^2 = 2(g/l)(\cos\theta - \cos\alpha),$$

which gives the angular velocity in any position.

This result may also be written down at once from Art. 193. (In this case the tension of the string replaces the normal reaction of the curve.) For the kinetic energy of the particle is $\frac{1}{2}m(l\omega)^2$, and the vertical distance it descends while the inclination of the string changes from α to θ is $l\cos\theta - l\cos\alpha$.

$$\therefore\quad \tfrac{1}{2}ml^2\omega^2 = mg(l\cos\theta - l\cos\alpha),$$

i.e.
$$\omega^2 = 2(g/l)(\cos\theta - \cos\alpha),\ \text{ as before.}$$

Returning now to the original equation, it may be written

$$\omega \frac{d\omega}{d\theta} = -\frac{g}{l} \sin\theta = -\frac{g}{l}\,\theta \times \frac{\sin\theta}{\theta}.$$

If θ be small, $(\sin\theta)/\theta$ is nearly 1, and therefore the motion is represented approximately by the equation

$$\omega\, d\omega/d\theta = -(g/l)\,\theta.$$

This is the same equation as was obtained and solved in Art. 187, with ω, θ, and g/l instead of v, x, and μ respectively. Using the result obtained there, we have $\theta = \alpha \cos t\sqrt{(g/l)}$. The particle moves along the arc with simple harmonic motion, and the time of a complete oscillation is $2\pi\sqrt{(l/g)}$.

If we try to find the time taken to swing through any angle θ, not very small, we get

$$d\theta/dt = \omega = \pm\sqrt{\{(2g/l)(\cos\theta - \cos\alpha)\}}.$$

Since (in the first swing) θ decreases as t increases, the $-$ sign must be taken. Using the formula $\cos 2A = 1 - 2\sin^2 A$, we get

$$d\theta/dt = -\sqrt{\{(2g/l)(2\sin^2\tfrac{1}{2}\alpha - 2\sin^2\tfrac{1}{2}\theta)\}}.$$

To simplify this, since $\theta \not> \alpha$, we may put $\sin\frac{1}{2}\theta = \sin\frac{1}{2}\alpha \sin\phi$;

$$\therefore\quad \tfrac{1}{2}\cos\tfrac{1}{2}\theta \frac{d\theta}{dt} = \sin\tfrac{1}{2}\alpha\cos\phi\frac{d\phi}{dt};$$

$$\therefore\quad \frac{2\sin\frac{1}{2}\alpha\cos\phi}{\cos\frac{1}{2}\theta}\frac{d\phi}{dt} = -\sqrt{\left[\frac{4g}{l}\sin^2\tfrac{1}{2}\alpha\,(1-\sin^2\phi)\right]}$$

$$= -2\sqrt{\frac{g}{l}}\sin\tfrac{1}{2}\alpha\cos\phi;$$

$$\frac{1}{\cos\frac{1}{2}\theta}\frac{d\phi}{dt} = -\sqrt{\frac{g}{l}};\quad \text{i.e.}\quad \frac{1}{\sqrt{(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)}}\frac{d\phi}{dt} = -\sqrt{\frac{g}{l}}.$$

When $\theta = \alpha$, $\sin\phi = 1$ and $\phi = \frac{1}{2}\pi$; therefore the time from the initial position to any position ϕ is given by the equation

$$t = \sqrt{\frac{l}{g}}\int_{\phi}^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)}}.$$

This cannot be integrated in finite terms of functions hitherto considered, but it can be expanded by the Binomial Theorem, and an approximate value of the integral can be found, as in Art. 160, in obtaining the length of an arc of an ellipse. Since $\phi = 0$ when $\theta = 0$, and $\phi = \frac{1}{2}\pi$ when $\theta = \alpha$, the time of a complete oscillation will be four times the value of the integral from $\phi = 0$ to $\phi = \frac{1}{2}\pi$. If we neglect $\sin^2\frac{1}{2}\alpha$, we get the approximation above, viz.:

$$t = 4\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi} d\phi = 4\sqrt{\frac{l}{g}}\cdot\frac{\pi}{2} = 2\pi\sqrt{\frac{l}{g}}.$$

If we expand by the Binomial Theorem, we get a closer approximation to the time of oscillation. This gives

$$t = 4\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi}(1-\sin^2\tfrac{1}{2}\alpha\sin^2\phi)^{-\frac{1}{2}}\,d\phi$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi}\left(1+\tfrac{1}{2}\sin^2\tfrac{1}{2}\alpha\sin^2\phi+\frac{1\,.\,3}{2\,.\,4}\sin^4\tfrac{1}{2}\alpha\sin^4\phi+\dots\right)d\phi$$

$$= 4\sqrt{\frac{l}{g}}\left(\tfrac{1}{2}\pi+\tfrac{1}{2}\sin^2\tfrac{1}{2}\alpha\,.\,\tfrac{1}{2}\,.\,\tfrac{1}{2}\pi+\frac{1\,.\,3}{2\,.\,4}\sin^4\tfrac{1}{2}\alpha\,.\,\frac{3\,.\,1}{4\,.\,2}\cdot\tfrac{1}{2}\pi+\dots\right)$$

$$= 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1^2}{2^2}\sin^2\tfrac{1}{2}\alpha+\frac{1^2\,.\,3^2}{2^2\,.\,4^2}\sin^4\tfrac{1}{2}\alpha+\dots\right).$$

If $\alpha = 30°$, the first two terms give the period as $2\pi\sqrt{(l/g)}\times 1{\cdot}016$.

195. The cycloidal pendulum.

The foregoing result $2\pi\sqrt{(l/g)}$ for the time of a complete oscillation is not exact, because it has been obtained only by taking $(\sin\theta)/\theta$ equal to unity; since this is only approximately true for small values of θ, the time of oscillation is only approximately constant. If, however, the particle be made to move along an arc of an inverted cycloid, instead of an arc of a circle as it does when suspended by an inextensible string, it can be shown that the time of oscillation is quite constant, whether θ be small or large.

For it has been shown (Art. 82) that, if s be the length of an arc of a cycloid measured from the vertex O,

$$ds/d\theta = -2a\sin\tfrac{1}{2}\theta,$$

$$\therefore\quad s = -2a\int\sin\tfrac{1}{2}\theta\,d\theta = 4a\cos\tfrac{1}{2}\theta + C;$$

$s = 0$ at the vertex where $\theta = \pi$, $\therefore\ 0 = C$, and $s = 4a\cos\frac{1}{2}\theta$. (i)

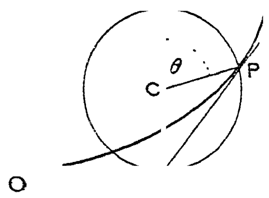


Fig. 152.

If the particle moves along the arc towards O, then resolving along the tangent at P (Fig. 152),

$$m\,d^2s/dt^2 = -mg\sin PTN = -mg\cos PTG = -mg\cos\tfrac{1}{2}\theta;$$

$$\therefore\quad d^2s/dt^2 = -g\cos\tfrac{1}{2}\theta = -(g/4a)\,s,\quad \text{from (i).}$$

This is the equation of simple harmonic motion again. Hence the particle moves along the arc with simple harmonic motion, and the time of a complete oscillation is $2\pi\sqrt{(4a/g)}$.

Here no approximation has been made, and the result is true whatever be the length of the arc in which the particle oscillates.

196. The compound pendulum.

The following investigation shows how the moment of inertia of a rigid body enters in dynamical problems:

A rigid body swings freely about a fixed horizontal axis; to find the equation of motion, and the time of a small oscillation.

Let Fig. 153 represent a section of the body by a plane through the centre of gravity G perpendicular to the axis of rotation which meets this plane in O. Consider the position in which the plane through G and the axis is inclined at an angle θ to the vertical. Let δm be an element of mass of the body situated at P, and let the perpendicular from P to the axis be of length r and make an angle ϕ with the vertical. Let $OG = h$.

The accelerations of δm at P are $r\ddot{\phi}$ perpendicular to OP in the direction in which ϕ increases, and $r\dot{\phi}^2$ along PO. (Art. 68.)

Hence the resultant forces on δm are

(i) $\delta m \,.\, r\ddot{\phi}$ perpendicular to PO,

(ii) $\delta m \,.\, r\dot{\phi}^2$ along PO;

therefore the sum of the moments about the axis of the forces on δm $= \delta m \,.\, r\ddot{\phi} \times r$ (since the moment of (ii) is zero) $= \delta m \,.\, r^2\ddot{\phi}$.

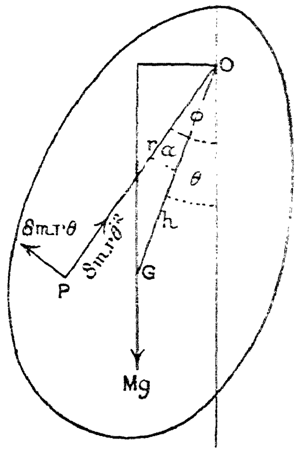


Fig. 153.

Therefore, for the whole body, the sum of the moments about the axis of the forces on all the elements of mass

$$= \Sigma(\delta m r^2 \ddot{\phi}).$$

Now, if α be the angle between the planes through OP and OG perpendicular to the plane of the paper, $\phi = \theta + \alpha$, and if the body be rigid, the angle α will be constant; therefore, differentiating twice with respect to the time, $\ddot{\phi} = \ddot{\theta}$, and is the same for every element δm.

Hence the sum of the moments about the axis of the forces on all the different elements of mass

$$= \ddot{\theta}\Sigma(r^2\delta m) = \ddot{\theta} \times \text{M.I. of the body about the axis} = \ddot{\theta} \,.\, Mk^2,$$

where k is the radius of gyration about the axis.

The aggregate of the forces on all the elements δm of the body consists of the external forces acting on the body and the mutual

actions and reactions of the elements among themselves. It may be taken as a consequence of the laws of motion that the latter are in equilibrium among themselves.* Assuming this, it follows that the quantity obtained above as the sum of the moments of all the forces about the axis is equal to the sum of the moments about the axis of the external forces on the body.

This gives $Mk^2 \ddot{\theta} = -Mgh \sin\theta, \quad \therefore \quad \ddot{\theta} = -(gh/k^2)\sin\theta.$

This is the same equation as was obtained in the case of the simple pendulum in Art. 194, with l replaced by k^2/h. Therefore, using the results of that article, a first integral gives

$$\dot{\theta}^2 = 2gh(\cos\theta - \cos\beta)/k^2,$$

if the body starts from rest with OG inclined at an angle β to the vertical; and if the oscillation be through a small angle only, the time of oscillation is approximately constant and equal to $2\pi\sqrt{(k^2/gh)}$.

A nearer approximation can be obtained exactly as in the case of the simple pendulum.

These results are the same as if the whole mass were concentrated at a point distant k^2/h from the axis, i.e. they are the same as in the case of a simple pendulum of length k^2/h. Hence k^2/h is called the length of the *simple equivalent pendulum*, k being the radius of gyration of the body about the axis round which the body rotates, and h the distance of the C. G. of the body from that axis.

We have, above, deduced the equation for $\dot{\theta}$ from the equation for $\ddot{\theta}$ by integration. This process can be reversed, if we assume the principle of energy for a rigid body. For the element δm is moving with velocity $r\dot{\phi}$, i.e. $r\dot{\theta}$, perpendicular to OP; therefore its kinetic energy is $\frac{1}{2}\delta m \,.\, r^2\dot{\theta}^2$, and the kinetic energy of the whole body

$$= \Sigma(\tfrac{1}{2}\delta m \,.\, r^2\dot{\theta}^2) = \tfrac{1}{2}\dot{\theta}^2\,\Sigma(r^2\,\delta m) = \tfrac{1}{2}Mk^2\,\dot{\theta}^2.$$

The only force which does work during the motion is the weight of the body, and the work done in turning from inclination β to inclination θ to the vertical

$$= Mg \times \text{vertical displacement of C. G.} = Mg\,(h\cos\theta - h\cos\beta).$$

Hence, since the increase in the kinetic energy is equal to the work done by the weight,

$$\tfrac{1}{2}Mk^2\,\dot{\theta}^2 = Mgh(\cos\theta - \cos\beta),$$

and $$\dot{\theta}^2 = 2gh(\cos\theta - \cos\beta)/k^2, \quad \text{as above.}$$

The equation of motion can now be obtained by differentiating this result, which gives

$$2\dot{\theta}\frac{d\dot{\theta}}{dt} = \frac{2gh}{k^2}(-\sin\theta \times \dot{\theta}),$$

i.e. $$\ddot{\theta} = -(gh/k^2)\sin\theta, \quad \text{as before.}$$

* From D'Alembert's Principle, which is fully explained in works on Dynamics.

Examples:

(i) A circular disc swings through a small angle about a tangent; find the time of oscillation.

In this case, $h = r$, and $k^2 = r^2 + \frac{1}{4}r^2 = \frac{5}{4}r^2$ (Art. 177. II);

$\therefore$ the period $= 2\pi\sqrt{(\frac{5}{4}r^2/gr)} = \pi\sqrt{(\frac{5}{32}r)}$.

(ii) If the disc swings about a line through a point on its edge perpendicular to its plane, $h = r$, and $k^2 = r^2 + \frac{1}{2}r^2 = \frac{3}{2}r^2$.

$\therefore$ in this case, the period $= 2\pi\sqrt{(\frac{3}{2}r^2/gr)} = \frac{1}{4}\pi\sqrt{(3r)}$.

(iii) A cube makes small oscillations about one edge which is fixed horizontally.

If a be the length of an edge, $h = a/\sqrt{2}$, and $k^2 = \frac{2}{3}a^2$ (Art. 177).

$\therefore$ the period $= 2\pi\sqrt{(\frac{2}{3}\sqrt{2}\,a^2/ga)} = \frac{1}{2}\pi\sqrt{(\frac{1}{3}\sqrt{2}\,a)}$.

(iv) An elliptic lamina of eccentricity $\frac{1}{2}$ makes small oscillations about a latus rectum which is fixed horizontally.

If C be the centre and S the focus through which the fixed latus rectum

$h = CS = ae = \frac{1}{2}a$ (p. 19), and the M.I. about the latus rectum

$= \text{M.I. about the minor axis} + M \cdot CS^2$

$= M \cdot \frac{1}{4}a^2 + Ma^2e^2 = \frac{1}{2}Ma^2$.

the period $= 2\pi\sqrt{(\frac{1}{2}a^2/\frac{1}{2}ga)} = 2\pi\sqrt{(a/g)}$.

Examples LXXIX.

1. A heavy particle is attached to a fixed point by a string a yard long; it is held with the string tight and horizontal, and then let go. Find its angular velocity in any subsequent position, and express as a definite integral the time it takes to fall into its lowest position.
2. A particle attached to a fixed point by a string 8 feet long is held with the string at 5° to the vertical and let go. Find the inclination of the string to the vertical (i) after $\frac{1}{3}\pi$ seconds, (ii) after 2 seconds.
3. A bead slides on a smooth wire in the form of an inverted cycloid with its base horizontal; the radius of the generating circle is 2 feet, and the bead starts from rest at the top. Find
   (i) the time of oscillation,
   (ii) the velocity at the lowest point,
   (iii) the velocity when half-way down (measured along the arc),
   (iv) the distance from the vertex after 1 second,
   (v) the time to reach a point distant 2 feet from the vertex,
   (vi) where it is when its velocity is 8 foot-seconds,
   (vii) the velocity after 1 second,
   (viii) when its velocity is first 12 foot-seconds downwards.
4. A rod of mass 2 lb. and length 4 feet swings freely about one end which is fixed; it is held in a horizontal position and let go. Determine its angular velocity in any position, and express as a definite integral the time it takes to reach the vertical position.
5. Answer the same questions in the case of an isosceles triangle of height 2 feet swinging about its base fixed horizontally.
6. Also in the case of the same triangle swinging about a line through the vertex parallel to the base, and starting with its plane horizontal.

7. Also in the case of a semicircular lamina swinging about its bounding diameter which is horizontal.

8. Also in the case of a cube swinging about one edge which is horizontal, and starting with the lower face through that edge vertical.

9. Find the time of a small oscillation of a rectangular lamina about
   (i) a side,
   (ii) an axis in its plane through an angular point,
   (iii) an axis through an angular point perpendicular to its plane,
   (iv) a horizontal line through the middle points of two adjacent sides.

10. A uniform solid sphere of radius 6 inches swings about a point 3 feet above its centre. Find the time of a small oscillation.

11. A circular disc of radius 1 inch swings about a horizontal axis perpendicular to its plane 9 inches from its centre. Find the time of a small oscillation.

12. Retaining the second term in the expansion at the end of Art. 194, find the time of oscillation of a uniform rod 4 feet long swinging about one end in a vertical plane through an angle 10° on either side of the vertical.

13. Using the same approximation, find the time of oscillation of an equilateral triangle swinging through 20° on either side of the vertical about one side which is horizontal.

14. For what value of h will the time of oscillation of a compound pendulum be a minimum?

15. A uniform rod of length 10 feet is bent into the form of the arc of one arch of a cycloid, and oscillates about a horizontal line joining its extremities. Find the length of the simple equivalent pendulum.

16. The motion of a magnetic needle is given by the equation $J\ddot{\phi} = -G \sin\phi$. Find the motion, and the time of oscillation when the magnet makes small oscillations.

THE CATENARY

197. The catenary.

A heavy uniform string or chain hangs in equilibrium in a vertical plane with its ends attached to two fixed points A *and* B; *to find the equation of the curve in which it hangs.*

Let the axis of x be parallel to the tangent at the lowest point C (Fig. 154), and let the vertical through C be the axis of y; let s be the length of the arc measured from C to any point P, and let w be the weight of the string per unit length.

Consider the equilibrium of the portion CP. The forces on it are the tension T at P along the tangent at P, the horizontal tension T_0 at C, and the weight ws. Therefore, resolving horizontally and vertically,

$$T\cos\psi = T_0, \quad \text{and} \quad T\sin\psi = ws,$$

whence, by division, $ws/T_0 = \tan\psi = dy/dx$.

If T_0 be written in the form wa, i.e. if the tension at the lowest point be equal to the weight of a length a of the string, we have

$$dy/dx = s/a.$$

Now $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]} = \sqrt{\left(1+\frac{s^2}{a^2}\right)} = \frac{\sqrt{(a^2+s^2)}}{a};$$

$$\therefore \quad \frac{1}{\sqrt{(a^2+s^2)}}\frac{ds}{dx} = \frac{1}{a}.$$

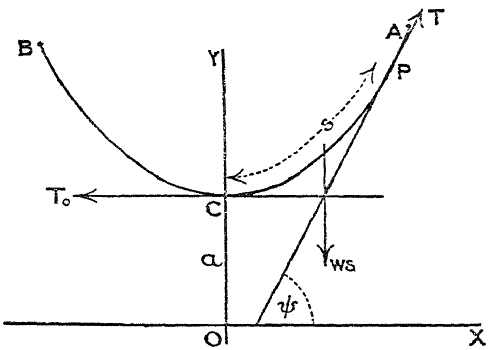


Fig. 154.

Integrating, $\sinh^{-1}(s/a) = x/a + A.$

Since $s = 0$ when $x = 0$, we have $A = 0$;

$$\therefore \quad \sinh^{-1}(s/a) = x/a, \quad \text{i.e. } s/a = \sinh(x/a);$$

$$\therefore \quad dy/dx = s/a = \sinh(x/a).$$

Integrating, $y = a\cosh(x/a) + A.$

The depth of the axis of x below C has not yet been chosen; it is convenient to take it so that A may be zero.

When $x = 0$, $\cosh(x/a) = 1$, and $y = a + A$. Therefore A will be 0 if $y = a$, i.e. if the axis of x be taken at a depth a below C.

The equation of the curve is then $y = a\cosh(x/a)$.

If a string of length $2l$ feet is suspended between 2 points A and B distant $2b$ apart in the same horizontal line, then, putting $x = b$ and $s = l$ in the preceding expressions for y and s, we have, if y_A denote the ordinate of A,

$$y_A = a\cosh(b/a), \text{ and } l = a\sinh(b/a).$$

These are two equations for y_A and a, whose difference is the depth below AB of the middle point of the string; they cannot be solved in finite terms however.

If the string is stretched tightly between A and B, the depth of C below AB is small, so that y_A and a are nearly equal; hence $\cosh(b/a)$ is nearly

equal to 1, and therefore b/a is small. In this case, an approximate solution of the equation $l = a \sinh(b/a)$ may be obtained, either graphically or by the use of Tables. [See also Art. 198.]

Example:

A chain 52 *feet long is suspended between two points* 50 *feet apart; find the depth of its middle point.*

In this case the equation for a is $26 = a \sinh(25/a)$.

Let $25/a = z$, and the equation becomes $\sinh z = 26/a = \frac{26}{25} z = 1{\cdot}04 z$.

The abscissa of the point of intersection of the graphs of $\sinh z$ and $1{\cdot}04z$ can be found by plotting these graphs carefully, and this will give an approximate solution.

If a table of hyperbolic functions be used, it is found, on tabulating values of $1{\cdot}04z$ and $\sinh z$, that when $z = {\cdot}5$, $1{\cdot}04z = {\cdot}520$, and $\sinh z = {\cdot}521$; hence $z = {\cdot}5$ is an approximate solution.

Hence, since $25/a = z = {\cdot}5$, $a = 50$;

$\therefore \quad y_A = a \cosh(b/a) = 50 \cosh \frac{1}{2} = 50 \times 1{\cdot}128 = 56{\cdot}4$.

Hence the depth of the middle point of the chain below $AB = y_A - a = 6{\cdot}4$ feet nearly.

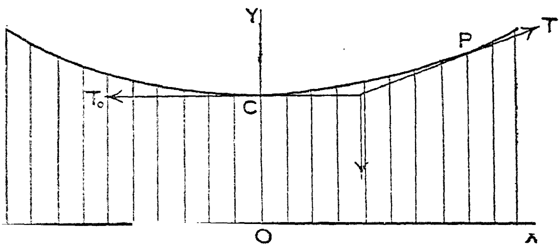


Fig. 155.

198. Suspension bridge.

Suppose that a uniform horizontal load is suspended from a chain by numerous vertical chains or rods, and that the weights of the chains and rods are small compared with the load.

Then, considering the equilibrium of a portion CP of the chain (Fig. 155), the only difference between this case and that of the preceding article is that the weight supported is wx instead of ws, where w is the weight of the horizontal load per unit length.

Hence, in this case, we get $dy/dx = x/a$, where wa is the tension at the lowest point.

$\therefore$ integrating, $\qquad y = \frac{1}{2} x^2/a + A$.

If C be taken as origin, $y = 0$ when $x = 0$;

$$\therefore \quad A = 0, \quad \text{and} \quad y = \tfrac{1}{2} x^2/a.$$

The form of the chain is in this case a parabola.

If a uniform heavy chain is suspended tightly between two fixed points (as in the case of a telegraph wire), then, in the preceding article, s and x are very nearly equal, and the equation $dy/dx = s/a$ there obtained may be replaced by $dy/dx = x/a$, so that in this case the form of the curve will differ but very little from the parabola $y = \frac{1}{2}x^2/a$. In this case the dip of the chain at any point is easily found, and thence the tension at the lowest or any other point.

The same result may be deduced from the equation of the catenary; for, using the expansion of Art. 92, we have

$$y = a\cosh\frac{x}{a} = a\left[1 + \tfrac{1}{2}\frac{x^2}{a^2} + \frac{x^4}{a^4 4!} \cdot\right.$$

Therefore, neglecting 4th and higher powers of x/a, $y = a + \frac{1}{2}x^2/a$, which, when the origin is moved to the point C (0, a), becomes $y = \frac{1}{2}x^2/a$.

Example:

If 200 *feet* 6 *inches of wire are stretched between two points* 200 *feet apart, find the maximum dip and the tension at any point.*

The equation of the curve assumed by the wire may be taken as $y = \frac{1}{2}x^2/a$.

Now $$\frac{dy}{dx} = \frac{x}{a}, \text{ and } \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = \sqrt{\left[1 + \frac{x^2}{a^2}\right]}.$$

Since the wire is nearly horizontal, dy/dx is very small; $\therefore$ x/a is small, and hence we may expand $\sqrt{(1+x^2/a^2)}$ by the Binomial Theorem, and neglect all terms after the first two (i. e. neglect the 4th and higher powers of x/a).

This gives $ds/dx = 1 + \frac{1}{2}x^2/a^2$, whence $s = x + \frac{1}{6}x^3/a^2 + A$.

Measuring from the vertex C, $s = 0$, when $x = 0$; $\therefore$ $A = 0$, and $s = x + \frac{1}{6}x^3/a^2$.

At an end of the wire, $s = 100\frac{1}{4}$, $x = 100$; $\therefore$ $100\frac{1}{4} = 100 + \frac{1}{6}\,.\,100^3/a^2$, which gives $a^2 = \frac{2}{3} \times 100^3$ and therefore $a = 816{\cdot}3$.

The maximum dip of the wire is evidently the value of y at one end, i. e. when $x = 100$, and is therefore equal to

$$100^2/2a = 100^2 \div 1632{\cdot}6 = 6{\cdot}1 \text{ feet}.$$

The tension at the lowest point $= wa = 816{\cdot}3\,w =$ the weight of $816{\cdot}3$ feet of the wire.

The tension at any other point, say at P, 40 feet from one of the posts [$\therefore$ $x = 60$], is found from the equation $T\cos\theta = T_0$.

$$\therefore\ T = T_0\sec\theta = T_0\,.\,\frac{ds}{dx} = T_0\left(1 + \tfrac{1}{2}\frac{x^2}{a^2}\right) = 816{\cdot}3\,w\left[1 + \frac{60^2}{\frac{4}{3} \times 100^3}\right] = 818\tfrac{1}{2}w.$$

Examples LXXX.

1. A chain 102 feet long is suspended from two points A and B, 100 feet apart; find the depth of the middle point of the chain below AB.
2. Three hundred and one feet of wire, weighing 1 lb. per yard, are suspended between two posts 300 feet apart. Find the tension (i) at the middle point, (ii) at one end, (iii) at a point 50 feet from a post.

3. Prove that the resultant tension at any point of a chain is equal to wy.

4. Prove that the C.G. of an arc of a catenary is vertically above the point of intersection of the tangents at the extremities of the arc.

5. Find the distance between the points where the ordinate $x = 4$ cuts the catenary $y = 8 \cosh \frac{1}{8} x$ and the parabola $y = 8 + \frac{1}{16} x^2$.

6. A wire hangs in the catenary $y = 200 \cosh \cdot 005\, x$ (x and y being measured in feet); find the length of the wire and the sag at the middle point, if the points of suspension be 100 feet apart.

7. Calculate the length and the sag if the form of the wire in the preceding question be taken as the parabola $y = 200 + \frac{1}{400} x^2$.

8. If l be the length of an arc of a catenary, show that the difference of the slopes at the extremities of the arc is equal to l/a, a being the parameter of the catenary.

9. A uniform string of length l is suspended from two points A and B in the same horizontal line at distance h apart; if h and l be nearly equal, prove that $a^2 = \frac{1}{24} h^3/(l-h)$.

10. A uniform chain of length 100 yards is stretched across a river so that the middle point just touches the surface of the water, and each end is 2 feet above the edge of the water. Find the difference between the length of the chain and the width of the river.

CHAPTER XX

CURVATURE

199. Radius and circle of curvature.

Let PT, QT be the tangents at two points P and Q on a continuous curve, and let them make angles ψ and $\psi+\delta\psi$ respectively with a given line, so that $\delta\psi$ is the angle between the tangents (Fig. 156).

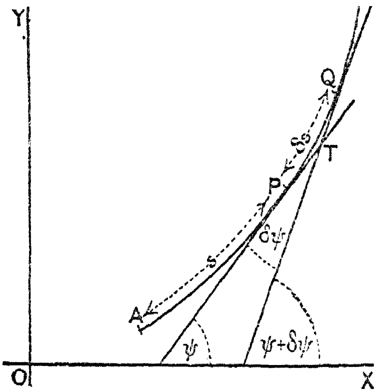


Fig. 156.

If δs be the length of the arc PQ, then $\delta\psi/\delta s$ is called the 'average curvature' of the arc PQ. The *curvature* at P is defined as the limit to which this quantity tends when δs is indefinitely diminished, i.e. the curvature at P is equal to $d\psi/ds$.

If PQ be an arc of a circle of radius r, the angle $\delta\psi$ between the tangents at P and Q is equal to the angle subtended at the centre of the circle by the arc PQ, and therefore $\delta s = r\delta\psi$; hence $\delta\psi/\delta s$, and ultimately $d\psi/ds$, $=1/r$. The curvature is constant at all points of a circle, and the radius $r = ds/d\psi$, the reciprocal of the curvature. In any curve, the value of $ds/d\psi$ at any point is called the length of the radius of curvature at that point; it is the reciprocal of the curvature, and is usually denoted by the letter ρ. It follows from the result immediately preceding that it is the radius of the circle which has the same curvature as the given curve at the point.

The circle with this radius, which has the same tangent at P and lies on the same side of that tangent as the given curve, is called the *circle of curvature* at P, its centre is called the *centre of curvature*, and its radius the *radius of curvature.*

The length of the radius of curvature at any point in terms of the rectangular coordinates of the point is obtained as follows:

If the angle ψ be measured from the axis of x, we have

$$\rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi} = \sec\psi\,\frac{dx}{d\psi}.$$

Also $\tan\psi = dy/dx$;

$\therefore$ differentiating with respect to x, $\sec^2\psi\,\dfrac{d\psi}{dx} = \dfrac{d^2y}{dx^2}.$

$$\rho = \sec\psi\,\frac{dx}{d\psi} = \sec\psi\,\frac{\sec^2\psi}{\dfrac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

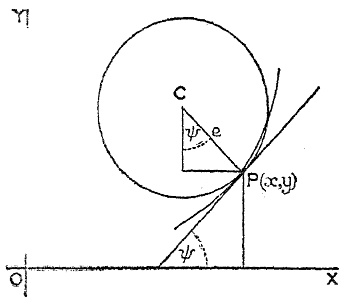


Fig. 157.

If the positive value of the root in the numerator be taken, the sign of ρ will be the same as the sign of d^2y/dx^2, i. e. positive if the curve is above the tangent and negative if below it (Art. 59). At a point of inflexion, d^2y/dx^2 is zero, and therefore ρ becomes infinite; i. e. the curvature at a point of inflexion is zero.

The coordinates of the centre of curvature can be obtained at once by drawing a figure. In Fig. 157, dy/dx and ρ are both positive.

Let ψ be measured from the axis of x, and let $(\xi,\ \eta)$ be the coordinates of C, the centre of curvature, then

$$\xi = x - \rho\sin\psi = x - \frac{ds}{d\psi}\,\frac{dy}{ds} = x - \frac{dy}{d\psi},$$

$$\eta = y + \rho\cos\psi = y + \frac{ds}{d\psi}\,\frac{dx}{ds} = y + \frac{dx}{d\psi}.$$

In terms of x, y, and dy/dx, we have,

$$\text{since}\quad \sin\psi = \frac{\dfrac{dy}{dx}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{1/2}}\quad\text{and}\quad \cos\psi = \frac{1}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{1/2}}$$

$$\xi = x - \frac{\dfrac{dy}{dx}\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}};\qquad \eta = y + \frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}$$

Examples:

(i) *Find the radius of curvature and the coordinates of the centre of curvature at the point* (3, 4) *of the rectangular hyperbola* $xy = 12$.

Here $\quad y = \dfrac{12}{x},\quad \dfrac{dy}{dx} = -\dfrac{12}{x^2},\quad \dfrac{d^2y}{dx^2} = \dfrac{24}{x^3};$

$\therefore$ at the given point (3, 5), $\quad \dfrac{dy}{dx} = -\dfrac{4}{3},\quad \dfrac{d^2y}{dx^2} = \dfrac{8}{9}.$

$$\therefore\quad \rho = (1+\tfrac{16}{9})^{3/2} \div \tfrac{8}{9} = \tfrac{125}{24}.$$

$$\xi = 3+\tfrac{4}{3}(1+\tfrac{16}{9})/\tfrac{8}{9} = \tfrac{43}{6};\quad \eta = 4+(1+\tfrac{16}{9})/\tfrac{8}{9} = \tfrac{57}{8}.$$

The centre of curvature is the point $(\frac{43}{6}, \frac{57}{8})$ and the radius of curvature is $\frac{125}{24}$. Hence the equation of the circle of curvature is

$$(x-\tfrac{43}{6})^2+(y-\tfrac{57}{8})^2 = (\tfrac{125}{24})^2.$$

(ii) *Find the radius of curvature at any point of the ellipse* $x^2/a^2+y^2/b^2=1$.

Here $\quad y = \dfrac{b}{a}\sqrt{(a^2-x^2)},\qquad \dfrac{dy}{dx} = -\dfrac{b}{a}\cdot\dfrac{x}{\sqrt{(a^2-x^2)}},$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\cdot\frac{\sqrt{(a^2-x^2)}+x\times x/\sqrt{(a^2-x^2)}}{a^2-x^2} = -\frac{b}{a}\cdot\frac{a^2}{(a^2-x^2)^{3/2}};$$

$$\therefore\ \rho = \left(1+\frac{b^2}{a^2}\cdot\frac{x^2}{a^2-x^2}\right)^{3/2} \Big/ \frac{-ab}{(a^2-x^2)^{3/2}} = -\frac{1}{a^4b}[a^4-(a^2-b^2)x^2]^{3/2}$$

$$= -\frac{1}{a^4b}\cdot a^3(a^2-e^2x^2)^{3/2}\ [\text{since } a^2-b^2 = a^2e^2\ (\text{p. } 19)] = -\frac{1}{ab}(a^2-e^2x^2)^{3/2}.$$

The result is negative, since we have taken the positive value of y, and for such values the curve at any point is below the tangent at the point.

(iii) *Find the radius of curvature at the point* (2, 1) *of the curve*

$$x(x+y) = x^3-2y^3$$

The evaluation of dy/dx and d^2y/dx^2 should be specially noticed in this example.

The given equation is $\quad x^2+xy = x^3-2y^3.$

Differentiating with respect to x, $\quad 2x+x\dfrac{dy}{dx}+y = 3x^2-6y^2\dfrac{dy}{dx};\qquad$ (i)

$\therefore$ at the point (2, 1), $\quad 4+2\dfrac{dy}{dx}+1 = 12-6\dfrac{dy}{dx}$, whence $\dfrac{dy}{dx} =$

Differentiating equation (i) again with respect to x,

$$2+\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)+\frac{dy}{dx}=6x-6\left(y^2\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot 2y\frac{dy}{dx}\right).$$

Substituting the values of x, y, and dy/dx, we have

$$2+2\frac{d^2y}{dx^2}+\tfrac{7}{8}+\tfrac{7}{8}=12-6\frac{d^2y}{dx^2}-12\cdot\tfrac{49}{64},\ \text{whence}\ \frac{d^2y}{dx^2}=-\tfrac{15}{128}$$

$$\therefore\quad \rho=(1+\tfrac{49}{64})^{3/2}/(-\tfrac{15}{128})=-20\ \text{nearly}.$$

(iv) *Find the radius of curvature at any point of a cycloid.*

It has been shown (Art. 50) that, in the cycloid, $s=4a\cos\frac{1}{2}\theta$, and $\frac{1}{2}\theta=\angle PTG=90-\psi$, if ψ be the inclination of the tangent to ON; therefore * $s=4a\sin\psi$.

From this equation the radius of curvature is obtained at once, for

$$\rho=ds/d\psi=4a\cos\psi=4a\sin PTG=2\,PG\ \text{(see Fig. 152)},$$

i. e. the radius of curvature at any point of a cycloid is double the length of the normal at that point.

If the equation of a curve is given by expressing x and y in terms of a third variable θ, we may proceed as in the following example:

(v) *The equation of an ellipse is given in the form* $x=a\cos\theta$, $y=b\sin\theta$; *find the radius of curvature at any point in terms of* θ.

We have $$\rho=\frac{ds}{d\psi}=\frac{ds}{d\theta}\frac{d\theta}{d\psi}.$$

$$\left(\frac{ds}{d\theta}\right)^2=\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\ \text{[Art. 82]}=a^2\sin^2\theta+b^2\cos^2\theta=a^2-(a^2-b^2)\cos^2\theta$$
$$=a^2(1-e^2\cos^2\theta).$$

Also $$\tan\psi=\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{b\cos\theta}{-a\sin\theta}=-\frac{b}{a}\cot\theta;$$

$\therefore$ differentiating with respect to ψ, $\sec^2\psi=\dfrac{b}{a}\operatorname{cosec}^2\theta\dfrac{d\theta}{d\psi}$,

whence $$\frac{d\theta}{d\psi}=\frac{a}{b}\sin^2\theta\sec^2\psi=\frac{a}{b}\sin^2\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)$$
$$=(a^2\sin^2\theta+b^2\cos^2\theta)/ab$$
$$=(a/b)(1-e^2\cos^2\theta),\ \text{as before};$$

$$\therefore\quad \rho=\pm a\sqrt{(1-e^2\cos^2\theta)}\times(a/b)(1-e^2\cos^2\theta)=\pm(a^2/b)(1-e^2\cos^2\theta)^{3/2},$$

which agrees with the result of Example (ii).

The sign depends upon the sign of $ds/d\theta$, i.e. upon the direction in which s is measured.

* The equation which connects s and ψ in any curve is called the *intrinsic equation* of the curve.

Examples LXXXI.

Find the radius of curvature in the following cases, 1–21:

1. At (1, 1) on the curve $y = x^2$. Find also the equation of the circle of curvature.
2. At (2, 4) on the curve $y^2 = 2x^3$. Find also the equation of the circle of curvature.
3. At $(\frac{1}{6}\pi, \frac{1}{2})$ on the curve $y = \sin x$.
4. At (3, 4) on the curve $x^2 + y^2 = 25$.
5. At any point (x, y) on the rectangular hyperbola $xy = c^2$.
6. At an end of a latus rectum of the ellipse $x^2 + 4y^2 = 4a^2$.
7. At an end of the latus rectum of the parabola $y^2 = 4ax$. Find also the equation of the circle of curvature. Find where this circle cuts the curve again.
8. At the vertex of the catenary $y = c\cosh(x/c)$.
9. At any point (x, y) on the rectangular hyperbola $x^2 - y^2 = a^2$.
10. At the point on the curve $a^2 y = x^3$ whose abscissa is $\frac{1}{2}a$.
11. At any point (x, y) of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.
12. At the point $(-4, 0)$ on the curve $xy^2 = 16(x+4)$.
13. At the origin on the curve $y^2 = x(x-3)^2$. Find also the equation of the circle of curvature.
14. At the point (2, 2) on the curve $x^3 + y^3 = 4xy$.
15. At any point of the cycloid, in terms of θ.
16. At the point $(0, a)$ on the curve $y(x^2+y^2) = a(y^2 - x^2)$.
17. At the origin on the curve $x^3 + y^3 + 2x^2 - 4y + 3x = 0$.
18. At any point of the curve $y = a\log\sin(x/a)$.
19. At any point of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$.
20. At any point of the catenary $s = c\tan\psi$, in terms of ψ.
    Prove that the radius of curvature is equal to the length of the normal between the curve and the axis of x.
21. At any point of the catenary $y = a\cosh(x/a)$. Where is it a minimum?
22. Show that in the curve in which $s = a\log\sin\psi$ (this curve is called the tractrix) the radius of curvature varies inversely as the normal.
23. If x and y are given as functions of a variable t, prove that
    $$\rho = (x'^2 + y'^2)^{3/2}/(x'y'' - x''y'),$$
    where the accents denote differential coefficients with respect to t.
24. Prove that the radius of curvature at an end of the major axis of an ellipse is equal to the semi-latus rectum.
25. Find the condition that the centre of curvature at one end of the minor axis of an ellipse may coincide with the other end.
26. Prove that the radius of curvature of a conic varies as the cube of the normal.
27. Find the radius of curvature of the curve given by the equations
    $$x = a\sin 2\theta(1+\cos 2\theta), \qquad y = a\cos 2\theta(1-\cos 2\theta).$$
28. Prove that the curvature
    $$= \frac{d}{dx}\left(\frac{dy}{ds}\right) = -\frac{d}{dy}\left(\frac{dx}{ds}\right) = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}^{\frac{1}{2}}.$$

29. Where is the curvature a maximum or minimum in the following curves? (i) $y = x^2$, (ii) $y = x^3$, (iii) $y^2 = x^3$.

30. Prove that the radius of curvature at any point $(a\cos\theta, b\sin\theta)$ of an ellipse is equal to CD^3/ab, where CD is the semi-diameter conjugate to CP.
[N.B. D is the point $(a\sin\theta, \ -b\cos\theta)$.]

31. Prove that in the equiangular spiral $r = ae^{\theta\cot\alpha}$ (Art. 163), the radius of curvature is equal to $r\operatorname{cosec}\alpha$, and hence show that it subtends a right angle at the origin.

32. Find the radius of curvature at any point of the curve

$$x = a(\log\cot\tfrac{1}{2}\theta - \cos\theta), \quad y = a\sin\theta.$$

BENDING OF BEAMS

200. Approximate value for the radius of curvature. Application to beams.

If at a point P on a curve the tangent is nearly parallel to the axis of x, dy/dx is small, and if dy/dx be regarded as a small quantity of the first order, $(dy/dx)^2$ will be of the second order (Art. 24); hence, neglecting it in the expression for ρ, we have approximately

$$\rho = 1 \Big/ \frac{d^2y}{dx^2}.$$

The same result may also be obtained directly from the definition of the radius of curvature as follows:

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\psi}{dx}\cdot\frac{dx}{ds} = \frac{d\psi}{dx}\cos\psi.$$

When ψ is very small, $\tan\psi$ is approximately equal to ψ, and $\cos\psi$ to unity.

$\therefore$ approximately, $\dfrac{1}{\rho} = \dfrac{d\psi}{dx} = \dfrac{d}{dx}(\tan\psi) = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$, as before.

This approximation is important in questions dealing with the deflection of beams. It is shown in the theory of bending of beams that, if ρ be the radius of curvature at any point of a deflected beam, the bending moment at that point is equal to EI/ρ, where E is Young's modulus, and I the moment of inertia of the section through the point about a line through its C. G. perpendicular to the plane of bending. Generally the deflection is so small that the approximation just mentioned for ρ is sufficient, and in this case we have

$$EI\, d^2y/dx^2 = \text{the bending moment at the point } (x, y),$$

where y is the vertical deflection of the point, and the axis of x is the horizontal straight line through a fixed point of the beam.

Examples:

(i) *A uniform beam of length* l *rests with its ends on two supports in the same horizontal line and has a weight* W *suspended from its middle point. Find the maximum deflection.*

Suppose the weight of the beam negligible compared with W. Then the upward pressure at each end will be $\frac{1}{2}W$ (Fig. 158), and therefore the bending moment at a point distant x ($<\frac{1}{2}l$) from one end is $\frac{1}{2}Wx$.

$$\therefore \quad EI\, d^2y/dx^2 = -\tfrac{1}{2}Wx.$$

The negative sign is taken, since at the point P the curve is above the tangent, and the positive direction of y is downwards; therefore d^2y/dx^2 is $-$ [cf. Art. 59].

Integrating, $$EI\, dy/dx = -\tfrac{1}{4}Wx^2 + C.$$

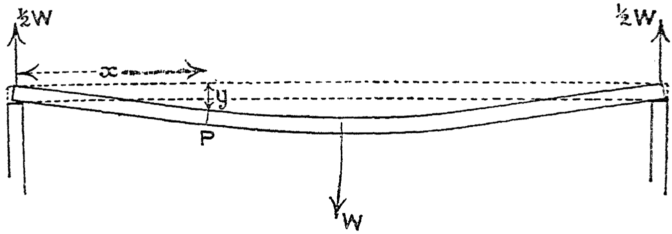


Fig. 158.

The tangent to the beam is, from symmetry, horizontal at the middle point, i.e. $dy/dx = 0$ when $x = \frac{1}{2}l$.

$$\therefore \quad 0 = -\tfrac{1}{4}W \cdot \tfrac{1}{4}l^2 + C, \text{ and } C = \tfrac{1}{16}Wl^2;$$

i.e. $$EI\, dy/dx = -\tfrac{1}{4}Wx^2 + \tfrac{1}{16}Wl^2.$$

Integrating again, $$EI\, y = -\tfrac{1}{12}Wx^3 + \tfrac{1}{16}Wl^2x + D.$$

Since $y = 0$ when $x = 0$, it follows that $D = 0$, and

$$y = \frac{W}{4EI}\left(\tfrac{1}{4}l^2x - \tfrac{1}{3}x^3\right) = \frac{W}{48EI}x(3l^2 - 4x^2).$$

The maximum deflection is at the centre where $x = \frac{1}{2}l$, and is therefore equal to $\frac{1}{48}Wl^3/EI$.

(ii) *Let the beam be fixed at one end and uniformly loaded.*

Let the load be w per unit length. (This includes the case of a heavy beam bending under its own weight.)

If P (Fig. 159) be a point distant x from the fixed end, the weight of the portion between P and the free end is $w(l-x)$, and therefore the bending moment is $w(l-x) \times \frac{1}{2}(l-x)$.

Hence in this case, $$EI\, d^2y/dx^2 = +\tfrac{1}{2}w(l-x)^2 = \tfrac{1}{2}w(l^2 - 2lx + x^2).$$

Integrating, $$EI\, dy/dx = \tfrac{1}{2}w(l^2x - lx^2 + \tfrac{1}{3}x^3) + C.$$

At the fixed end, where $x = 0$, the beam has no slope, i.e. $dy/dx = 0$;

$$\therefore \quad C = 0, \text{ and } EI\, dy/dx = \tfrac{1}{2}w(l^2x - lx^2 + \tfrac{1}{3}x^3).$$

Integrating again, $EI\,y = \frac{1}{2}w\left(\frac{1}{2}l^2x^2 - \frac{1}{3}lx^3 + \frac{1}{12}x^4\right) + D,$
and $y = 0$ when $x = 0$; $\therefore$ $D = 0$, and

$$y = \frac{w}{24EI}x^2(6l^2 - 4lx + x^2).$$

The greatest deflection is at the free end where $x = l$, and is equal to

$$\frac{}{24EI}l^2(3l^2) = \frac{Wl^3}{8EI},$$ if W be the total weight wl.

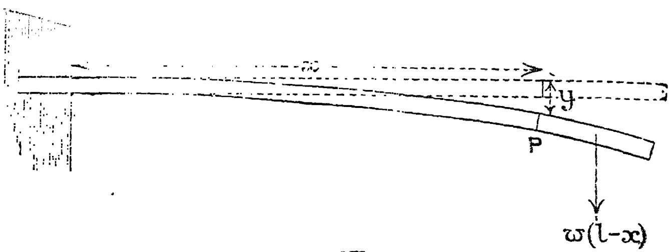


Fig. 159.

If a beam be either supported at both ends or clamped at one end or both ends, and subject only to a load and the reactions at the ends, the result of differentiating the fundamental equation ($EI\,d^2y/dx^2$ = the bending moment) twice with respect to x is always

$$EI\,d^4y/dx^4 = w,$$

where w is the load per unit length.

For the only term in the bending moment which contains x^2 is (as in the preceding example) $\frac{1}{2}wx^2$; this, when differentiated twice, gives w, and the other terms of the bending moment disappear after two differentiations.

From this equation, the form assumed by the beam and the deflection at any point of the beam under given conditions can be found. This is a very good illustration of the part played by the constants of integration. In all the various cases the equation we start with is the same, but the different initial conditions in the several cases give us different constants and, of course, quite different final results.

We here work out two cases.

(iii) *A uniformly loaded beam rests upon supports at its extremities; to find the equation of the curve assumed by the beam and the maximum deflection.*

Let the line joining the ends of the beam and its perpendicular bisector be taken as axes of x and y respectively, and let l be the length of the beam.

The initial conditions are

(i) $y = 0$ at each end, i.e. when $x = \pm\frac{1}{2}l$;

(ii) since the ends are free, there is no curvature there, i.e. $d^2y/dx^2 = 0$ when $x = \pm\frac{1}{2}l$.

Integrating the equation $EI\, d^4y/dx^4 = w$,

we have $EI\, d^3y/dx^3 = wx + A$;

and integrating again, $EI\, d^2y/dx^2 = \frac{1}{2}wx^2 + Ax + B$.

Substituting initial values (ii), $0 = \frac{1}{2}w \cdot \frac{1}{4}l^2 + \frac{1}{2}lA + B$,

$$0 = \frac{1}{2}w \cdot \frac{1}{4}l^2 - \frac{1}{2}lA + B;$$

whence, subtracting, $A = 0$, and adding, $B = -\frac{1}{8}wl^2$;

$$\therefore\quad EI\, d^2y/dx^2 = \frac{1}{2}wx^2 - \frac{1}{8}wl^2 = \frac{1}{2}w(x^2 - \frac{1}{4}l^2).$$

Integrating twice again, $EI\, dy/dx = \frac{1}{2}w(\frac{1}{3}x^3 - \frac{1}{4}l^2x) + C$,

$$EI\, y = \frac{1}{2}w(\frac{1}{12}x^4 - \frac{1}{8}l^2x^2) + Cx + D.$$

Substituting initial values (i), $0 = \frac{1}{2}w(\frac{1}{12} \cdot \frac{1}{16}l^4 - \frac{1}{32}l^4) + C \cdot \frac{1}{2}l + D$,

$$0 = \frac{1}{2}w(\frac{1}{12} \cdot \frac{1}{16}l^4 - \frac{1}{32}l^4) - C \cdot \frac{1}{2}l + D;$$

whence, subtracting, $C = 0$, and adding, $D = -\frac{1}{2}w(\frac{1}{192}l^4 - \frac{1}{32}l^4) = \frac{5}{384}wl^4$;

$$\therefore\quad EI\, y = \frac{1}{2}w(\frac{1}{12}x^4 - \frac{1}{8}l^2x^2) + \frac{5}{384}wl^4 = \frac{1}{384}w(16x^4 - 24x^2l^2 + 5l^4).$$

This gives the deflection at any point, and is the equation of the curve taken by the beam. The maximum deflection is at the centre where $x = 0$; therefore $EI\, y = \frac{5}{384}wl^4$, i.e. the maximum deflection is $\frac{5}{384}wl^4/EI$.

(iv) *Let the beam be clamped at both ends; to find the form it takes and the maximum deflection.*

The initial conditions are in this case

(i) $y = 0$ at both ends, i.e. when $x = \pm\frac{1}{2}l$;

(ii) since the beam is now horizontal at both ends, $dy/dx = 0$ when $x = \pm\frac{1}{2}l$.

Integrating the general equation three times, we have

$$EI\, d^3y/dx^3 = wx + A,$$
$$EI\, d^2y/dx^2 = \frac{1}{2}wx^2 + Ax + B,$$
$$EI\, dy/dx = \frac{1}{6}wx^3 + \frac{1}{2}Ax^2 + Bx + C.$$

Substituting $dy/dx = 0$ when $x = \pm\frac{1}{2}l$, we have

$$0 = \frac{1}{48}wl^3 + \frac{1}{8}Al^2 + \frac{1}{2}Bl + C,$$
$$0 = -\frac{1}{48}wl^3 + \frac{1}{8}Al^2 - \frac{1}{2}Bl + C;$$

whence, on subtracting, $0 = \frac{1}{24}wl^3 + Bl$, and $B = -\frac{1}{24}wl^2$,

and adding, $0 = \frac{1}{4}Al^2 + 2C$; $\quad\therefore\quad C = -\frac{1}{8}Al^2$.

Integrating again (after substituting the values of B and C),

$$EI\, y = \frac{1}{24}wx^4 + \frac{1}{6}Ax^3 - \frac{1}{24}wl^2 \cdot \frac{1}{2}x^2 - \frac{1}{8}Al^2x + D.$$

Substituting $y = 0$ when $x = \pm\frac{1}{2}l$,

$$0 = \frac{1}{24}w \cdot \frac{1}{16}l^4 + \frac{1}{6}A \cdot \frac{1}{8}l^3 - \frac{1}{48}wl^2 \cdot \frac{1}{4}l^2 - \frac{1}{16}l^3A + D,$$
$$0 = \frac{1}{24}w \cdot \frac{1}{16}l^4 - \frac{1}{6}A \cdot \frac{1}{8}l^3 - \frac{1}{48}wl^2 \cdot \frac{1}{4}l^2 + \frac{1}{16}l^3A + D.$$

Subtracting, $0 = \frac{1}{24}Al^3 - \frac{1}{8}Al^3$, whence $A = 0$;

and therefore $0 = \frac{1}{384}wl^4 - \frac{1}{192}wl^4 + D$; i.e. $D = \frac{1}{384}wl^4$.

Substituting in the integrated equation, we have

$$EIy = \frac{1}{24}wx^4 - \frac{1}{48}wl^2x^2 + \frac{1}{384}wl^4 = \frac{1}{384}w(16x^4 - 8l^2x^2 + l^4) = \frac{1}{384}w(4x^2 - l^2)^2.$$

This gives the deflection at any point, and is the equation of the curve assumed by the beam. The maximum deflection is at the centre, where $x = 0$; therefore $EIy = \frac{1}{384}wl^4$, i.e. the maximum deflection $= \frac{1}{384}wl^4/EI$.

Comparing this result with that of the preceding example, it follows that the maximum deflection when the beam is free at the ends is five times as great as when it is fixed at the ends.

Examples LXXXII.

1. A uniform beam of length l and negligible weight is fixed at one end and has a weight W suspended from the other end. Find the equation of the curve assumed by the beam and its maximum deflection.

2. Obtain the result for a beam uniformly loaded and supported at both ends from the equation $EI\,d^2y/dx^2$ = the bending moment.

3. Obtain the result for a beam uniformly loaded, fixed at one end and free at the other, from the equation $EI\,d^4y/dx^4 = w$.

4. A uniform heavy beam is fixed at one end and free at the other; a weight equal to the total weight of the beam is suspended from the free end. Find the deflection at any point, and the maximum deflection.

5. Compare the deflections of two beams of the same material and length and similarly loaded, one with a square section of side a, the other with a circular section of diameter a.

6. In the case of a light beam with a weight at the end or at the middle point, prove that if the length of the beam be doubled, the maximum deflection is increased eightfold.

7. In the case of a heavy beam under the action of its own weight, prove that if the length be doubled, the maximum deflection is increased sixteenfold.

8. Find the deflection at a point distant one quarter of the length from one end in the case of a heavy beam supported at the ends.

9. Find the deflection at the same point if the beam is clamped horizontally at the ends.

10. Find the deflection at any point in the case of a heavy beam clamped horizontally at one end and supported at the other. Where is the deflection greatest?

11. Find the deflection at the centre of a light beam with a weight W suspended at the middle point, the beam being supported at one end and at a point distant one quarter of the length from the other end.

    [Find expressions for y on both sides of the centre, and notice that both must give the same values of y and dy/dx at the centre.]

12. A bar 1 yard long and cross-section 1 inch square is fixed at one end and loaded at the other with 2 cwt.; find the deflection of the free end, neglecting the weight of the bar, and taking Young's modulus as 3×10^7 lb. weight per square inch.

201. Intersection of consecutive normals.

The centre of curvature is the limiting position of the point of intersection of the normals at two points when one point approaches indefinitely near to the other.

Let the normals at P and Q meet at C. The angle PCQ is equal to the angle $\delta\psi$ between the tangents at P and Q (Fig. 160). Join PQ.

Then $$\frac{CP}{PQ} = \frac{\sin CQP}{\sin PCQ};$$

$$CP = PQ \times \frac{\sin CQP}{\sin \delta\psi} = \frac{PQ}{\delta s} \times \frac{\delta s}{\delta\psi} \times \frac{\delta\psi}{\sin \delta\psi} \times \sin CQP$$

$$= 1 \times ds/d\psi \times 1 \times 1, \text{ when } \delta s \text{ and } \delta\psi \to 0 \; [\text{since } CQP \to \tfrac{1}{2}\pi],$$

$$= \rho, \text{ the radius of curvature.}$$

C is the centre of curvature at P.

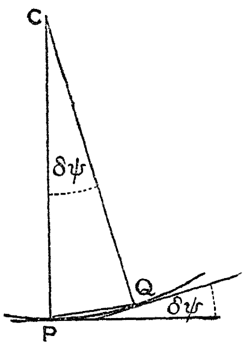


Fig. 160.

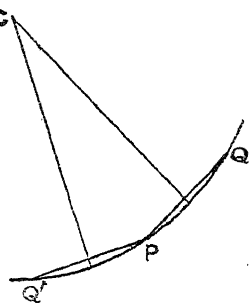


Fig. 161.

Also the circle of curvature at P is the limiting position of the circle which passes through three points Q, P, Q' on the curve when Q and Q' approach indefinitely near to P, or, as it is often expressed, the circle of curvature is the circle which passes through three consecutive points on the curve.

Let Q and Q' be two points on the curve near P (Fig. 161), one on either side of it, and let the perpendicular bisectors of PQ, PQ' meet in C, so that C is the centre of the circle through QPQ'. Let Q and Q' move indefinitely near to P; then PQ and PQ' become ultimately two consecutive tangents to the curve, and their perpendicular bisectors become two consecutive normals; hence their point of intersection C is ultimately the centre of curvature, and the circle becomes the circle of curvature.

The circle which passes through three consecutive points on a curve is called the *osculating circle*; since it cuts the curve in *three* points, it follows that in general it will cross the curve at the point of contact.

202. Radius of curvature in tangential-polar coordinates.

If the equation of a curve be given in tangential-polar coordinates (Art. 165), a very simple expression can be found for the radius of curvature, viz.:

$$\rho = r\frac{dr}{dp}.$$

For, let two consecutive normals PC, $P'C$ meet in C (Fig. 162).

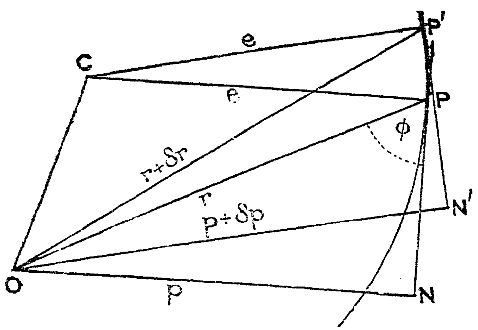


Fig. 162.

Ultimately, as P' approaches P, C is the centre of curvature at P, and PC, $P'C$ are each of length ρ.

From the triangle OCP we have

$$\begin{aligned} OC^2 &= \rho^2 + r^2 - 2\rho r \cos OPC \\ &= \rho^2 + r^2 - 2\rho r \sin\phi \\ &= \rho^2 + r^2 - 2\rho p. \end{aligned}$$

Similarly, if $r+\delta r$ be the radius vector of P', and $p+\delta p$ the perpendicular from O to the tangent at P', we have

$$OC^2 = \rho^2 + (r+\delta r)^2 - 2\rho(p+\delta p).$$

Subtracting, we get $\quad 0 = 2r\delta r + (\delta r)^2 - 2\rho\delta p$;

$$\therefore \quad \rho = \frac{\delta r}{\delta p}(r + \tfrac{1}{2}\delta r).$$

Hence, in the limit when P' is indefinitely near P,

$$\rho = r\,dr/dp.$$

Examples:

In the cardioid, $r^3 = 2ap^2$ (Art. 165);

$\therefore$ differentiating with respect to p, $3r^2\, dr/dp = 4ap$.

$$\rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4a}{3r}\cdot\sqrt{\frac{r^3}{2a}} = \frac{2}{3}\sqrt{(2ar)}.$$

In the lemniscate, $r^3 = a^2 p$;

$\therefore$ differentiating, $3r^2\, dr/dp = a^2$.

$$\therefore \quad \rho = r\, dr/dp = a^2/3r.$$

A formula can be found also for the radius of curvature in polar coordinates (see Ex. LXXXIII. 10), but it is not very often used. If the equation of a curve is given in polar coordinates, it is often advisable to obtain the tangential-polar equation as in Art. 165, and then use the simple expression obtained above.

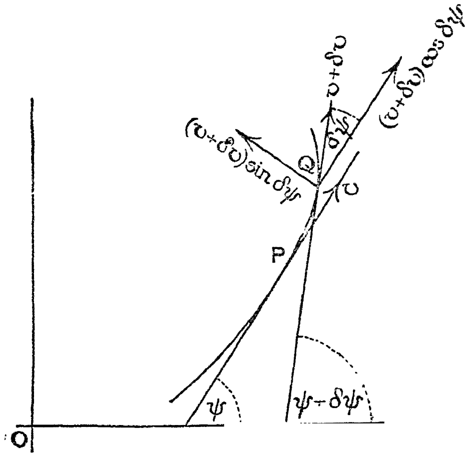


Fig. 163.

203. Application to mechanics.

If a point is moving in a plane curve, it is often convenient to resolve its velocity and acceleration along the tangent and normal to the curve.

Let v be the velocity when the moving point is at P, where the tangent makes an angle ψ with a given line, and let s be the length of the arc measured from a fixed point of the curve to P: let $v+\delta v$ be the velocity at Q, where the inclination of the tangent and the length of the arc are $\psi+\delta\psi$ and $s+\delta s$ (Fig. 163).

The velocity v is the rate at which the point is describing the arc s, and therefore is equal to ds/dt or $\dot{s}$. The components of the velocity at Q in the direction of the tangent and normal at P are $(v+\delta v)\cos\delta\psi$ and $(v+\delta v)\sin\delta\psi$.

Hence the acceleration in the direction of the tangent at P

$= $ rate of change of velocity along the tangent at P

$$= \underset{t}{\mathrm{L}} \frac{(v+\delta v)\cos\delta\psi - v}{\delta t} \text{ when } \delta t \to 0$$

$$= \underset{t}{\mathrm{L}} \frac{dv}{dt}, \text{ since } \cos\delta\psi \text{ differs from 1 by a small quantity of the second order, as } \delta\psi \to 0$$

$= dv/dt$, i.e. $\dot{v}$ or $\ddot{s}$ or $v\,dv/ds$.

The acceleration in the direction of the normal at P

$$= \underset{t}{\mathrm{L}} \frac{(v+\delta v)\sin\delta\psi}{\delta t} \text{ as } \delta t \to 0$$

$$= \underset{t}{\mathrm{L}} (v+\delta v) \times \frac{\sin\delta\psi}{\delta\psi} \times \frac{\delta\psi}{\delta s} \times \frac{\delta s}{\delta t}$$

$= v \times 1 \times (1/\rho) \times v$

$= v^2/\rho$, where ρ is the radius of curvature at P.

Of course, in the case of the circle, ρ is equal to the radius r, and we have the well-known result that the acceleration towards the centre in circular motion is v^2/r (Art. 68).

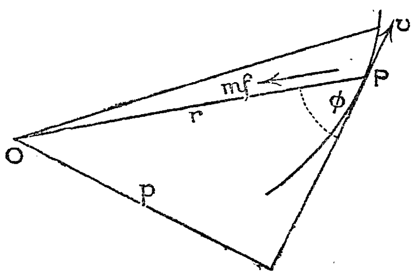


Fig. 164.

204. Motion in an orbit.

An important application of this result is to the motion of a particle which describes an orbit about a fixed point under the action of a force to that point which is a function of the distance.

Let m be the mass of the particle, and mf the force towards O under the influence of which the particle is moving (Fig. 164)

Then, resolving along the tangent and normal,

$$mv\,dv/ds = -mf\cos\phi = -mf\,dr/ds,$$

$$mv^2/\rho = mf\sin\phi = mf\,.\,p/r.$$

The first equation gives $\quad f = -v\dfrac{dv}{ds} \Big/ \dfrac{dr}{ds} = -v\dfrac{dv}{dr}.$ (i)

The second gives $\quad v^2 = f\,.\,\dfrac{p}{r}\,\rho = f\,.\,\dfrac{p}{r}\,.\,r\dfrac{dr}{dp}$, from Art. 202,

$$= fp\frac{dr}{dp}. \qquad \text{(ii)}$$

eliminating f, $\quad v^2 = -v\dfrac{dv}{dr}\,.\,p\dfrac{dr}{dp} = -vp\dfrac{dv}{dp},$

i.e. $$v + p\frac{dv}{dp} = 0.$$

The left-hand side is the d.c. of pv with respect to p. Therefore, integrating, $\qquad pv = h$, a constant.

Substituting this value for v in equation (ii), we have

$$f = \frac{v^2}{p}\frac{dp}{dr} = \frac{h^2}{p^3}\frac{dp}{dr}.$$

From this equation, we can, if the tangential-polar equation of a curve be given, find the value of f, i.e. the equation gives the 'law of force' under the influence of which the particle would describe the given curve.

If the law of force be given, i.e. the expression for f in terms of r, then by integration we obtain the tangential-polar equation of the path in which the particle travels under the influence of the force.

It should be noticed that, by integrating equation (i), we get an expression for the velocity of the particle in any position when under the action of a given force, viz.:

$$\int f dr = -\tfrac{1}{2}v^2 + \tfrac{1}{2}C, \qquad \therefore \quad v^2 = C - 2\int f dr.$$

When f is given in terms of r, this can be integrated, and the constant C will be determined from the initial conditions.

If A be the area swept out in time t by the radius vector starting from some fixed position, and if Q be a point on the path very near P,

$$\delta A = \triangle\, OPQ = \tfrac{1}{2}PQ \times p, \quad \text{ultimately } = \tfrac{1}{2}p\,\delta s.$$

$$\frac{\delta A}{\delta t} = \tfrac{1}{2}p\frac{\delta s}{\delta t}; \text{ and when } \delta t \to 0,\ \frac{dA}{dt} = \tfrac{1}{2}p\frac{ds}{dt} = \tfrac{1}{2}pv = \tfrac{1}{2}h.$$

Hence $A = \frac{1}{2}ht$, since h is constant, and $A = 0$ when $t = 0$.

Hence the constant h is twice the area described by the radius vector in unit time, and we have the important law: *the radius vector describes equal areas in equal times.*

Examples:

(i) *Let the force vary inversely as the square of the distance.*

In this case, $f = \mu/r^2$, and the above equation gives $\frac{\mu}{r^2} = \frac{h^2}{p^3}\frac{dp}{dr}$.

Integrating, $$-\mu/r = -\tfrac{1}{2}h^2/p^2 + C.$$

This is the tangential-polar equation of a conic referred to its focus as pole, and represents an ellipse, parabola, or hyperbola according as C is negative, zero, or positive. [Cf. this equation, $\frac{1}{2}h^2/p^2 = \mu/r + C$, with the tangential-polar equations of the conics obtained in Art. 165.]

Since this is the law of gravitation obeyed by the heavenly bodies, it follows that the orbit of the earth relative to the sun is a conic (it is an ellipse) with the sun at a focus.

The velocity at any point of the orbit is obtained from the equation

$$C - 2\int \frac{\mu}{r^2}\,dr = C + \frac{2\mu}{r}.$$

If the particle have velocity v_0 when at distance r_0, $v_0^2 = C + 2\mu/r_0$.

$$\therefore \quad v^2 - v_0^2 = 2\mu\,(1/r - 1/r_0).$$

(ii) *Find the law of force to the pole under which a particle will describe an equiangular spiral.*

In the equiangular spiral (Art. 163, Ex. ii), $p = r\sin\alpha$.

$$f = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^3\sin^3\alpha}\cdot\sin\alpha = \frac{h^2}{r^3\sin^2\alpha}.$$

Hence the force varies inversely as the cube of the distance from the pole.

205. Differential equation of the orbit in polar coordinates.

This equation can be deduced from the tangential-polar equation by aid of the Theorem of Art. 165, viz.:

$$1/p^2 = u^2 + (du/d\theta)^2, \quad \text{where } u = 1/r.$$

For, differentiating this equation with respect to r, we get

$$-\frac{2}{p^3}\frac{dp}{dr} = \frac{d}{dr}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{d}{du}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]\frac{du}{dr}$$

$$= \left[2u + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} \times \frac{d\theta}{du}\right] \times -\frac{1}{r^2}$$

$$= -2u^2\left(u + \frac{d^2u}{d\theta^2}\right).$$

$$\therefore \quad f = \frac{h^2}{p^3}\frac{dp}{dr} = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right),$$

i.e. $$\frac{d^2u}{d\theta^2} + u = \frac{f}{h^2u^2},$$ which is the equation required.

Substituting the value of f in terms of u and integrating, the polar equation of the orbit is obtained.

Examples LXXXIII.

[See Art. 165 for Tangential-Polar Equations].

1. Find the radius of curvature at any point of the parabola $p^2 = ar$.
2. Find the radius of curvature in the lemniscate (i) at the vertex, (ii) at a point where $\theta = 30^\circ$.
3. Find the radius of curvature in the cardioid (i) at the vertex, (ii) at the point furthest from the axis, (iii) at a point of contact of the double tangent perpendicular to the axis.
4. Find the radius of curvature at any point of an ellipse in terms of the distance of the point from a focus.
5. Prove that the radius of curvature at any point of the rectangular hyperbola $r^2 \cos 2\theta = a^2$ varies as the cube of the radius vector. What is its value at the end of the latus rectum?

Find the radius of curvature at any point of the four curves:

6. $r = a\theta$.
7. $r = a/\theta$.
8. $r^n = a^n \cos n\theta$.
9. $r = a \sin^3 \frac{1}{3}\theta$.
10. Deduce the formula for radius of curvature in polar coordinates from
$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\phi}{ds} + \frac{d\theta}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right),$$
together with $\left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2$, and $\tan\phi = r \Big/ \dfrac{dr}{d\theta}$.
11. Prove that $\rho = p + d^2p/d\psi^2$.
12. In Art. 204, prove that $r^2\dot{\theta} = h$.
13. Taking the tangential-polar equation of an ellipse, prove that the force to a focus under the influence of which a particle describes the curve varies inversely as the square of the distance.
14. Find the law of force to a point on the circumference of a circle under which the particle describes that circle.
15. Find the velocity at any point of a particle which is describing an equiangular spiral under the action of a force to the pole.
16. Find the law of force to the pole under which a particle describes a cardioid.
17. Find the law of force to the pole under which a particle describes a lemniscate.
18. A particle is moving in a curve under the action of a force to a fixed point which produces an acceleration $\mu/(\text{distance})^7$; initially, $p = r = a$ and $\mu = 3a^4h^2$. Find the curve which the particle is describing.
19. In the case of an ellipse described under the action of a force to the focus, prove that $h^2 = \mu$ (semi-latus rectum), and that the velocity at any point is given by the equation $v^2 = \mu(2/r - 1/a)$.
20. Find the corresponding results in the case of a hyperbola described under the action of a repulsive force varying inversely as the square of the distance from a focus.

206. Envelopes.

Let $f(x, y, \alpha) = 0$ be the equation of a curve, where x and y are rectangular coordinates of a point and α a constant, depending it may be on the size or position of the curve. If we take different values for α, we shall get different curves of the same kind: the equation $f(x, y, \alpha) = 0$, when different values are assigned to the constant α, is said to represent a *system* or *family of curves.*

For instance, the equation $y^2 = 4ax$, for different values of a, represents a family of parabolas with a common vertex and axis: a variation in the value of a alters the length of the latus rectum.

The equation $(x-h)^2 + y^2 = r^2$, for different values of h, r remaining constant, represents a family of equal circles (radius r) with their centres at points on the axis of x; if h is fixed and r varied, the equation represents a family of concentric circles, centre $(h, 0)$, with different radii. If only one of the two constants h and r be varied, we get a singly-infinite system of curves; if both h and r be varied, we get a doubly-infinite system, consisting of all circles which have their centres on the axis of x.

If, in $f(x, y, \alpha) = 0$, we take the curves corresponding to two values of α which only differ by a small amount, these curves will in general intersect.* If one of these two values of α be made to approach indefinitely near the other, the points of intersection will generally tend to limiting positions; and the locus of these limiting positions of the points of intersection is called the *envelope* of the family of curves.

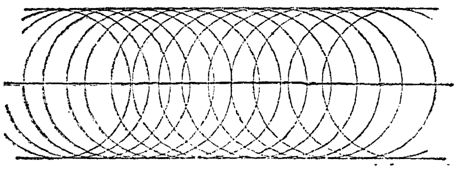


Fig. 165.

For instance, in the case of the circles mentioned above, when r is constant and h varies, the points of intersection of consecutive circles tend to coincide with the ends of diameters perpendicular to the axis of x, and the envelope consists of two straight lines parallel to the axis of x and distant r from it (Fig. 165).

Again, if the equation of a straight line be written in the form

$$x \cos\theta + y \sin\theta = a,$$

it is easily seen geometrically that, whatever the value of θ, the perpendicular distance of the straight line from the origin is a, therefore all the

* It does not always happen that such curves intersect, e.g. in the system of concentric circles obtained above, by keeping h constant and varying r, two consecutive curves do not intersect.

straight lines of the family are tangents to a circle whose centre is the origin and radius a; hence, any two consecutive lines being consecutive tangents to this circle, their point of intersection tends to coincide with a point on the circle, and the circle is therefore the envelope of the lines

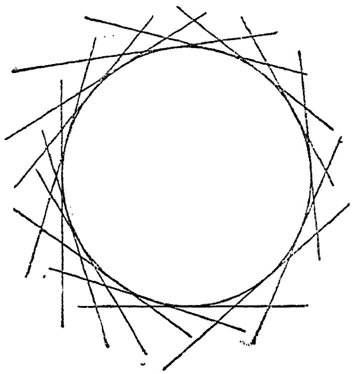


Fig. 166.

(Fig. 166). If the lines are drawn for values of θ which differ by only small amounts, it will be seen that the points of intersection and the parts of the tangents between them are almost indistinguishable to the eye from a circle of radius a.

The property which is seen to be true in these cases is true generally, viz. *the envelope of a system of curves touches at each of its points the corresponding curve of the system.*

For, of three consecutive curves of the family, let the first and second meet in P_1 and the second and third in P_2 (Fig. 167). Then ultimately P_1 and P_2 are consecutive points on the envelope, and

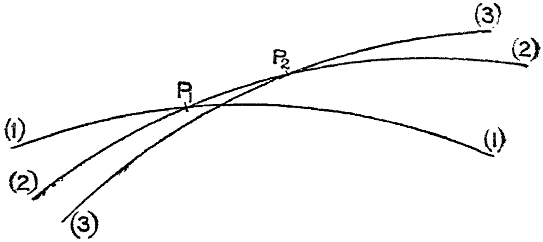


Fig. 167.

they are also on the second curve; therefore, when they move up indefinitely near together, $P_1 P_2$ becomes a tangent both to the envelope and to the second curve. Hence, since they have a common

tangent at a common point, the envelope touches the second curve, and similarly it touches each other curve of the system.

207. Analytical method of finding envelopes.

Let $f(x, y, \alpha)$ and $f(x, y, \alpha+h)$ be two curves of the system for which the values of α differ by a small amount h.

The second equation may, from the mean-value theorem of Art. 117, be written in the form

$$f(x, y, \alpha)+hf'(x, y, \alpha+\theta h)=0,$$

where $|\theta|<1$, and f' denotes the differential coefficient with respect to α, x and y being regarded as constants.

At a point of intersection both equations are satisfied, therefore by subtraction

$$hf'(x, y, \alpha+\theta h)=0.$$

Therefore, since h is not 0 (it is very small, but not zero, otherwise the two curves would coincide altogether), it follows that

$$f'(x, y, \alpha+\theta h)=0.$$

Therefore in the limit, at the points of ultimate intersection, when $h \rightarrow 0$,

$$f'(x, y, \alpha)=0.$$

Hence, to find the locus of these points for different values of α, we have to eliminate α from the two equations

$$f(x, y, \alpha)=0, \quad f'(x, y, \alpha)=0.$$

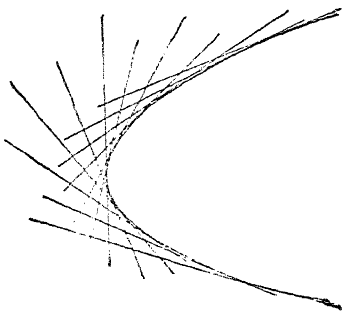


Fig. 168.

Examples:

(i) *Find the envelope of the straight lines* $y=mx+a/m$, *for different values of* m.

Differentiate with respect to m (regarding x and y as constants).

$$0=x-a/m^2, \quad \text{whence} \quad m=\pm\sqrt{(a/x)}.$$

Substituting this value of m in the given equation,

$$y = \pm x\sqrt{(a/x)} \pm a\sqrt{(x/a)} = \pm 2\sqrt{(ax)}.$$

$$\therefore \quad y^2 = 4\,ax, \text{ a parabola.}$$

Hence the given family of straight lines consists of the tangents to the parabola $y^2 = 4\,ax$ (Fig. 168).

(ii) *Find the envelope of the concentric ellipses which have their axes coincident in direction, and the sum of the axes constant.*

Taking the axes of the ellipses as axes of coordinates, and the sum of the semi-axes as c, the lengths of the semi-axes may be written a and $c-a$, a being the variable parameter.

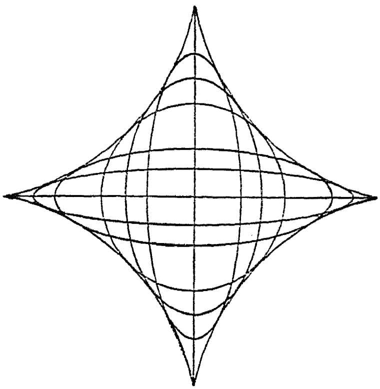


Fig. 169.

The equation of the ellipses is $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{(c-a)^2} = 1 \quad$ [p. 19].

Differentiating with respect to a, $\quad -\dfrac{2\,x^2}{a^3} + \dfrac{2\,y^2}{(c-a)^3} = 0\,;$

$$\therefore \quad \frac{a^3}{(c-a)^3} = \frac{x^2}{y^2}, \text{ and } \frac{a}{c-a} = \frac{x^{2/3}}{y^{2/3}}\,;$$

whence $\quad \dfrac{a}{c} = \dfrac{x^{2/3}}{x^{2/3}+y^{2/3}}, \text{ and } \dfrac{c-a}{c} = \dfrac{y^{2/3}}{x^{2/3}+y^{2/3}}$

Substituting these values of a and $c-a$ in the equation of the ellipse, it becomes

$$x^2\,\frac{(x^{2/3}+y^{2/3})^2}{c^2 x^{4/3}} + y^2\,\frac{(x^{2/3}+y^{2/3})^2}{c^2 y^{4/3}} = 1\,;$$

$$\therefore \quad x^{2/3}\,(x^{2/3}+y^{2/3})^2 + y^{2/3}\,(x^{2/3}+y^{2/3})^2 = c^2,$$

i.e.

$$(x^{2/3}+y^{2/3})^3 = c^2,$$

$$x^{2/3}+y^{2/3} = c^{2/3}.$$

Hence the envelope is the curve called the astroid (Fig. 169).

203. Evolute of a curve.

The locus of the centres of curvature of a curve is called the *evolute* of the curve. The coordinates (ξ, η) of the centre of curvature have been obtained in Art. 199. If ξ and η can be expressed in terms of a single variable, then, by eliminating this variable, the equation of the evolute will be obtained, as in the example below.

The normals to a curve are tangents to its evolute, for, if R, P, Q (Fig. 170) be three points very near together on a curve, and if the normals at R, P meet in C' and the normals at P, Q in C, then in the limit when R and Q move indefinitely near P, C and C' become two consecutive centres of curvature, i.e. two consecutive points on the evolute, and both are on the normal at P; hence the normal at P goes through two consecutive points on the evolute, and therefore touches the evolute.

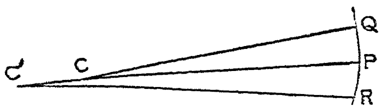


Fig. 170.

Therefore *the evolute of a curve is the envelope of the normals to the curve.* It is generally easier to deduce the equation of the evolute as the envelope of the normals rather than as the locus of the centres of curvature. The following example illustrates both methods in the case of the parabola.

Example:

Find the equations of the circle of curvature and the evolute of a parabola.

The coordinates of any point on the parabola $y^2 = 4ax$ may be written in the form $(am^2, 2am)$ (Art. 50).

Hence $$\frac{dy}{dx} = \frac{dy}{dm} \Big/ \frac{dx}{dm} = \frac{2a}{2am} = \frac{1}{m}.\text{*}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{m}\right) = -\frac{1}{m^2}\frac{dm}{dx} = -\frac{1}{m^2}\cdot\frac{1}{2am} = -\frac{1}{2am^3}.$$

The radius of curvature $$\rho = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \frac{d^2y}{dx^2}$$

$$= \left(1 + \frac{1}{m^2}\right)^{3/2} \Big/ \left(\frac{-1}{2am^3}\right) = -2a(1+m^2)^{3/2}.$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{m^2}; \quad \left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + m^2.$$

Since, if s be measured from the vertex, ds/dx and ds/dy are both +, we have $$ds/dx = \sqrt{(1+m^2)}/m; \quad ds/dy = \sqrt{(1+m^2)}.$$

* Since $dy/dx = \tan\psi$, where ψ is the inclination of the tangent to the axis of x, it follows from this result that $m = \cot\psi$, i.e. m is the tangent of the angle which the tangent to the curve makes with the axis of y.

The coordinates of the centre of curvature are (Art. 199)

$$(x-\rho\, dy/ds, \quad y+\rho\, dx/ds),$$

i.e. $$\left\{am^2+2a(1+m^2)^{3/2}\frac{1}{\sqrt{(1+m^2)}}; \quad 2am-2a(1+m^2)^{3/2}\frac{m}{\sqrt{(1+m^2)}}\right\},$$

i.e. $$\{am^2+2a(1+m^2); \quad 2am-2am(1+m^2)\},$$

i.e. $$\{a(3m^2+2); \quad -2am^3\}.$$

Hence the equation of the circle of curvature at any point is

$$(x-3am^2-2a)^2+(y+2am^3)^2 = 4a^2(1+m^2)^3.$$

To find the evolute as the locus of the centres of curvature, we have to eliminate m from $\quad x = 3am^2+2a; \quad y = -2am^3$,

which gives $\quad (x-2a)^3 = 27a^3m^6 = 27a^3 \cdot y^2/4a^2 = \frac{27}{4}ay^2.$

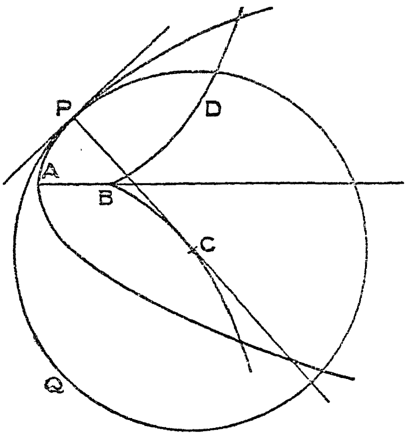


Fig. 171.

To find the evolute as the envelope of the normals, the equation of the normal at $(am^2, 2am)$ is (Art. 47), since $dy/dx = 1/m$,

$$x-am^2+(y-2am)/m = 0,$$

i.e. $$y+mx-2am-am^3 = 0.$$

Differentiating this with respect to m, $\quad x-2a-3am^2 = 0.$

Eliminating m, we have from the last equation, $m = \left(\dfrac{x-2a}{3a}\right)^{1/2}$;

$$\therefore\ y = am^3-m(x-2a) = am^3-m \cdot 3am^2 = -2am^3 = -2a\left(\frac{x-2a}{3a}\right)^{3/2},$$

whence, squaring, $\quad y^2 = \frac{4}{27}(x-2a)^3/a,$

which is the same equation as before.

The parabola and its evolute are shown in Fig. 171; PA is the parabola, DBC the evolute, PQ the circle of curvature at P, PC the normal at P

touching the evolute at C, which is the centre of curvature for the point P; the length of PC is the radius of curvature.

Examples LXXXIV.

Find and draw the envelope of the following, 1–26:

1. Chords of a circle of constant length.
2. A system of equal circles with their centres on the circumference of a given circle.
3. A straight line which moves so that the sum of its perpendicular distances from two fixed points is constant.
4. A straight line which moves so that the product of its intercepts on the coordinate axes is constant.
5. A straight line which moves so that the sum of the intercepts on the axes is constant.
6. A straight line which moves so that the part intercepted between the axes is of constant length.
7. A system of concentric ellipses, with their axes along the coordinate axes, and of constant area.
8. The circles on double ordinates of a fixed parabola as diameters.
9. The parabolas $y^2 = 4m(x-m)$.
10. The parabolas $y^2 = m^2(x-m)$.
11. The straight lines $m^3x - my = a$, for different values of m.
12. The straight lines $y = mx + a\sqrt{(1+m^2)}$.
13. The straight lines $x + y\sin\theta = a\cos\theta$, for different values of θ.
14. The straight lines $y = m^2x + am$, for different values of m.
15. The straight lines $x\sin\theta + y\cos\theta = \frac{1}{2}c\sin 2\theta$, for different values of θ.
16. The parabolas $m^2x^2 + 2my + 1 = 0$.
17. The conics $x^2\sin\alpha + y^2\cos\alpha = a^2$, for different values of α.
18. The circles whose diameters are chords of a fixed circle through a fixed point on its circumference.
19. The circles on central radii of a rectangular hyperbola as diameters.
20. The circles described with double ordinates of an ellipse as diameters.
21. A straight line which rotates with uniform angular velocity about one of its points which moves uniformly along a fixed straight line.
22. The paths, for different angles of elevation, of particles projected from a fixed point with given velocity.
    [If θ $(=\tan^{-1} m)$ be the elevation and $\sqrt{(2gh)}$ the velocity, the equation of the path, referred to horizontal and vertical axes through the fixed point, is $y = mx - \frac{1}{4}x^2(1+m^2)/h$.]
23. Ellipses with their axes along two fixed straight lines and the sum of squares of the axes constant.
24. Circles which have their centres on a fixed circle and which pass through a fixed point.

25. Circles which touch the axis of x and have their centres on the parabola $y = x^2$.

26. Circles through the origin which have their centres on $xy = 1$.

27. Parallel rays of light fall on the inner surface of a cylindrical mirror in a plane perpendicular to its axis; find the envelope of the reflected rays (which make the same angle with the normal as the incident rays). This envelope is called the *caustic by reflexion* at a circle.

28. Rays of light proceed from a point on the inner surface of a bright circular ring, and are reflected from the surface; find the envelope of the reflected rays.

Find the evolute of the following curves, 29-34:

29. The cycloid.

30. The ellipse.

31. The astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

32. The rectangular hyperbola $xy = c^2$.

33. The hyperbola $x = a\cosh u$, $y = b\sinh u$.

34. The curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

35. S is a fixed point and P any point on a fixed straight line; find the envelope of lines drawn from P perpendicular to SP.

36. Find the envelope of a straight line which moves so that the product of its perpendicular distances from two fixed points is constant.

CHAPTER XXI

ELEMENTARY DIFFERENTIAL EQUATIONS

209. Definitions.

A relation between two variables x, y, and differential coefficients of y with respect to x is called an ordinary differential equation.

The *order* of the differential equation is that of the highest differential coefficient which occurs in the equation.

The *degree* of the differential equation is the degree of the highest power of the highest differential coefficient in the equation when rationalized and cleared of fractions.

E. $\quad x\dfrac{dy}{dx}+y = a$ is of the first order and of the first degree,

$x\dfrac{d^2y}{dx^2}+y=\left(\dfrac{dy}{dx}\right)^3$ is of the second order and of the first degree,

$x\left(\dfrac{dy}{dx}\right)^2+y$ is of the first order and of the second degree,

$\left(\dfrac{d^2y}{dx^2}\right)^2+y\dfrac{dy}{dx}=0$ is of the second order and of the second degree,

and, generally, $\left(\dfrac{d^ny}{dx^n}\right)^r$ + any function of x, y, and lower d. c.'s than the n^{th} is of the n^{th} order and of the r^{th} degree.

210. Formation of differential equations.

Let us consider one of the ways in which differential equations can be formed.

Examples:

(i) If $y=mx+c$, we have, by differentiating, $dy/dx=m$,
and, by differentiating again, $d^2y/dx^2=0$.

The first differentiation eliminates c, and therefore gives a result which is true for all values of c. The second differentiation eliminates m, and gives a result true for all values of m and c.

The geometrical interpretation of this should be carefully noticed. $y=mx+c$ is the equation of any straight line. The first equation $dy/dx=m$ expresses a property common to all the straight lines obtained

by taking different values of c, viz. that their inclination to the axis of x is $\tan^{-1} m$. The second equation $d^2y/dx^2 = 0$ expresses a property true for all values of m and c, i.e. a property common to all straight lines, viz. (Art. 199) that the curvature is zero.

The equation $d^2y/dx^2 = 0$ is said to be the differential equation of all straight lines.

(ii) If $y^2 = 4\,ax + c$, we have, by differentiating,

$$2\,y\,dy/dx = 4\,a,$$

and, by differentiating again,

$$y\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dy}{dx} = 0.$$

The given equation represents a system of parabolas with their axes along the axis of x. The first equation $y\,dy/dx = 2\,a$ states that all these parabolas have their subnormal equal to $2\,a$, whatever be the value of c. The second equation states that the differential coefficient of this subnormal is zero, i.e. that, for any individual parabola of the family $y^2 = 4\,ax + c$, the portion of the axis of x intercepted between the normal and the ordinate at any point is constant.

The second equation is called the differential equation of all parabolas which have their axes along the axis of x.

(iii) If $(x-a)^2 + (y-b)^2 = r^2$, then, differentiating and dividing by 2, we have

$$x - a + (y-b)\,dy/dx = 0. \tag{i}$$

This eliminates r. Differentiating again,

$$1 + (y-b)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dy}{dx} = 0.$$

This eliminates a, and gives

$$y - b = -\left[1 + \left(\frac{dy}{dx}\right)^2\right] \Big/ \frac{d^2y}{dx^2}. \tag{ii}$$

If this be now differentiated again, we get a differential equation of the third order, from which all the three constants a, b, r which occurred in the original equation have disappeared.

If the result of (ii) be substituted in (i), we have

$$x - a = -(y-b)\frac{dy}{dx} = \frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right] \Big/ \frac{d^2y}{dx^2}.$$

If these results be now substituted in the original equation, it becomes

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 \Big/ \left(\frac{d^2y}{dx^2}\right)^2 = r^2 \tag{iii}$$

If this be differentiated again, we shall again get the differential equation of the third order which contains none of the three constants a, b, r.

Geometrically, the original equation represents any circle.

Equation (i), which does not contain r, expresses a property common to all circles with centre (a, b), whatever the radius, viz. that

$$dy/dx = -(x-a)/(y-b);$$

i.e. if P be any point (x, y) on a circle, centre A (a, b), the inclination of the tangent at P to the axis of x exceeds by 90° the inclination of AP to the axis of x. This is obvious geometrically, since (Fig. 172)

$$dy/dx = \tan\psi = -\cot MPT = -\cot MAP = -AM/MP = -(x-a)/(y-b).$$

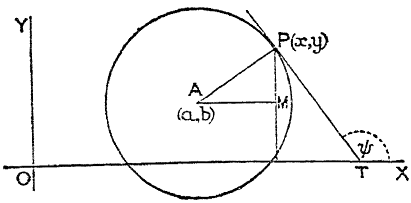


Fig. 172.

Equation (ii), which does not contain a or r, expresses a property common to all circles, whatever the radius and the abscissa of the centre. Comparing it with the expression for η in Art. 199, it gives $\eta = b$, which is obviously true for all such circles since the centre of curvature is the centre of the circle.

Equation (iii), which does not contain a or b, expresses a property common to all circles of radius r, viz. that the radius of curvature [the left-hand side of (iii) is the square of the value obtained for ρ in Art. 199] is at all points on such circles equal to r, which again is obvious geometrically.

The result of differentiating (iii), which reduces to

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3} = 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2$$

and which contains neither a, b, nor r, expresses a property common to all circles, viz. that the d.c. of the radius of curvature is zero, and hence is equivalent to the statement that for any individual circle the radius of curvature, and therefore also the curvature, is constant.

These examples show that differential equations may be formed by eliminating the constants from a given equation. The given equation, by taking different values for the constants, represents a family of curves. The successive differential equations express geometrical properties common to certain sets of these curves, and the final differential equation from which all the constants are eliminated expresses some property common to all curves of the family.

It will be noticed that, in these examples, the order of the differential equation when all the constants are eliminated is equal

to the original number of constants. This is always the case, for if the original equation contains n constants, then by differentiating it n times a differential equation of the n^{th} order is finally obtained; the results of these n successive differentiations together with the given equation form a system of $n+1$ equations, and it is proved in works on Algebra that, in general, from $n+1$ equations, n of the quantities they contain can be eliminated. Hence the n constants can be eliminated, and the result is a differential equation of the n^{th} order.

211. Solution of a differential equation.

Conversely, in finding the integral of a differential equation of the n^{th} order, we should expect the most general solution to be a relation between the variables containing n arbitrary constants, and it can be proved that, in general, this is the case. Reversing the above process, and finding the most general relation between the variables x and y, which leads to a given differential equation, is called 'integrating' or 'solving the equation'. The result, which must contain a number of arbitrary constants equal to the order of the differential equation, is called the *complete* or the *general integral* or the *complete primitive.* Any simpler solution which satisfies the equation is called a *particular solution,* e.g. the general integral of the equation $d^2y/dx^2 = 0$ is $y = Ax+B$, containing the arbitrary constants A and B; $y = 2x$, $y = -3$, $y = 4x-5$, &c., are particular solutions (obtained by giving definite numerical values to A and B).

Geometrically, the process of solving a differential equation consists in finding a system of curves which possess a specified property. Since the general solution contains n arbitrary constants, a curve of the system can be made to satisfy n conditions.

If the differential equation be of the first order, the solution will contain one arbitrary constant c, and will be of the form $f(x, y, c) = 0$, which for different values of c represents a family of curves. If in this equation we substitute for x and y the coordinates of some definite point, we have an equation to find c, which determines the curves of the family that pass through the given point. If the differential equation be $F(x, y, dy/dx) = 0$, then, on substituting in this equation the coordinates of the same point as before, we have an equation for dy/dx, which gives the *directions* at the point of those curves of the family that pass through the given point. Hence the differential equation specifies the curves of the system which pass through a given point by means of their slope: the integral equation specifies the same curves by means of the parameter c.

Examples:

(i) *Find the equation of the straight line which goes through* (3, 2) *and makes an angle* $\tan^{-1}\frac{3}{5}$ *with the axis of* x.

The differential equation of all straight lines is $d^2y/dx^2 = 0$ (since this expresses that the curvature is zero).

The first integration gives $dy/dx = A$. Since dy/dx is given to be $\frac{3}{5}$, we have $A = \frac{3}{5}$;

$$\therefore \quad 5\, dy/dx = 3.$$

Integrating again, $$5y = 3x + C.$$

Since the line is to go through (3, 2), it follows that $10 = 9 + C$, and $C = 1$.

$\therefore$ the equation is $$5y = 3x + 1.$$

This is a 'particular solution' of the equation $d^2y/dx^2 = 0$.

(ii) *Find the equation of the parabola which has its axis along the axis of* x, *passes through the point* (4, 2), *and has the slope* $\frac{1}{3}$ *at that point.*

The subnormal of such a parabola is constant, $\therefore \quad y\, dy/dx = a$.

Integrating, $$y^2 = 2ax + b.$$

Substituting the given values of y and dy/dx in the first equation, we have $2 \times \frac{1}{3} = a$; substituting the coordinates of the given point in the second equation, we have $4 = 8a + b = \frac{16}{3} + b$, whence $b = -\frac{4}{3}$.

$\therefore$ the required equation is $y^2 = \frac{4}{3}x - \frac{4}{3}$, i.e. $3y^2 = 4x - 4$.

Before proceeding to the various methods of solving differential equations, the student should work some examples in the formation of differential equations by eliminating constants.

Examples LXXXV.

1. Eliminate c from the equation $xy = c^2$.
   Give the geometrical meaning of the result (see p. 103, Ex. v).
2. Eliminate m from the equation $y = mx + a/m$.
   Explain the result geometrically. [$y - x\, dy/dx$ is the intercept on the axis of y.]
3. Eliminate (i) p alone, (ii) α alone, (iii) both p and α from the equation $x\cos\alpha + y\sin\alpha = p$. [This equation represents a straight line such that the perpendicular to it from the origin is of length p and inclined to the axis of x at an angle α.]
   Explain each result geometrically.
4. Eliminate (i) A, (ii) both A and b from the equation $y = Ae^{bx}$.
   What is the geometrical meaning of the first result?
5. If $y = c\cosh(x/c) + A$, prove that $dy/dx = \sqrt{(y^2 - c^2)}/c$.
   What is the geometrical meaning of this result? [See Art. 197.]
6. Eliminate m from the equation $y = mx \pm a\sqrt{(1 + m^2)}$.
   Explain the result geometrically.
7. Eliminate the constants from $y = Ax^2 + Bx + C$.
8. Prove that if $y = A\cos mx + B\sin mx$, or if $y = A\sin(mx + \alpha)$, then $d^2y/dx^2 + m^2y = 0$. [See Art. 192.]
9. Eliminate A and B from $y = Ae^{mx} + Be^{-mx}$, and from $y = A\cosh(mx + \alpha)$.

10. Eliminate the constants from the equation $Ax^2+By^2=1$.

11. Prove that, if $y=e^{-\frac{1}{2}kt}(A\cos nt+B\sin nt)$, then
$$\ddot{y}+k\dot{y}+(n^2+\tfrac{1}{4}k^2)y=0.$$

12. Eliminate A and B from the equation $y=e^{-\frac{1}{2}kt}(Ae^{nt}+Be^{-nt})$.

13. Eliminate A and B from the equation $Ax+B=xy$.

14. Verify that $y=A\log x+B$ is the solution of the differential equation $x\,.\,d^2y/dx^2+dy/dx=0$.

15. Verify that $y=(A+Bt)e^{nt}$ is the solution of $\ddot{y}-2n\dot{y}+n^2y=0$.

16. Show that the differential equation of all parabolas which have their axes parallel to the axis of y is $d^3y/dx^3=0$.

17. If $y=A\sin^{-1}x+B$, prove that $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=0$.

18. If $y=(\sin^{-1}x)^2+A\sin^{-1}x+B$, prove that $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2$.

19. If $y=(A+Bx)\sin mx+(C+Dx)\cos mx$, prove that
$$\frac{d^4y}{dx^4}+2m^2\frac{d^2y}{dx^2}+m^4y=0.$$

20. Find the differential equation of all conics which have their axes along the axes of coordinates.

21. Find the differential equation of all circles which touch both coordinate axes.

22. Find the differential equation of all circles which have their centres on the axis of y.

23. If $y=(Ae^{mx}+Be^{-mx})/x$, prove that $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}-m^2y=0$.

24. Eliminate A and B from the equation $y=A\cos(\log x)+B\sin(\log x)$.

212. Differential equations of the first order.

We have several times in the preceding chapters, especially in Chapter XIX, met with differential equations, and have solved them. We now proceed to collect together and consider the more common methods of solving such equations, and commence with equations of the first order and of the first degree. Such equations involve dy/dx and one or both of the quantities x and y, and the solution will involve one arbitrary constant. There is no method which will solve the equation in its most general form, but various particular cases will be considered.

I. Let y be absent.

We have then $dy/dx=f(x)$, $\therefore\ y=\int f(x)\,dx+A$.

This is merely the evaluation of an ordinary indefinite integral, which, as already pointed out (Art. 72), involves an arbitrary constant.

Example. $x^2\,.\,dy/dx=1+x$.

Here $dy/dx=1/x^2+1/x$; $\therefore\ y=-1/x+\log x+A$.

II. **Let x be absent.** In this case $dy/dx = f(y)$,

which may be written $$\frac{1}{f(y)}\frac{dy}{dx} = 1.$$

Integrating with respect to x, and remembering that

$\int F(y)\frac{dy}{dx}\,dx = \int F(y)\,dy$ [Art. 131], we have $\int \frac{1}{f(y)}\,dy = x + A$.

Example. Find a function of x *which has the values* 10 *and* 20 *when* $x = 0$ *and* 1 *respectively, and such that its rate of change is proportional to the square of its value.*

Here $\frac{dy}{dx} = ky^2$, $\therefore \frac{1}{y^2}\frac{dy}{dx} = k$.

Integrating, $$-1/y = kx + C.$$

Substituting the given values,

$$-\tfrac{1}{10} = C, \text{ and } -\tfrac{1}{20} = k + C, \quad \therefore C = -\tfrac{1}{10}, \quad k = +\tfrac{1}{20};$$

and the equation is $-1/y = \frac{1}{20}x - \frac{1}{10} = \frac{1}{20}(x-2)$,

whence $y = 20/(2-x)$, which is the required function.

III. **Let the variables be separable.** This is the case if dy/dx is equal to an expression which can be resolved into factors containing x only or y only. It includes the two preceding forms as particular cases.

The factors which involve y only can be put on one side of the equation with the dy/dx, and those that contain x only on the other side, so that the equation takes the form

$$f(y)\frac{dy}{dx} = F(x).$$

Integrating with respect to x, $\int f(y)\,dy = \int F(x)\,dx + A$.

Examples:

(i) $x \, . \, dy/dx + y^2 = 1$.

i.e. $x\frac{dy}{dx} = 1 - y^2$, which may be written $\frac{1}{1-y^2}\frac{dy}{dx} = \frac{1}{x}$.

Integrating, $$\frac{1}{2}\log\frac{1+y}{1-y} = \log x + \log A. \qquad \text{(i)}$$

It should be noticed that, if all or most of the terms of the integral are logarithms, it is best to take the constant in the form $\log A$ instead of A (since the logarithm admits of all values from $-\infty$ to $+\infty$, this is just as general as A). A simpler result is then obtained if we pass to the equation which yields the preceding equation on taking logarithms, viz. in this case,

$$\sqrt{\left(\frac{1+y}{1-y}\right)} = Ax.$$

[If logarithms of both sides be taken, equation (i) is obtained.]

Therefore $(1+y)/(1-y) = A^2x^2$, which is the required solution.

(ii) *In what curves does the subtangent bear a constant ratio to the abscissa?*

The subtangent $= y \cot \psi = y \Big/ \dfrac{dy}{dx}$.

$$\therefore \quad y \Big/ \frac{dy}{dx} = nx, \qquad \text{i.e. } \frac{n}{y}\frac{dy}{dx} = \frac{1}{x}.$$

Integrating, $$n \log y = \log x + \log C,$$

whence $$y^n = Cx.$$

These are the curves which possess the property mentioned.

If $n = 2$, we have the parabolas $y^2 = Cx$, showing that such parabolas are the only curves which possess the property proved in Art. 46, Ex. iv.

Many equations which are not of this type can be reduced to it by making a simple substitution.

For example, in the equation $dy/dx = x+y$, the variables are not separable, but, if we put $x+y=z$, the variables in the resulting equation for z are separable; for since $y = z - x$, we have $dy/dx = dz/dx - 1$, and the equation becomes $\dfrac{dz}{dx} - 1 = z$; i. e. $\dfrac{1}{1+z}\dfrac{dz}{dx} = 1$.

Therefore $$\log(1+z) = x + \log A,$$

whence $1 + z = Ae^x$, or, returning to y, $1 + x + y = Ae^x$.

Most of the equations we have hitherto met with have belonged to one or other of these three types.

Examples LXXXVI.

1. Find the curves in which the subnormal is constant, and equal to a.
2. Find the curves in which the subtangent is constant, and equal to a.
3. Find the function of x whose rate of change with respect to x is always proportional to its own value.
4. In what curves is the subtangent double the abscissa?
5. In what curves is the subnormal three times the abscissa?
6. In what curves is the portion of the tangent between the axes bisected at the point of contact?
7. In what curves is the portion of the tangent between the axes divided in a given ratio $m : n$ at the point of contact?
8. In what curves are the lengths of the normal and of the radius vector always numerically equal?
9. Find the curves in which (i) the polar subtangent, (ii) the polar subnormal, is constant.
10. Find the general equation of all curves in which the tangent makes a constant angle α with the radius vector.

Solve the equations:

11. $(x+a)\dfrac{dy}{dx} = y + b$.
12. $x^3 \dfrac{dy}{dx} = 2x + 3$.
13. $xy\dfrac{dy}{dx} = 1 + y^2$.
14. $\dfrac{dy}{dx} + ay + b = 0$.

15. $\frac{dy}{dx} + ax + b = 0,$

16. $\frac{dy}{dx} = 2x(y+b).$

17. $\frac{dy}{dx} = ax + by.$

18. $x(y+2) + y(x+2)\frac{dy}{dx} = 0.$

19. $xy(1+x^2)\frac{dy}{dx} - y^2 = 1.$

20. $x\frac{dy}{dx} - y = xy.$

21. $x + y\frac{dy}{dx} = x^2 + y^2.$

22. $\frac{dy}{dx} = \tan y \cot x.$

23. $\frac{dy}{dx} = \cos(x+y).$

24. $\frac{dy}{dx} + 1 = xy + x - y.$

25. Find a function which is equal to 1 when $x = 0$, and to 2 when $x = 1$, and whose rate of change is proportional to the cube of its value.

26. Find a function which is equal to 0 when $x = 1$, and to 1 when $x = 4$, and whose rate of change is inversely proportional to its value.

213. IV. Homogeneous equations.

The equation $P\frac{dy}{dx} = Q$ is said to be homogeneous if P and Q are homogeneous functions of x and y of the same degree.

The equation may be reduced to the preceding form by substituting $y = zx$, and therefore

$$\frac{dy}{dx} = z + x\frac{dz}{dx}.$$

It will be found that, after dividing out by x^n, where n is the degree of P and Q, the variables are separable.

Example. $\qquad (x^2 - xy)\,dy/dx = xy + y^2.$

Making the substitution just mentioned, the equation becomes

$$(x^2 - zx^2)(z + x\,.\,dz/dx) = zx^2 + z^2x^2.$$

Therefore, after removing the factor x^2 from both sides,

$$z + x\frac{dz}{dx} = \frac{z+z^2}{1-z},$$

i.e.

$$x\frac{dz}{dx} = \frac{z+z^2}{1-z} - z = \frac{2z^2}{1-z},$$

which may be written

$$\frac{1-z}{z^2}\frac{dz}{dx} = \frac{2}{x}.$$

Integrating,

$$-1/z - \log z = 2\log x + \log A.$$

$$\therefore\quad e^{-1/z} \div z = Ax^2,$$

i.e.

$$e^{-x/y} = Ax^2 \times y/x = Axy.$$

The equation $(ax + by + c)\frac{dy}{dx} = a'x + b'y + c'$ is not homogeneous, but it can be reduced to one or other of the preceding forms in the following manner:

(i) Let $a'/a = b'/b = k$, so that $a' = ka$, $b' = kb$.
Then the equation can be written

$$(ax+by+c)\,dy/dx = k(ax+by)+c'.$$

Let $ax+by = z$; therefore $a+b\,dy/dx = dz/dx$,

and the equation becomes $\quad (z+c)\dfrac{1}{b}\left(\dfrac{dz}{dx}-a\right) = kz+c'$,

i.e. $$\frac{dz}{dx}-a = b\,.\,\frac{kz+c'}{z+c},$$ which is of the form II.

(ii) Let $a'/a \neq b'/b$.

The equation may be written $\dfrac{dy}{dx} = \dfrac{a'x+b'y+c'}{ax+by+c}$.

Let $a'x+b'y+c' = X$ and $ax+by+c = Y$, so that the equation becomes $dy/dx = X/Y$.

Then $$\frac{dY}{dX} = \frac{dY}{dx}\bigg/\frac{dX}{dx} = \frac{a+b\,dy/dx}{a'+b'\,dy/dx} = \frac{a+b\,X/Y}{a'+b'\,X/Y} = \frac{aY+bX}{a'Y+b'X};$$

and this equation is homogeneous in X and Y and therefore can be solved as above.

Example. Solve $\quad (2x+y-1)\,dy/dx = 2x-2y+1$.

Let $2x+y-1 = X$ and $2x-2y+1 = Y$. Then the given equation takes the form $dy/dx = Y/X$.

$$\frac{dY}{dX} = \frac{dY}{dx}\bigg/\frac{dX}{dx} = \frac{2-2\,dy/dx}{2+dy/dx} = \frac{2-2\,Y/X}{2+Y/X},$$ which is homogeneous.

Let $Y = Xz$, $\qquad \therefore\ \dfrac{dY}{dX} = z+X\dfrac{dz}{dX}$.

The preceding equation now becomes

$$z+X\frac{dz}{dX} = \frac{2-2z}{2+z};$$

$$X\frac{dz}{dX} = \frac{2-2z}{2+z} - z = \frac{2-4z-z^2}{2+z},$$

i.e. $$\frac{2+z}{2-4z-z^2}\,\frac{dz}{dX} = \frac{1}{X}.$$

Since $2+z = -\frac{1}{2}$ (d.c. of $2-4z-z^2$), this equation gives on integration

$$\log(2-4z-z^2) = -2\log X+\log C;$$

$$\therefore\ 2-4z-z^2 = C/X^2$$

or $$2X^2-4zX^2-z^2X^2 = C,$$

i.e. $$2X^2-4XY-Y^2 = C,$$

i.e. $$2(2x+y-1)^2-4(2x+y-1)(2x-2y+1)-(2x-2y+1)^2 = C.$$

This reduces to $\qquad 2x^2-4xy-y^2+2x+2y = A$,

[after multiplying out, dividing by -6, and writing A instead of $\frac{1}{6}(5-C)$].

It should be noticed that in this particular case the solution can be obtained more readily as follows:

If all the terms are collected on the left-hand side, the given equation becomes

$$2x\frac{dy}{dx}+2y+y\frac{dy}{dx}-\frac{dy}{dx}-2x-1=0.$$

The first two terms together are the d.c. of $2xy$, and all the other terms can be integrated at once. Hence, on integration, we get

$$2xy+\tfrac{1}{2}y^2-y-x^2-x=C,$$

i.e., changing the signs,

$$2x^2-4xy-y^2+2x+2y=-2C,$$

as before.

The integral can be obtained in this simple manner whenever $b'=-a$ in the general equation. [See also Art. 216.]

214. V. Linear equation of the first order.

A differential equation is said to be linear when it is of the first degree in y and the differential coefficients of y with respect to x.

Hence the general linear equation of the first order can be written in the form

$$\frac{dy}{dx}+Py=Q,$$

where P and Q are functions of x only, since the coefficient of dy/dx can always be made unity by division.

First take the particular case when $Q=0$. The variables are then separable, and the equation may be put in the form

$$\frac{1}{y}\frac{dy}{dx}+P=0.$$

Integrating, $\log y+\int Pdx=\log C$, i.e. $ye^{\int Pdx}=C$.

If we test this by differentiation, we get the original differential equation with the addition of the factor $e^{\int Pdx}$, for we have, on differentiating with respect to x,*

$$y\times e^{\int Pdx}\times P+e^{\int Pdx}\times\frac{dy}{dx}=0,$$

i.e. $$e^{\int Pdx}(dy/dx+Py)=0.$$

This gives the clue to the solution in the general case when $Q\neq 0$. The left-hand side, when multiplied by the 'integrating factor' $e^{\int Pdx}$, becomes the differential coefficient of $ye^{\int Pdx}$, and the right-hand side becomes $Qe^{\int Pdx}$.

Hence, on integration, we have

$$ye^{\int Pdx}=\int Qe^{\int Pdx}dx+C,$$

* The d.c. with respect to x of $e^{\int Pdx}$ = the d.c. of e^u, where $u=\int Pdx$

$$=e^u\times du/dx=e^{\int Pdx}\times P.$$

which is the required solution. The equation is now to be regarded as solved, whether we are able to perform the actual integrations or not.

In particular cases, the results should not be written down by substituting in this general solution, but by finding in each case the integrating factor $e^{\int P dx}$; this often turns out to be a simple algebraical or trigonometrical function, which in many cases can be seen by inspection. The following examples illustrate the process.

Examples:

(i) $\sin x \,.\, dy/dx + y\cos x = x^2$.

In this case it is evident that the left-hand side is, as it stands, the d. c. of $y\sin x$;

$\therefore$ we have $$y\sin x = \int x^2\,dx = \tfrac{1}{3}x^3 + C.$$

(ii) $x^2 \,.\, dy/dx + 3xy = 1$.

Since $3x^2$ is the d. c. of x^3, it is evident that the left-hand side, if multiplied by x, will become the d. c. of x^3y.

Then $$x^3 \,.\, dy/dx + 3x^2y = x,$$
$$\therefore\quad x^3y = \int x\,dx = \tfrac{1}{2}x^2 + C.$$

(iii) $$(1+x^2)\frac{dy}{dx} + xy = x, \quad \text{i. e.}\quad \frac{dy}{dx} + \frac{x}{1+x^2}y = \frac{x}{1+x^2}. \qquad \text{(i)}$$

Here $P = x/(1+x^2)$; therefore
$$\int P dx = \int \frac{x}{1+x^2}dx = \tfrac{1}{2}\int\frac{2x}{1+x^2}dx = \tfrac{1}{2}\log(1+x^2) = \log\sqrt{(1+x^2)};$$
$$\therefore\quad e^{\int P dx} = \sqrt{(1+x^2)}.$$

Hence, multiplying (i) by $\sqrt{(1+x^2)}$, the equation becomes
$$\sqrt{(1+x^2)}\,\frac{dy}{dx} + \frac{x}{\sqrt{(1+x^2)}}y = \frac{x}{\sqrt{(1+x^2)}}.$$

Integrating, $$y\sqrt{(1+x^2)} = \int\frac{x}{\sqrt{(1+x^2)}}dx = \sqrt{(1+x^2)} + C,$$

i. e. $$(y-1)\sqrt{(1+x^2)} = C, \quad\text{or}\quad (y-1)^2(1+x^2) = C^2.^*$$

(iv) *A particle moves horizontally in a medium whose resistance varies as the velocity, and is also subject to another retarding force which is proportional to the time; find the velocity at the end of time* t.

If v be the velocity at time t, the equation of motion is $dv/dt = -kv - bt$, where k and b are constants [i. e. $dv/dt + kv = -bt$].

To integrate, multiply by $e^{\int k dt}$, i.e. e^{kt}. Then
$$e^{kt} \,.\, dv/dt + v \,.\, ke^{kt} = -bte^{kt};$$
$$\therefore\quad ve^{kt} = -b\int te^{kt}\,dt + C.$$

* The variables are separable in the given differential equation, and this result can be obtained more readily by the method of Art. 212.

Integrating by parts, $\int te^{kt}\,dt = t\,.\frac{e^{kt}}{k} - \int 1\,.\frac{e^{kt}}{k}\,dt = \frac{t}{k}e^{kt} - \frac{1}{k^2}e^{kt}$;

$$\therefore\quad ve^{kt} = -be^{kt}(t/k - 1/k^2) + C.$$

Initially, when $t=0$, $v=u$, the velocity of projection;

$$\therefore\quad u = -b(-1/k^2) + C, \text{ and } C = u - b/k^2.$$

$\therefore$ dividing by e^{kt} and inserting the value of C,

$$v = \frac{b}{k^2}(1-kt) + \left(u - \frac{b}{k^2}\right)e^{-kt}.$$

The example from Electricity given in Art. 182 is also an example of this type of equation.

The more general equation

$$\frac{dy}{dx} + Py = Qy^n,$$

where P and Q are functions of x, can be reduced to the preceding form by dividing by y^n and putting $1/y^{n-1} = z$; the resulting equation is linear in z.

Example. Solve the equation $x\,.\,dy/dx + y = xy^3$.

Dividing by y^3x, $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{xy^2} = 1.$

Let $\frac{1}{y^2} = z$; $\quad\therefore\quad -\frac{2}{y^3}\frac{dy}{dx} = \frac{dz}{dx}$,

and the equation becomes $-\frac{1}{2}\frac{dz}{dx} + \frac{1}{x}z = 1$, i. e. $\frac{dz}{dx} - \frac{2}{x}z = -2.$

This is linear in z. In this case $P = -2/x$,

$$\int P\,dx = \int -\frac{2}{x}\,dx = -2\log x = \log\frac{1}{x^2};\qquad e^{\int P dx} = \frac{1}{x^2}.$$

Multiplying by the integrating factor $\frac{1}{x^2}$, $\quad \frac{1}{x^2}\frac{dz}{dx} - \frac{2}{x^3}z = -\frac{2}{x^2}.$

Integrating, $\quad \frac{1}{x^2}z = \frac{2}{x} + C$, i. e. $z = 2x + Cx^2$,

which, since $z = 1/y^2$, gives $1 = 2xy^2 + Cx^2y^2$.

215. Another method of solution.

In both of the cases considered in the preceding article, the solution can also be obtained by substituting $y = uv$, and choosing u so that the coefficient of v in the resulting equation may be zero; we shall then have for v an equation in which the variables are separable. As an illustration, let us solve the last equation of the preceding article by this method:

Example. Solve $x\,.\,dy/dx + y = xy^2$.

Let $y = uv$; therefore, since $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

the equation becomes $$xu\frac{dv}{dx} + \left(x\frac{du}{dx} + u\right)v = xu^2v^2. \qquad \text{(i)}$$

The function u may be *any* function, and is quite at our disposal; hence we are at liberty to choose u so that the coefficient of v may be zero,

i.e. so that $$x\frac{du}{dx} + u = 0, \qquad \therefore\ \frac{1}{u}\frac{du}{dx} + \frac{1}{x} = 0.$$

Integrating, $\log u + \log x = 0$, $\therefore\ ux = 1$, and $u = 1/x$.

[We are not finding the general solution of the equation at this stage, but we want the simplest form of u which will satisfy our object of making the coefficient of v zero, hence we take the constant of integration as 0 instead of in the arbitrary form C.]

Now substituting $u = \dfrac{1}{x}$ in (i), we get $\dfrac{dv}{dx} = \dfrac{1}{x^2}v^2$.

$$\therefore\ \frac{1}{v^2}\frac{dv}{dx} = \frac{1}{x^2}, \qquad -\frac{1}{2v^2} = -\frac{1}{x} + C.$$

Since $v = y/u = yx$, we have

$$-\frac{1}{2y^2x^2} = -\frac{1}{x} + C, \quad \text{i.e.} \quad 1 = 2xy^2 - 2Cx^2y^2.$$

This is the same solution as before, except that the arbitrary constant occurs as $-2C$ instead of $+C$, which is immaterial.

Examples LXXXVII.

Solve the following equations:

1. $xy\dfrac{dy}{dx} + x^2 + y^2 = 0$.

2. $x^2\dfrac{dy}{dx} + xy = y^2$.

3. $x\dfrac{dy}{dx} + y + x = 0$.

4. $\dfrac{dy}{dx} = \dfrac{2x+y}{2y-x}$.

5. $x\dfrac{dy}{dx} + \sqrt{(x^2+y^2)} = y$.

6. $(x^2-y^2)\dfrac{dy}{dx} = 2xy$.

7. $\dfrac{dy}{dx} = \dfrac{y(y-2x)}{x(x-2y)}$.

8. $(y^3 - 3xy^2)\dfrac{dy}{dx} = y^3 + x^3$.

9. $(x+y+1)\dfrac{dy}{dx} = x - y + 1$.

10. $(3x+y-5)\dfrac{dy}{dx} = 2x + 2y - 2$.

11. $(x+y)\dfrac{dy}{dx} = x + y - 2$.

12. $(3x-5y)\dfrac{dy}{dx} = x - 3y + 2$.

13. $x\dfrac{dy}{dx} + y = x^3$.

14. $x\dfrac{dy}{dx} + 4y = x$.

15. $\dfrac{dy}{dx} + y\cot x = \operatorname{cosec} x$.

16. $\dfrac{dy}{dx} - y\tan x = \cos x$.

17. $\dfrac{dy}{dx} + y = e^x$.

18. $x\dfrac{dy}{dx} - y = xy^2$.

19. $\dfrac{dy}{dx} = y \tan x - y^2 \sec x$.

20. $\dfrac{dy}{dx} + y =$

21. $x\dfrac{dy}{dx} + y = x^3 y^4$.

22. $\dfrac{ay}{dx} + ay = \cos bx$.

23. In what curves is the subnormal at any point equal to half the sum of the coordinates of the point?

24. The current i in an electric circuit of resistance R, self-induction L, and E.M.F. E satisfies the equation $L\,di/dt + Ri = E$. Find i in terms of t when $L = \cdot 05$, $R = 10$, $E = 100 \sin 500t$.

25. A particle of mass 1 lb. moving horizontally in a medium whose resistance is $\cdot 1v$ lb. weight is subject to an accelerating force which at time t is equal to $4t$ lb. weight. Find its velocity after 1 second, if it starts from rest.

216. VI. Exact equations.

The equation

$$P\frac{dy}{dx} + Q = 0$$

is said to be an exact equation if the left-hand side is the differential coefficient of some function $f(x, y)$ with respect to x. When this is the case, the integral is obviously $f(x, y) = C$.

The condition which must be satisfied by P and Q in order that the equation may be exact will be investigated in Chapter XXIII, where we deal with partial differentiation. In the meantime, it can often be seen by inspection whether the equation is exact or not. In some cases, too, an integrating factor which will render the equation exact can be seen by inspection. Such a factor always exists, and there are various rules for finding it, but it is frequently very difficult to find.

Examples:

(i) $ax + hy + g + (hx + by + f)\dfrac{dy}{dx} = 0$.

This equation may be written $ax + h\left(y + x\dfrac{dy}{dx}\right) + by\dfrac{dy}{dx} + g + f\dfrac{dy}{dx} = 0$.

Integrating, $\quad \frac{1}{2}ax^2 + hxy + \frac{1}{2}by^2 + gx + fy = C$,

i.e. $\quad ax^2 + 2hxy + by^2 + 2gx + 2fy = 2C$.

(ii) $2y + x\,dy/dx = x^2$.

This becomes exact if multiplied by x, for then $2xy + x^2 dy/dx = x^3$. The left-hand side is now the d. c. of yx^2; $\therefore$ integrating, $yx^2 = \frac{1}{4}x^4 + C$.

(iii) $1 + x^2 y + x^3\,dy/dx = 0$.

This becomes exact if multiplied by $1/x^2$, for then $1/x^2 + y + x\,dy/dx = 0$, whence, on integration, $\quad -1/x + xy = C$.

(iv) $x\dfrac{dy}{dx} - y = 2x^2y\dfrac{dy}{dx}$.

This becomes exact if multiplied by $\dfrac{1}{x^2}$; for then $\left[x\dfrac{dy}{dx} - y\right] \Big/ x^2 = 2y\dfrac{dy}{dx}$.

The left-hand side is the d. c. of y/x with respect to x;

$\therefore$ integrating, $y/x = y^2 + C$, i.e. $y = xy^2 + Cx$.

217. VII. Equations of the first order, but not of the first degree.

If the equation is of the second degree in dy/dx, it can be solved as a quadratic for dy/dx, and the resulting simpler equations may be integrable by one of the preceding methods.

Examples:

(i) Solve the equation $\quad x(dy/dx)^2 = y + a$.

Taking the square root, $\quad \sqrt{x} \,.\, dy/dx = \pm\sqrt{(y+a)}$,

$$\therefore \quad \frac{1}{\sqrt{(y+a)}}\frac{dy}{dx} = \pm\frac{1}{\sqrt{x}}.$$

Integrating, $\quad 2\sqrt{(y+a)} = \pm 2\sqrt{x} + 2C$,

$$\therefore \quad y + a = (C \pm \sqrt{x})^2.$$

Since C is arbitrary, and may be $+$ or $-$, there is no need to retain the double sign in this case. This also follows from the fact that $(C \pm \sqrt{x})^2$ is the same as $(\sqrt{x} \pm C)^2$, from which it is obvious that the double sign is unnecessary.

Hence the solution is $\quad y + a = (C + \sqrt{x})^2$.

(ii) Solve $\quad \dfrac{dy}{dx}\left(\dfrac{dy}{dx} + x\right) = y(y + x)$,

i.e. $\quad \left(\dfrac{dy}{dx}\right)^2 - y^2 = -x\left(\dfrac{dy}{dx} - y\right)$.

On factorizing, either $\dfrac{dy}{dx} - y = 0$ or $\dfrac{dy}{dx} + y = -x$.

The first of these equations when solved gives $y = Ce^x$; the second gives $1 + x + y = Ae^x$ [see the last example of Art. 212].

These two equations constitute the complete solution; whatever value be assigned to C or A in either of these two equations, the resulting function satisfies the differential equation.

Geometrically, if we assign values to x and y, the differential equation, being a quadratic in dy/dx, gives two values of dy/dx, i.e. at any point (x, y) there are two directions for the tangent; in other words, two curves or two branches of one curve out of the system of curves given by the complete solution pass through any specified point in the plane (provided the values of dy/dx are real at that point).

In the example just worked out, one curve of the system $y = Ce^x$ and one curve of the system $1+x+y = Ae^x$ will pass through any given point (x, y); the values of dy/dx are always real in this case. For example, if we take the point $(0, 1)$, we find, on substituting these values in the solutions, $1 = C$ and $2 = A$; hence the curves $y = e^x$ and $1+x+y = 2e^x$ pass through the point $(0, 1)$. The values of dy/dx at this point are (i) $dy/dx = 1$, (ii) $dy/dx+1 = 0$; hence the tangents to the two curves are inclined to the axis of x at angles 45° and 135° respectively. The two curves therefore cut at right angles at the point $(0, 1)$.

(iii) *Find the equation of the curve which goes through the point* $(a, 0)$ *and has a normal of constant length* c.

The normal $= y \sec\psi = y\sqrt{[1+(dy/dx)^2]}$ [Art. 48],

$$\therefore \quad y^2[1+(dy/dx)^2] = c^2.$$

Hence $$\left(\frac{dy}{dx}\right)^2 = \frac{c^2}{y^2} - 1, \text{ and } \frac{y}{\sqrt{(c^2-y^2)}}\frac{dy}{dx} = \pm 1.$$

Integrating, $$-\sqrt{(c^2-y^2)} = \pm(x+A);$$

$$\therefore \quad c^2 - y^2 = (x+A)^2, \quad \text{i.e.} \quad (x+A)^2 + y^2 = c^2,$$

which represents a family of circles with their centres on the axis of x, and of radius c.

The fact that the curve is to go through $(a, 0)$ enables the value of A to be found, for substituting $x = a$, $y = 0$, we have $a+A = \pm c$ and $A = \pm c - a$.

Hence there are *two* circles satisfying the given conditions, viz.: $(x \pm c - a)^2 + y^2 = c^2$, agreeing with what was stated in the preceding example, since the differential equation is of the second degree in dy/dx. In this case, the result is obvious geometrically.

218. VIII. Clairaut's form.

This is the name given to the equation which takes the form

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right), \tag{i}$$

where $f(dy/dx)$ denotes any function of dy/dx only, i.e. a function not containing x or y explicitly.

It is usual, in differential equations, to denote dy/dx by p, so that the equation may be written $y = xp + f(p)$.

If the equation be differentiated with respect to x, we have

$$p = x\frac{dp}{dx} + p + f'(p)\frac{dp}{dx},$$

i.e. $$\frac{dp}{dx}[x + f'(p)] = 0.$$

Therefore either $dp/dx = 0$ or $x + f'(p) = 0$. (ii)

In the first case, $p =$ a constant c, and therefore, substituting in the given equation, we get $y = cx + f(c)$.

In the second case, another solution is obtained by eliminating p between equations (i) and (ii). Since the p is to be eliminated, it is immaterial what value it has, and therefore the result of the elimination is the same as the result of eliminating c between the equations $y = cx + f(c)$ and $x + f'(c) = 0$. But this result gives the envelope of the system $y = cx + f(c)$ [Art. 207] (since the second of the two equations is obtained by differentiating the first with respect to c).

Hence, the first solution represents a family of straight lines $y = cx + f(c)$, obtained by varying the arbitrary parameter c; the second solution represents their envelope. The latter is called a *singular solution*; it contains no arbitrary constant, neither can it be obtained from the general solution by assigning a particular value to the arbitrary constant c.

Geometrically, it is easily seen from Fig. 173 that $y - x\,dy/dx$ is the intercept made by the tangent on the axis of y. For

$$OT' = NP - KP = y - T'K \tan PT'K$$
$$= y - x \tan\theta = y - x\,dy/dx.$$

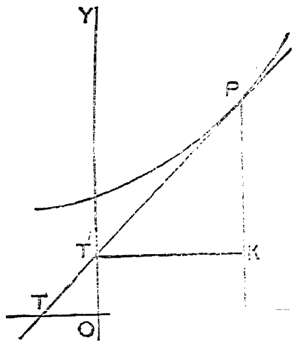


Fig. 173.

Hence the given differential equation may be interpreted geometrically as expressing the length of this intercept in terms of the slope. It is obvious that the given property is, at any point P, equally true for the curve itself and for the tangent to it at P (since the tangent and the curve have the same slope at P), i.e. it is true for the family of straight lines formed by the tangents and for the curve, their envelope.

Example. Solve the equation $y\dfrac{dy}{dx} = x\left(\dfrac{dy}{dx}\right)^2 + a$,

i.e. $$\frac{dy}{dx} + a \Big/ \frac{dy}{dx} = xp + a/p.$$

Differentiating with respect to p, we have $p = \dfrac{dy}{dx} = x\dfrac{dp}{dx} + p - \dfrac{a}{p^2}\dfrac{dp}{dx}$;

$$\frac{dp}{dx}\left(x - \frac{a}{p^2}\right) = 0, \text{ whence } \frac{dp}{dx} = 0 \text{ or } x \text{ :}$$

i.e. $$p = c \text{ or } p = \pm\sqrt{(a/x)}.$$

Eliminating p, in the first case we have $y = cx + a/c$, and in the second case $y = \pm x\sqrt{(a/x)} \pm a\sqrt{(x/a)} = \pm 2\sqrt{(ax)}$, i.e. $y^2 = 4ax$.

This is the singular solution; it is a parabola which is touched by all the straight lines of the family $y = cx + a/c$, which constitutes the general solution (Art. 211). The equation being of the second degree, two curves of the family pass through a given point, viz. the two tangents to the parabola from that point.

Examples LXXXVIII.

Solve the equations:

1. $x\dfrac{dy}{dx} + y = x^2.$ 2. $x\dfrac{dy}{dx} + y = y\dfrac{dy}{dx}.$

3. $(x^2+y^2)\dfrac{dy}{dx} + 2xy = 0.$ 4. $x\dfrac{dy}{dx} + ny = x.$

5. $y^2 + 2xy\dfrac{dy}{dx} = x.$ 6. $x^3\dfrac{dy}{dx} + x^2 y = 1.$

7. $x\dfrac{dy}{dx} - y = x^3.$ 8. $x\dfrac{dy}{dx} = y + x^2.$

9. $x\dfrac{dy}{dx} = x^2\dfrac{dy}{dx} + y.$ 10. $3x^2 - 2xy + y + (x - x^2 - 2y^2)\dfrac{dy}{dx}$

11. $(x + 2y^2)\dfrac{dy}{dx} = y.$ 12. $y - x\dfrac{dy}{dx} = x^2 y + xy^2\dfrac{dy}{dx}.$

13. In Ex. (i) of Art. 217, find the equations of the two curves of the system which go through the point $(4a, 8a)$, and find their slopes.

Solve the equations:

14. $\left(\dfrac{dy}{dx}\right)^2 - 7\dfrac{dy}{dx} + 12 = 0.$ 15. $\left(\dfrac{dy}{dx}\right)^2 = a^2$

16. $\left(\dfrac{dy}{dx}\right)^2 = x^2.$ 17. $\left(\dfrac{dy}{dx}\right)^2 = x\dfrac{dy}{dx}.$

18. $x\left(\dfrac{dy}{dx}\right)^2 - 2y\dfrac{dy}{dx} - x = 0.$ 19. $(2x+3)\left(\dfrac{dy}{dx}\right)^2 = 1.$

20. Solve the equation $xy\left(\dfrac{dy}{dx}\right)^2 - (x^2+y^2)\dfrac{dy}{dx} + xy = 0.$ [Factorize.]

Give the geometrical meaning of the answer. Find the equations of the curves which go through (3, 5), and find their slopes at that point. Prove that the tangent to the curve and the straight line of the system which go through any point make complementary angles with the axis.

21. Solve the equation $\dfrac{dy}{dx}\left(\dfrac{dy}{dx} + y\right) = x(x+y).$

Find the equations of the two curves of the system which pass through the origin, and their slopes at the origin.

22. Solve the equation $xy\left(\dfrac{dy}{dx}\right)^2 - (x^2-y^2)\dfrac{dy}{dx} - xy = 0.$

Give the geometrical meaning of the answer, and find the curves which go through the point (3, 2). Prove that the two curves which go through any point cut at right angles.

23. Solve $x\left(\dfrac{dy}{dx}\right)^2 + y\dfrac{dy}{dx} = x - y$, and explain the geometrical meaning of the result; at what angle do the two curves of the system intersect which pass through the point (2, 5)?

24. Find the curve in which the tangent cuts off from the coordinate axes a triangle of constant area A.

25. Find the curves in which the perpendicular from the foot of the ordinate to the tangent is constant and equal to a.

26. Find the curve in which the perpendicular from the origin to a tangent is constant and equal to a.

27. Find the curve, which goes through the origin, in which the square of the subnormal is equal to the rectangle contained by the abscissa and a line of given length a.

28. Solve the equation $\qquad y = x\dfrac{dy}{dx} + a\sqrt{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}.$

Explain the result geometrically.

29. Solve $y - x\dfrac{dy}{dx} = \left(\dfrac{dy}{dx}\right)^2.$ $\qquad$ 30. Solve $\left(y - x\dfrac{dy}{dx}\right)^2 = 4\dfrac{dy}{dx}.$

31. Solve $y\dfrac{dy}{dx} + a = x\left(\dfrac{dy}{dx}\right)^2.$

32. Find the curve in which the rectangle contained by the intercepts made on the axes by a tangent is constant (a^2).

33. Find the curve in which the sum of the squares of these intercepts is constant (a^2).

34. Find the curve in which the sum of these intercepts is constant (a).

35. Find the curve in which the intercept made by the tangent on the axis of y varies inversely as the slope.

36. Find the curve in which the intercept made by the tangent on the axis of x varies as the slope.

219. Equations of the second order.

We will now consider some of simpler types of equations of the second order.

I. $$\frac{d^2y}{dx^2} = f(x), \text{ a function of } x \text{ only.}$$

In this case two direct integrations give the solution.

The first gives $\quad dy/dx = \int f(x)\,dx + A = F(x) + A$, say.

The second gives $\quad y = \int F(x)\,dx + Ax + B.$

This is the general solution containing two arbitrary constants A and B.

Example. If $d^2y/dx^2 = \sin x$,

then $dy/dx = -\cos x + A$, and $y = -\sin x + Ax + B$.

II. $$\frac{d^2y}{dx^2} = f(y), \text{ a function of } y \text{ only.}$$

Simple equations of this type have already been solved in Chapter XIX.

Denoting dy/dx by p, d^2y/dx^2 will be dp/dx, which may be written

$$\frac{dp}{dy} \times \frac{dy}{dx}, \text{ i.e. } p\frac{dp}{dy}.$$

Hence the equation may be written $p\,dp/dy = f(y)$, which, on being integrated with respect to y, gives $\frac{1}{2}p^2 = \int f(y)\,dy + A$.

$$\therefore \quad dy/dx = p = [2\int f(y)\,dy + 2A]^{\frac{1}{2}} = F(y), \text{ say,}$$

i.e. $$\frac{1}{F(y)}\frac{dy}{dx} = 1.$$

Integrating again with respect to x, this gives

$$\int \frac{dy}{F(y)} = x + B.$$

This is the complete solution, containing two arbitrary constants, the A involved in the $F(y)$, and B.

The first stage of the solution may be put in the following form:

Since the d. c. of $\left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx} \times$ d. c. of $\frac{dy}{dx} = 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}$, multiply the given equation by $2\,dy/dx$; this gives

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx}.$$

Therefore, integrating with respect to x,

$$(dy/dx)^2 = 2\int f(y)\,dy + A, \text{ as before.}$$

Example. $d^2y/dx^2 + a^2y = 0$. [See Arts. 187, 192.]

Multiply by $2\frac{dy}{dx}$; then $2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + a^2.2y\frac{dy}{dx} = 0$.

Integrating, $\quad (dy/dx)^2 + a^2y^2 = C = a^2c^2$, say.

This is a more convenient form, c being now the arbitrary constant;

$$\therefore \quad dy/dx = \pm a\sqrt{(c^2 - y^2)},$$

i.e. $$\frac{1}{\sqrt{(c^2-y^2)}}\frac{dy}{dx} = \pm a.$$

Integrating again $\quad \sin^{-1}(y/c) = \pm ax + \alpha,$

$$\therefore \quad y = c\sin(\pm ax + \alpha).$$

This is the general solution containing the two arbitrary constants c and α.

The solution may be written in the form

$$y = \pm c\sin ax\cos\alpha + c\cos ax\sin\alpha,$$

i.e. $$y = A\sin ax + B\cos ax,$$

replacing the two arbitrary constants $\pm c\cos\alpha$ and $c\sin\alpha$ by A and B.

III. A differential equation containing d^2y/dx^2, dy/dx, and *one* only of the variables x and y can be reduced to one of the first order (which may then admit of being integrated by one of the methods of Arts. 212–218) by making use of the substitution mentioned above, p for dy/dx, as shown in the following examples:

Examples:

(i) Let y be absent.

Solve $$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+1=0.$$

Putting $\dfrac{dy}{dx}=p$, $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}$, the equation becomes $x\dfrac{dp}{dx}+p+1=0$, an equation between p and x of the first order with the variables separable.

It may be written $$\frac{1}{p+1}\frac{dp}{dx}=-\frac{1}{x};$$

$$\therefore\ \log(p+1)=-\log x+\log C,\quad\text{or}\quad p+1=C/x.$$

$$\frac{dy}{dx}=p=\frac{C}{x}-1,$$

and integrating again, $$y=C\log x-x+A.$$

(ii) Let x be absent.

Solve $$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+a^2=0.$$

Putting $\dfrac{dy}{dx}=p$, and $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}=\dfrac{dp}{dy}\cdot\dfrac{dy}{dx}=p\dfrac{dp}{dy}$, the equation becomes $yp\dfrac{dp}{dy}+p^2+a^2=0$, in which the variables are separable.

It may be written $$\frac{2p}{p^2+a^2}\cdot\frac{dp}{dy}=-\frac{2}{y};$$

$$\log(p^2+a^2)=\log C-2\log y,\quad\text{or}\quad p^2+a^2=C/y^2.$$

$$\therefore\ \frac{dy}{dx}=p=\pm\sqrt{\left(\frac{C}{y^2}-a^2\right)}=\pm\frac{\sqrt{(C-a^2y^2)}}{y},$$

i.e. $$\frac{y}{\sqrt{(C-a^2y^2)}}\frac{dy}{dx}=\pm 1.$$

Integrating again, $\quad -\sqrt{(C-a^2y^2)}/a^2=\pm x+A;$

$\therefore\ C-a^2y^2=a^4(x+A)^2$, the $\pm$ sign being unnecessary;

or, if the first constant be written as Ca^2 instead of C,

$$C=y^2+a^2(x+A)^2.$$

(iii) *Find the curves in which the radius of curvature is double the normal and on the same side of the curve.*

The length of the normal $=y\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{1/2}$ [Art. 48]; the radius of curvature, ρ, $=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}\Big/\dfrac{d^2y}{dx^2}$, and is positive when the curve is

above the tangent. Since the radius of curvature and the normal are on the same side of the curve, the curve must be concave towards the axis of x. If the curve is on the positive side of the axis of x, y and the length of the normal will be positive (taking the positive root), and ρ, being below the tangent, will be negative; if the curve be on the negative side of the axis of x, y and the length of the normal will be negative, and the radius of curvature will be above the tangent and positive. Hence in both cases the signs of the lengths of the normal and the radius of curvature are different, and we have

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2} = -2y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}.$$

Dividing by $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$, and putting $\frac{dy}{dx} = p$, $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$, we get

$$1+p^2 = -2yp\frac{dp}{dy}; \quad \therefore \quad \frac{2p}{1+p^2}\frac{dp}{dy} = -\frac{1}{y}.$$

Integrating, $\log(1+p^2) = -\log y + \log a$, whence $1+p^2 = a/y$,

and
$$\frac{dy}{dx} = p = \sqrt{\left(\frac{a}{y}-1\right)} = \sqrt{\left(\frac{a-y}{y}\right)}.$$

To rationalize this, put $y = a\sin^2\theta$; $\quad \frac{dy}{dx} = 2a\sin\theta\cos\theta\frac{d\theta}{dx}$,

and the equation becomes

$$2a\sin\theta\cos\theta\frac{d\theta}{dx} = \sqrt{\left(\frac{a\cos^2\theta}{a\sin^2\theta}\right)} = \pm\frac{\cos\theta}{\sin\theta},$$

$$2a\sin^2\theta\frac{d\theta}{dx} = \pm 1, \qquad a(1-\cos 2\theta)\frac{d\theta}{dx} = \pm 1.$$

Integrating again, $\quad a(\theta - \frac{1}{2}\sin 2\theta) = \pm x + A$;

therefore $\quad \pm x + A = \frac{1}{2}a(2\theta - \sin 2\theta)$;

also $\quad y = a\sin^2\theta = \frac{1}{2}a(1-\cos 2\theta)$.

These two equations give the coordinates of any point on a cycloid [Art. 50]. Hence the curves which possess the given property are cycloids. [See Art. 199, Ex. (iv).]

The θ in these equations is half the angle turned through by the radius through the tracing-point on the rolling circle. A change in the value of a, the first arbitrary constant, alters the radius of this circle and therefore the size of the cycloid and, with it, the actual lengths of the normal and radius of curvature. A change in the value of the other arbitrary constant A slides the cycloid along the axis of x, an operation which obviously would not affect the lengths of the normal and the radius of curvature.

Examples LXXXIX.

Solve the following equations:

$\frac{d^2y}{dx^2} = x^n$. $\qquad$ 2. $x\frac{d^2y}{dx^2} = 1$. $\qquad$ 3. $\frac{d^2y}{dx^2} = a^2\sin^2 x$.

$\frac{d^2y}{dx^2} = 4y$. $\qquad$ 5. $e^{2y}\frac{d^2y}{dx^2} = a^2$. $\qquad$ 6. $\frac{d^2y}{dx^2} = \frac{a^2}{y^3}$.

7. $x\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}$.

8. $y\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx}\right)^2$.

. $\dfrac{d^2V}{dr^2} + \dfrac{1}{r}\dfrac{dV}{dr} = 0$.

10. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 2x$.

11. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 =$

12. $y\dfrac{d^2y}{} + \left(\dfrac{dy}{a}\right)^2 = 2y^2$.

13. $1 + \left(\dfrac{dy}{dx}\right)^2 = x\dfrac{dy}{dx} \quad \dfrac{d^2y}{dx^2}$.

14. $\left(\dfrac{d^2y}{dx^2}\right)^2 = 4\dfrac{dy}{dx}$.

15. $\dfrac{d^2y}{dx^2} : \dfrac{dy}{dx}$.

16. $a\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$.

17. $y^3\dfrac{d^2y}{dx^2} = -1$.

18. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 1 = x$.

19. $8\dfrac{dy}{dx}\cdot\dfrac{d^2y}{dx^2} = 9$.

20. $\dfrac{d^2u}{d\theta^2} + u = \dfrac{k}{h^2u^3}$.

21. Find the curves in which the radius of curvature is constant.

22. Find the curves in which the radius of curvature is equal to the normal, but on the opposite side of the curve.

23. Find the curves in which the radius of curvature varies as the cube of the normal.

24. Find the curves in which the radius of curvature is double the normal, and on the opposite side of the curve.

220. Linear equation of the second order, with constant coefficients.

This equation is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = P,$$

where a, b, c are constants, and P is a function of x.

We shall commence by proving one or two general theorems about the solutions of such equations. These theorems express properties which are true for linear equations of any order, and which are also true when the coefficients are functions of x. It will be obvious that the following proofs will hold when a, b, c are functions of x just as when they are constants; no assumption is made as to their nature in the working except that they do not contain y. The method of proof is also exactly the same for equations of higher order, but in that case the equations will contain more terms.

I. If $u+v$ be substituted for y in the equation, it becomes

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu + a\frac{d^2v}{dx^2} + b\frac{dv}{dx} + cv = P.$$

If v be *any* solution of the given equation (not the general solution, but any particular solution, the simpler the better), so that

$$a\frac{d^2v}{dx^2}+b\frac{dv}{dx}+cv=P,$$

then we have, by subtraction,

$$a\frac{d^2u}{dx^2}+b\frac{du}{dx}+cu=0;$$

i. e. u satisfies the original equation with the right-hand side P replaced by zero. The general solution of this equation will contain two arbitrary constants. If then this general solution u can be found, and also the particular solution v of the original equation just mentioned, $y=u+v$ will give the general solution of the original equation.

Of these two functions, v is called the *particular integral*, and u the *complementary function*.

The problem is now reduced to finding any solution whatever of the given equation, and the general solution of this equation with its right-hand side replaced by zero.

II. Next, if u_1 and u_2 be any two independent particular solutions of the equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cu=0$, then $y=Au_1+Bu_2$, where A and B are arbitrary constants, will also be a solution. For, substituting Au_1+Bu_2 for y in the left-hand side of the equation, it becomes

$$a\left(A\frac{d^2u_1}{dx^2}+B\frac{d^2u_2}{dx^2}\right)+b\left(A\frac{du_1}{dx}+B\frac{du_2}{dx}\right)+c(Au_1+Bu_2),$$

i.e. $$A\left(a\frac{d^2u_1}{dx^2}+b\frac{du_1}{dx}+cu_1\right)+B\left(a\frac{d^2u_2}{dx^2}+b\frac{du_2}{dx}+cu_2\right).$$

Since u_1 and u_2 are both particular solutions, the contents of each bracket are equal to zero, and therefore the equation is satisfied;

$$\therefore\quad y=Au_1+Bu_2$$

is a solution. This will be the complementary function.

Hence, summing up the results of these two theorems, it follows that, if v be a particular solution of the given equation, and u_1, u_2 particular independent solutions of the equation when the right-hand side P is replaced by zero, the complete solution is $y=Au_1+Bu_2+v$, where A and B are arbitrary constants.

Similarly, in the linear equation of the n^{th} order, if v be a particular solution of the given equation, and if $u_1, u_2 \ldots u_n$ be n independent

particular solutions of the equation with the right-hand side zero, the complete solution will be

$$y = Au_1 + Bu_2 + \ldots + Ku_n + v.$$

The methods of finding the complementary function and the particular integral will now be considered.

221. Method of finding the complementary function [C. F.].

The equation to be solved is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Two particular solutions are needed. Try $y = e^{mx}$.

[This substitution is suggested by noticing that each term of the equation will then become a multiple of e^{mx}, and by suitably choosing m, the sum of the coefficients of e^{mx} after the substitution may be made to vanish, and the equation will then be satisfied.

Moreover, in the corresponding equation of the first degree,

$$a\frac{dy}{dx} + by = 0, \text{ we have } \frac{1}{y}\frac{dy}{dx} + \frac{b}{a} = 0.$$

$\therefore \log y + bx/a = \log C$, whence $ye^{bx/a} = C$, and $y = Ce^{-bx/a}$, a solution of the above type.]

The equation becomes $(am^2 + bm + c)e^{mx} = 0$, which is satisfied if $am^2 + bm + c = 0$.

This equation for m is often referred to as *the auxiliary equation*. In this, and in the general case of an equation of the n^{th} order, it can be written down at once by substituting $1, m, m^2, \ldots m^n$ for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}$ respectively in the given differential equation.

In this case we have a quadratic which gives two values of m for which e^{mx} is a solution. There are three cases which may arise.

(i) *Let the roots be real and different*, m_1 and m_2, say.

Then e^{m_1x} and e^{m_2x} are particular solutions of the equation; hence the C. F. is $\qquad y = Ae^{m_1x} + Be^{m_2x}$.

(ii) *Let the roots be real and equal*, each m_1.

The preceding result becomes $y = Ae^{m_1x} + Be^{m_1x} = (A + B)e^{m_1x}$, which is no longer the general solution, because $A + B$ is equivalent merely to a single arbitrary constant C; but it suggests that e^{m_1x} may be a factor of the solution. Therefore try the substitution $y = e^{m_1x}z$, where z of course is a function of x.

The quadratic whose roots are both m_1 is $m^2-2m_1 m+m_1^2=0$; hence the corresponding differential equation is

$$\frac{d^2y}{dx^2}-2m_1\frac{dy}{dx}+m_1^2y=0.$$

Substitute $y=e^{m_1x}z$; $\therefore$ $\dfrac{dy}{dx}=e^{m_1x}\dfrac{dz}{dx}+zm_1e^{m_1x}$,

and $$\frac{d^2y}{dx^2}=e^{m_1x}\frac{d^2z}{dx^2}+\frac{dz}{dx}m_1e^{m_1x}+\frac{dz}{dx}m_1e^{m_1x}+zm_1^2e^{m_1x}.$$

The differential equation becomes

$$e^{m_1x}\left[\frac{d^2z}{dx^2}+2m_1\frac{dz}{dx}+m_1^2z-2m_1\frac{dz}{dx}-2m_1^2z+m_1^2z\right]=0,$$

i. e. $$e^{m_1x}\frac{d^2z}{dx^2}=0,\quad\text{whence}\quad\frac{d^2z}{dx^2}=0.$$

Therefore, integrating twice, $z=Ax+B$;
and the C. F. is $y=e^{m_1x}z=e^{m_1x}(Ax+B)$.

Similarly, if, in solving a differential equation of higher order, the auxiliary equation has three equal roots, three of the particular solutions will coalesce, and it follows, by exactly similar reasoning, that the corresponding part of the C.F. is

$$e^{m_1x}(Ax^2+Bx+C).$$

Similarly for any number of equal roots.

(iii) *Let the roots be imaginary*, $m_1\pm m_2i$ (where $i=\sqrt{-1}$), say.

Then the C. F. takes the form

$$y=Ae^{m_1x+m_2ix}+Be^{m_1x-m_2ix}.$$

This expression, involving imaginaries, is an inconvenient form, especially in practical applications, and the result can be expressed otherwise as follows:

It is clear that e^{m_1x} is a factor of this solution, therefore, as in the preceding case, put $y=e^{m_1x}z$; z will be a function of x.

The quadratic whose roots are $m_1\pm m_2i$ is

$$m^2-2m_1m+m_1^2+m_2^2=0;$$

hence the corresponding differential equation is

$$\frac{d^2y}{dx^2}-2m_1\frac{dy}{dx}+(m_1^2+m_2^2)y=0.$$

Substituting $y=e^{m_1x}z$, and the values of dy/dx and d^2y/dx^2, as in the preceding case, we get

$$e^{m_1x}\left[\frac{d^2z}{dx^2}+2m_1\frac{dz}{dx}+m_1^2z-2m_1\frac{dz}{dx}-2m_1^2z+m_1^2z+m_2^2z\right]=0,$$

i.e. $$e^{m_1 x}\left[\frac{d^2 z}{dx^2}+m_2^2 z\right]=0,$$

$$\therefore \quad \frac{d^2 z}{dx^2}+m_2^2 z=0.$$

The general solution of this equation is (Art. 219 II)

$$z=C\sin(m_2 x+\alpha), \quad \text{or} \quad z=A\sin m_2 x+B\cos m_2 x;$$

and therefore the C.F. is

$$y=e^{m_1 x}z=Ce^{m_1 x}\sin(m_2 x+\alpha).$$

Hence, summing up these results:

(i) If the roots of the auxiliary equation are real and different, m_1 and m_2, the C.F. is $y=Ae^{m_1 x}+Be^{m_2 x}$;

(ii) If the roots of the auxiliary equation are equal, each m_1, the C.F. is $$y=e^{m_1 x}(Ax+B);$$

(iii) If the roots of the auxiliary equation are imaginary, $m_1 \pm m_2 i$, the C. F. is $$y=e^{m_1 x}(A\sin m_2 x+B\cos m_2 x)$$
$$=Ce^{m_1 x}\sin(m_2 x+\alpha) \text{ or } C'e^{m_1 x}\cos(m_2 x-\alpha').$$

Examples:

(i) $\dfrac{d^2 y}{dx^2}-7\dfrac{dy}{dx}+12y=0.$

The auxiliary equation is $m^2-7m+12=0$, whence $m=3$ or 4, and the solution is $$y=Ae^{3x}+Be^{4x}.$$

(ii) $\dfrac{d^2 y}{dx^2}+4\dfrac{dy}{dx}+4y=0.$

The auxiliary equation is $m^2+4m+4=0$, which has two roots, each -2. Hence the solution is $y=e^{-2x}(A+Bx)$.

(iii) $\dfrac{d^2 y}{dx^2}+2\dfrac{dy}{dx}+5y=0.$

The auxiliary equation is $m^2+2m+5=0$, whence $m=-1\pm 2i$.
Hence the solution is $y=e^{-x}(A\sin 2x+B\cos 2x)=Ce^{-x}\cos(2x-\alpha)$.

(iv) $2\dfrac{d^2 y}{dx^2}-3\dfrac{dy}{dx}+4y=0.$

The auxiliary equation is $2m^2-3m+4=0$, which has the roots

$$\tfrac{3}{4}\pm\tfrac{1}{4}\sqrt{-23}, \quad \text{i.e. } \tfrac{3}{4}\pm\tfrac{1}{4}i\sqrt{23};$$

hence the solution is $y=Ce^{\frac{3}{4}x}\cos(\frac{1}{4}\sqrt{23}\,x-\alpha)$

(v) $\dfrac{d^3 y}{dx^3}+2\dfrac{d^2 y}{dx^2}+\dfrac{dy}{dx}=0.$

The auxiliary equation is $m^3+2m^2+m=0$, which has roots $0, -1, -1$; hence the solution is $y=Ae^{0x}+e^{-x}(B+Cx)=A+e^{-x}(B+Cx)$.

(vi) $\dfrac{d^4y}{dx^4}+4\dfrac{d^2y}{dx^2}=0.$

The auxiliary equation is $m^4+4m^2=0$, which has roots 0, 0, $\pm 2i$. Hence, combining the results of (ii) and (iii), the solution is

$$\begin{aligned} y &= e^{0x}(A+Bx)+e^{0x}(C\sin 2x+D\cos 2x) \\ &= A+Bx+E\cos(2x-\alpha). \end{aligned}$$

Examples XC.

Solve, by aid of the summary given above, the equations:

1. $\dfrac{d^2y}{dx^2}-10\dfrac{dy}{dx}+16y \quad 0.$

2. $\dfrac{d^2y}{dx^2}-16y=0.$

3. $\dfrac{d^2y}{dx^2}-10\dfrac{dy}{dx}+25y=0.$

4. $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+9y =$

5. $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+10y=0.$

6. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=0.$

7. $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-10y=0.$

8. $4\dfrac{d^2y}{dx^2}-9y=0.$

9. $4\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-5y=0.$

10. $4\dfrac{d^2y}{dx^2}+12\dfrac{dy}{dx}+9y=0.$

11. $\dfrac{d^3y}{dx^3}=\dfrac{dy}{dx}.$

12. $\dfrac{d^3y}{dx^3}+\dfrac{dy}{dx}=0.$

13. $\dfrac{d^4y}{dx^4}=16y.$

14. $\dfrac{d^4y}{dx^4}=\dfrac{dy}{dx}.$

15. $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}=0.$

16. $\dfrac{d^2y}{dx^2}+6y=2\dfrac{dy}{dx}.$

222. Method of finding the particular integral [P.I.].

We have to find *any* particular solution of the equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=P.$$

Frequently a solution can be found by trial, as shown in the following examples, which include some of the simplest and most useful cases.

(i) Let P be a constant, C.

Then a particular solution is obviously $y=C/c$, since all the d.c.'s of this are zero.

(ii) Let P be a rational integral function of x, i.e. let P be of the form $p+qx+rx^2+\ldots$, where $p, q, r \ldots$ are constants.

The only functions of x whose differential coefficients are positive integral powers of x are themselves positive integral powers of x. Hence assume

$$y=A+Bx+Cx^2+\ldots.$$

Clearly, the degree of the expression assumed for y must (if $c \neq 0$) be equal to the degree of P, since dy/dx, d^2y/dx^2, ... are of lower degree than y, and hence the highest power of x in y cannot be cancelled out, and therefore must occur in P. Substituting this value of y in the differential equation and comparing the coefficients of the different powers of x, equations are obtained from which the values of the coefficients A, B, ... can be found. If $c = 0$, dy/dx must be of the same degree as P, and so on.

Example. Solve the equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 3 - 2x^2$.

Since the right-hand side is of the second degree in x, take

$$y = A + Bx + Cx^2.$$

Substituting in the differential equation, we get

$$2C - 3(B + 2Cx) + 2(A + Bx + Cx^2) = 3 - 2x^2.$$

Comparing (i) coefficients of x^2: $\quad 2C = -2, \quad C = -1;$

(ii) coefficients of x: $\quad -6C + 2B = 0, \quad B = -3;$

(iii) constant terms: $\quad 2C - 3B + 2A = 3, \quad A = -2.$

Hence $\quad y = -2 - 3x - x^2$

satisfies the equation, and this is the particular integral. The C.F. is, by Art. 221, $Ae^x + Be^{2x}$, and therefore the complete solution of the given equation is

$$y = Ae^x + Be^{2x} - 2 - 3x - x^2.$$

(iii) Let P be of the form Ce^{mx}.

Since all the d.c.'s of e^{mx} are multiples of e^{mx}, we assume $y = ke^{mx}$. Substituting in the differential equation, it becomes

$$(am^2 + bm + c)\,ke^{mx} = Ce^{mx},$$

whence $\quad k = C/(am^2 + bm + c).$

This fails if e^{mx} be a term of the C.F., for then the coefficient of k in the preceding equation becomes zero. In this case, substitute $y = kxe^{mx}$; this fails in a similar manner if xe^{mx} be a term of the C.F. If this be so, substitute $y = kx^2e^{mx}$, and so on.

Examples:

(i) Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2e^{-2x}$.

The C.F. is $y = Ae^x + Be^{2x}$, and the right-hand side of the given equation is not a term of this expression; therefore put $y = ke^{-2x}$.

This gives, on substitution in the differential equation,

$$4ke^{-2x} - 3(-2k)\,e^{-2x} + 2ke^{-2x} = 2e^{-2x};$$

$$\therefore \quad 12k = 2, \quad \text{and} \quad k = \tfrac{1}{6}.$$

Hence the P.I. is $\frac{1}{6}e^{-2x}$, and the complete solution is

$$y = Ae^x + Be^{2x} + \tfrac{1}{6}e^{-2x}.$$

(ii) Let the right-hand side of the preceding equation be $2e^{2x}$.

Since this is a term of the C. F., we put $y = kxe^{2x}$;

$\therefore\ dy/dx = k(2xe^{2x}+e^{2x})$; $d^2y/dx^2 = k(4xe^{2x}+2e^{2x}+2e^{2x})$.

Substituting these in the differential equation, it becomes

$$ke^{2x}[4x+4-6x-3+2x] = 2e^{2x},$$

whence $k=2$, and the P. I. is $2xe^{2x}$.

The complete solution of the equation is

$$y = Ae^{x}+Be^{2x}+2xe^{2x}.$$

(iii) Solve the equation $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y = e^{x}$.

The C. F. is $y = e^{x}(A+Bx)$.

Since e^x and xe^x are both terms of the C.F., to find the P.I. we must substitute $y = kx^2e^x$; therefore

$dy/dx = k(x^2e^x+2xe^x)$, and $d^2y/dx^2 = k(x^2e^x+2xe^x+2xe^x+2e^x)$.

The equation becomes

$$ke^x[x^2+4x+2-2x^2-4x+x^2] = e^x, \text{ whence } k=\tfrac{1}{2}.$$

The P.I. is $\frac{1}{2}x^2e^x$, and the complete solution is

$$y = e^x(A+Bx)+\tfrac{1}{2}x^2e^x = e^x(A+Bx+\tfrac{1}{2}x^2).$$

(iv) Let P be of the form $C\sin nx + D\cos nx$ [either C or D may be zero].

Since all the differential coefficients of $\sin nx$ and $\cos nx$ are multiples of either $\sin nx$ or $\cos nx$, we assume $y = k\sin nx + l\cos nx$. Substituting in the differential equation and comparing coefficients of $\sin nx$ and $\cos nx$ on both sides, we get two equations to determine k and l. As in the preceding case, if the C.F. contains terms of the form $A\sin nx + B\cos nx$, the substitution fails and, as before, we then put $y = kx\sin nx + lx\cos nx$.

Examples:

(i) Solve $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y = 5\sin 2x$.

The C. F. is Ae^x+Be^{2x}.

To find the P. I., put $y = k\sin 2x + l\cos 2x$. The equation becomes

$$-4k\sin 2x-4l\cos 2x-3[2k\cos 2x-2l\sin 2x]+2[k\sin 2x+l\cos 2x] = 5\sin 2x,$$

e. $\sin 2x[-4k+6l+2k]+\cos 2x[-4l-6k+2l] = 5\sin 2x$.

Comparing coefficients of $\sin 2x$ and $\cos 2x$,

$$-2k+6l=5, \qquad -2l-6k = 0; \text{ whence } k=-\tfrac{1}{4} \text{ and } l=\tfrac{3}{4}.$$

The P. I. is $-\frac{1}{4}\sin 2x+\frac{3}{4}\cos 2x$, and the complete solution is

$$= Ae^x+Be^{2x}-\tfrac{1}{4}\sin 2x+\tfrac{3}{4}\cos 2x.$$

(ii) Solve $d^2y/dx^2+4y = \cos 2x$.

The C. F. is $A\cos 2x+B\sin 2x$.

Since $\cos 2x$ is a term of the C. F., put $y = kx\sin 2x+lx\cos 2x$.

$\therefore\quad dy/dx = 2kx\cos 2x+k\sin 2x-2lx\sin 2x+l\cos 2x,$

and $\quad d^2y/dx^2 = -4kx\sin 2x+4k\cos 2x-4lx\cos 2x-4l\sin 2x.$

Substituting in the differential equation, it becomes

$$-4kx\sin 2x+4k\cos 2x-4lx\cos 2x-4l\sin 2x+4kx\sin 2x+4lx\cos 2x = \cos 2x.$$

Comparing coefficients of $\sin 2x$ and $\cos 2x$, $4l=0$, $4k=1$; $l=0$ and $k=\frac{1}{4}$.

Hence the P. I. is $\frac{1}{4}x\sin 2x$, and the complete solution is

$$y = A\cos 2x+B\sin 2x+\tfrac{1}{4}x\sin 2x.$$

(v) Let P be the sum of several terms of the preceding types. To find the P. I. in this case, find the part of it corresponding to each term separately, and add the results together.

223. Applications of the preceding results. Damped harmonic motion.

The equation

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+\mu x = a\cos(nt+\alpha)$$

is of considerable importance in dynamics and electricity. It is a linear equation of the second order with constant coefficients.

If the right-hand side is zero, it is the equation of motion of a simple pendulum making small oscillations under gravity in a medium of which the resistance varies as the velocity. For if, in Art. 194, the motion be supposed to take place in such a medium, and if the resistance to a particle of mass m be written in the form kmv, i.e. $kml\,d\theta/dt$, the equation there given becomes

$$ml\frac{d^2\theta}{dt^2} = -kml\frac{d\theta}{dt}-mg\sin\theta,$$

i.e.

$$\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+\frac{g}{l}\sin\theta = 0;$$

which, when θ is so small that $(\sin\theta)/\theta$ may be taken as unity, becomes

$$\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+\frac{g}{l}\theta = 0.$$

It is convenient to write n^2 for g/l, so that the equation may be written

$$\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+n^2\theta = 0.$$

The same equation also represents the motion of the needle of a galvanometer, the resistance of the air being supposed proportional to the velocity, which is approximately true when the velocity is not very large.

Since the right-hand side is zero, there is no P.I. To find the C.F., the auxiliary equation is

$$m^2+km+n^2=0, \quad \text{of which the roots are } m=-\tfrac{1}{2}k\pm\sqrt{(\tfrac{1}{4}k^2-n^2)}.$$

The nature of the motion depends upon the value of $\frac{1}{4}k^2-n^2$.

(i) If $\frac{1}{4}k^2>n^2$, the roots are real and different; denoting them by m_1 and m_2, the solution of the equation is

$$\theta=Ae^{m_1 t}+Be^{m_2 t}.$$

Since k is + and $\frac{1}{4}k^2-n^2<\frac{1}{4}k^2$, and therefore $\sqrt{(\frac{1}{4}k^2-n^2)}<\frac{1}{2}k$, it follows that m_1 and m_2 are both −.

$\therefore$ $e^{m_1 t}$ and $e^{m_2 t}$ both $\to 0$ as t increases, and therefore $\theta\to 0$ as t increases. The particle does not oscillate but gradually approaches the position of equilibrium.*

Suppose the particle starts from rest with θ initially equal to α.

Then, since $\theta=\alpha$ when $t=0$, $\alpha=A+B$.

Now $$d\theta/dt=Am_1e^{m_1t}+Bm_2e^{m_2t};$$

and $d\theta/dt=0$ when $t=0$. $\quad\therefore\quad 0=Am_1+Bm_2$.

From these two equations, $A=\dfrac{m_2\alpha}{m_2-m_1}$ and $B=\dfrac{-m_1\alpha}{m_2-m_1}$.

$$\frac{\alpha}{m_2-m_1}(m_2e^{m_1t}-m_1e^{m_2t}).$$

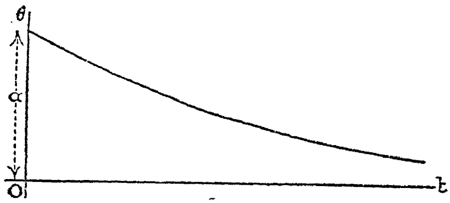


Fig. 174.

If we denote $+\sqrt{(\frac{1}{4}k^2-n^2)}$ by p, $m_2=-\frac{1}{2}k+p$, $m_1=-\frac{1}{2}k-p$, and $m_2-m_1=2p$.

$$\therefore\quad \theta=\frac{\alpha}{2p}[(-\tfrac{1}{2}k+p)e^{-\frac{1}{2}kt-pt}-(-\tfrac{1}{2}k-p)e^{-\frac{1}{2}kt+pt}]$$

$$=\frac{\alpha e^{-\frac{1}{2}kt}}{2p}[(\tfrac{1}{2}k+p)e^{pt}-(\tfrac{1}{2}k-p)e^{-pt}].$$

Since p and k are +, it follows that $\frac{1}{2}k+p>\frac{1}{2}k-p$ and $e^{pt}>e^{-pt}$; therefore the first term in the bracket is greater than the second, so that θ is always +, and $\to 0$. In this case the particle never passes through the position of equilibrium given by $\theta=0$. It gradually approaches it but never quite reaches it.

* It may first pass through it once, since, under some initial conditions, there may be one value of t for which $\theta=0$.

If we draw the displacement-time graph of the motion, the curve starts at the point $(0, \alpha)$ and constantly descends towards the axis of t, which is an asymptote (Fig. 174).

The motion is of this nature if $k^2 > 4n^2$, i.e. if $k > 2n$; i.e., in the case of the pendulum mentioned above, if $k > 2\sqrt{(g/l)}$. This is also the character of the motion in the case of a dead-beat galvanometer.

(ii) If $\frac{1}{4}k^2 = n^2$, the roots are real and equal; each is $-\frac{1}{2}k$.

$$\therefore \quad \theta = e^{-\frac{1}{2}kt}(A + Bt).$$

If the particle starts from rest with $\theta = \alpha$, then substituting α for θ and 0 for t in this equation, we have $\alpha = A$.

Also $\quad \dot{\theta} = e^{-\frac{1}{2}kt}B - \frac{1}{2}ke^{-\frac{1}{2}kt}(A + Bt) = e^{-\frac{1}{2}kt}[B - \frac{1}{2}kA - \frac{1}{2}kBt]$.

Since $\quad \dot{\theta} = 0$ when $t = 0$, $\quad 0 = B - \frac{1}{2}kA$; $\quad \therefore \quad B = \frac{1}{2}k\alpha$.

Therefore $\quad \theta = e^{-\frac{1}{2}kt}(\alpha + \frac{1}{2}k\alpha t) = \alpha e^{-\frac{1}{2}kt}(1 + \frac{1}{2}kt)$.

Hence, in this case too, θ tends to the value 0, but never reaches it, since $e^{-\frac{1}{2}kt}$ and $1 + \frac{1}{2}kt$ are always $+$.

The graph of the motion is similar to that in the preceding case.

(iii) If $\frac{1}{4}k^2 < n^2$, the roots are imaginary. Denoting $n^2 - \frac{1}{4}k^2$ by p^2, they may be written in the form $-\frac{1}{2}k \pm \sqrt{-p^2}$, i.e. $-\frac{1}{2}k \pm pi$, where $i = \sqrt{-1}$.

In this case the solution of the equation is (Art. 221)

$$\theta = Ce^{-\frac{1}{2}kt}\cos(pt - \epsilon). \tag{i}$$

The particle passes through the position of equilibrium, $\theta = 0$, when $\cos(pt - \epsilon) = 0$, i.e. when $pt - \epsilon$ is equal to any odd multiple of $\frac{1}{2}\pi$.

When $pt - \epsilon = \frac{1}{2}\pi$, the position of equilibrium is reached for the first time; when $pt - \epsilon = \frac{3}{2}\pi$, for the second time, moving in the opposite direction; when $pt - \epsilon = \frac{5}{2}\pi$, for the third time, moving in the original direction, and so on. Hence, after passing through the position of equilibrium, the particle reaches the position of equilibrium again, moving in the same direction, when $pt - \epsilon$ increases by 2π, i.e. when t increases by $2\pi/p$. Therefore it makes oscillations about the position of equilibrium in the periodic time $2\pi/p$, and, owing to the presence of the continually decreasing factor $e^{-\frac{1}{2}kt}$, the amplitudes of the successive oscillations continually diminish. In order to find their values, we have θ a maximum or minimum when $\dot{\theta}$ is zero, i.e. when the velocity is zero.

Now
$$\begin{aligned}\dot{\theta} &= Ce^{-\frac{1}{2}kt}\{-p\sin(pt - \epsilon)\} - \tfrac{1}{2}kCe^{-\frac{1}{2}kt}\cos(pt - \epsilon) \\ &= -Ce^{-\frac{1}{2}kt}[p\sin(pt - \epsilon) + \tfrac{1}{2}k\cos(pt - \epsilon)].\end{aligned}$$

This is zero when $\quad p\sin(pt - \epsilon) = -\frac{1}{2}k\cos(pt - \epsilon)$,

i.e. when $\quad \tan(pt - \epsilon) = -\frac{1}{2}k/p = \tan\beta$, say,

i.e. when $\quad pt - \epsilon = n\pi + \beta$, where n is any integer,

i.e. when $\quad t = \dfrac{n\pi}{p} + \dfrac{\epsilon + \beta}{p} = \dfrac{n\pi}{p} + \gamma$, if γ denote $\dfrac{\epsilon + \beta}{p}$.

Taking first $n=0$, i.e. $t=\gamma$, we have $\theta = Ce^{-\frac{1}{2}k\gamma}\cos\beta = \alpha$, say. (iii)

Next, when $n=1$, i.e. $t=\pi/p+\gamma$,

$$\theta = Ce^{-\frac{1}{2}k(\gamma+\pi/p)}\,.\cos(\pi+\beta) = -Ce^{-\frac{1}{2}k\gamma}\,.\,e^{-k\pi/2p}\cos\beta = -\alpha e^{-k\pi/2p};$$

when $n=2$, i.e. $t=2\pi/p+\gamma$,

$$\theta = Ce^{-\frac{1}{2}k(\gamma+2\pi/p)}\,.\cos(2\pi+\beta) = Ce^{-\frac{1}{2}k\gamma}\,.\,e^{-k\pi/p}\cos\beta = \alpha e^{-k\pi/p};$$

when $n=3$, i.e. $t=3\pi/p+\gamma$, θ becomes $= -\alpha e^{-3k\pi/2p}$, and so on.

Hence the successive amplitudes form a descending geometrical progression whose common ratio is $-e^{-k\pi/2p}$.

If the particle starts from rest * with $\theta=\alpha$, then substituting in (i) we get

$$\alpha = C\cos(-\epsilon),\quad\text{whence}\quad C=\alpha\sec\epsilon.$$

Also since $\dot\theta = 0$ when $t=0$, we have, on substituting in (ii),

$$0 = -C[-p\sin\epsilon + \tfrac{1}{2}k\cos\epsilon],\quad\text{whence}\quad \tan\epsilon = \tfrac{1}{2}k/p.$$

$$\therefore\quad \theta = \alpha\sec\epsilon\,.\,e^{-\frac{1}{2}kt}\cos(pt-\epsilon),\quad\text{where}\quad \epsilon=\tan^{-1}(\tfrac{1}{2}k/p).$$

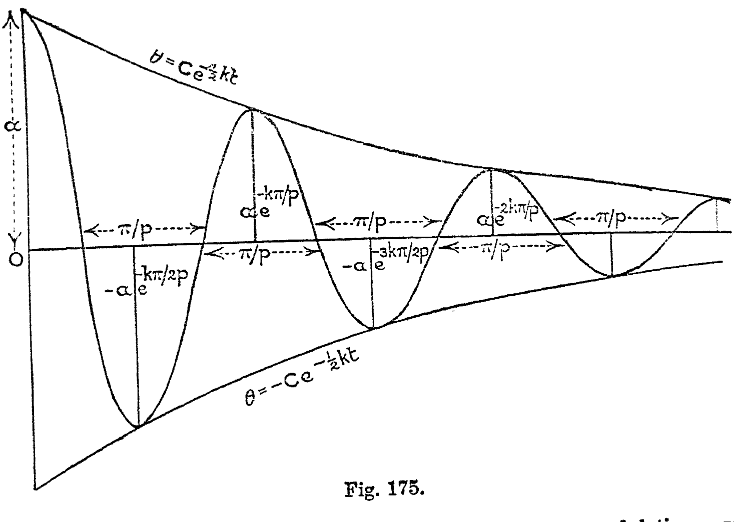


Fig. 175.

In this case, the displacement-time graph consists of an undulating curve like a sine-curve with constantly decreasing maximum and minimum ordinates (Fig. 175). It meets the curve $y=Ce^{-\frac{1}{2}kt}$ at points where $\cos(pt-\epsilon)=1$, i.e. $pt-\epsilon=2n\pi$; and it meets $y=-Ce^{-\frac{1}{2}kt}$ where $pt-\epsilon=(2n+1)\pi$. It can easily be proved that at their common points, which do *not* coincide with the maxima and minima, the curves have the same slope, and therefore touch each other. (Cf. Art. 105, Ex. iii.)

If there be no damping, i.e. if k be zero, the equation of motion is $\ddot\theta + n^2\theta = 0$, and the period is $2\pi/n$. It has just been proved that in the

* In this case, the 'γ' of the preceding investigation, since it denotes the time when the particle is first at rest, is zero, so that $\beta=-\epsilon$, and equation (iii) becomes $C\cos(-\epsilon)=\alpha$.

present case the period is $2\pi/p$. Since $p < n$ $[p^2 = n^2 - \frac{1}{4}k^2]$, it follows that the effect of the damping is to increase the time of oscillation.

If k be small,

$$\frac{2\pi}{p} = \frac{2\pi}{(n^2 - \frac{1}{4}k^2)^{\frac{1}{2}}} = \frac{2\pi}{n}\left(1 - \frac{k^2}{4n^2}\right)^{-\frac{1}{2}} = \frac{2\pi}{n}\left(1 + \frac{k^2}{8n^2}\right) \text{ nearly [Art. 28];}$$

hence the time of oscillation is increased by a small quantity of the second order.

If the oscillations be forced, i.e. if the pendulum be subject to a force which prevents the oscillations from dying away, there will be an additional term on the right-hand side of the equation. For instance, the general equation at the beginning of this article is the equation of motion of a pendulum in a medium whose resistance varies as the velocity, and acted upon by a force which is a periodic function of the time. In this case, the C.F. is as before, and it remains to find the P.I. This is obtained by the method of Art. 222 (iv), as in the example which follows.

224. An example from Electricity.

If an electromotive force E is applied to a circuit of resistance R and coefficient of self-induction L, containing a condenser of capacity K, then it is proved in books on Electricity that the charge q in the condenser at time t satisfies the equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{K} = E.$$

This is a linear equation of the second order with constant coefficients.

First, taking a numerical example, let the condenser be initially uncharged, and let $R = 100$ ohms, $L = \cdot 005$ henry, $K = 1$ microfarad $= 10^{-6}$ farad, and $E = 1000$ volts.

The equation becomes $\quad \cdot 005\dfrac{d^2q}{dt^2} + 100\dfrac{dq}{dt} + 10^6 q = 1000.$

By Art. 222 (i), the P.I. $= 1000/10^6 = \cdot 001$.

To find the C.F., we have the auxiliary equation

$$\cdot 005\, m^2 + 100\, m + 10^6 = 0,$$

whence $\quad m = \dfrac{-100 \pm \sqrt{(10^4 - 4 \times \cdot 005 \times 10^6)}}{\cdot 01} = \dfrac{-100 \pm \sqrt{-10^4}}{\cdot 01} = 10^4(-1 \pm i).$

Therefore the C.F. is $q = Ae^{-10^4 t}\cos(10^4 t - \epsilon)$, and the complete solution is

$$q = \cdot 001 + Ae^{-10^4 t}\cos(10^4 t - \epsilon).$$

To find the constants A and ϵ, we have $q = 0$ and $\dot{q} = 0$ when $t = 0$, since the charge and the current are both initially zero.

$$\dot{q} = Ae^{-10^4 t}[-10^4(\sin 10^4 t - \epsilon)] - A \,.\, 10^4 e^{-10^4 t}\cos(10^4 t - \epsilon).$$

Substituting the initial conditions, $0 = \cdot 001 + A\cos\epsilon$;

$$0 = A \,.\, 10^4 \sin\epsilon - A \,.\, 10^4\cos\epsilon,$$

whence $\quad \tan\epsilon = 1; \quad \therefore \quad \epsilon = \frac{1}{4}\pi, \quad$ and $\quad A = -\cdot 001\sec\epsilon = -\cdot 001414.$

Hence $\qquad q = \cdot 001 - \cdot 001414\, e^{-10^4 t}\cos(10^4 t - \frac{1}{4}\pi).$

This gives the charge in coulombs at the end of time t.

Next, returning to the general equation, let $E=0$, and let us find the condition that the discharge may be oscillatory.

The auxiliary equation is $Lm^2+Rm+1/K=0$, the roots of which are

$$m=\frac{-R\pm\sqrt{(R^2-4L/K)}}{2L}.$$

As in the preceding article, the discharge will be oscillatory if the roots are imaginary, i.e. if $4L/K>R^2$, i.e. if $4L>KR^2$. In this case, if we denote $(R^2-4L/K)/4L^2$ by $-\mu^2$, the roots take the form $-\frac{1}{2}R/L\pm\mu\sqrt{-1}$, and the C. F. is

$$q=Ae^{-tR/2L}\cos(\mu t-\epsilon),$$

as in Case (iii) of the last article. The time of a complete oscillation is

$$\frac{2\pi}{\mu}=2\pi\div\frac{\sqrt{(4L/K-R^2)}}{2L}=\frac{4\pi L}{\sqrt{(4L/K-R^2)}}.$$

If the initial charge in the condenser be q_0, the constants A and ϵ are found from the conditions that when $t=0$, $q=q_0$ and the current $\dot{q}=0$.

If the right-hand side of the general equation be $E\sin pt$, i.e. if the E. M. F. be a periodic function of the time, then we have in addition to find the P. I.

Assuming $q=A\sin pt+B\cos pt$ [Art. 222 (iv)], and substituting in the differential equation, we get

$$L[-Ap^2\sin pt-Bp^2\cos pt]+R[pA\cos pt-pB\sin pt]$$
$$+[A\sin pt+B\cos pt]/K=E\sin pt;$$

whence, comparing coefficients of $\sin pt$ and $\cos pt$, we have

$$\left.\begin{array}{r}-A(Lp^2-1/K)-B\,.\,Rp=E\\ A\,.\,Rp-B(Lp^2-1/K)=0\end{array}\right\}.$$

Multiplying by Lp^2-1/K and Rp respectively and subtracting, we get

$$-A[(Lp^2-1/K)^2+R^2p^2]=E(Lp^2-1/K);$$

$$\therefore\ A=\frac{-E(Lp^2-1/K)}{(Lp^2-1/K)^2+R^2p^2},\ \text{and}\ B=\frac{Rp}{Lp^2-1/K}\,.\,A=\frac{-E\,.\,Rp}{(Lp^2-1/K)^2+R^2p^2}.$$

Hence the particular integral is

$$\frac{-E}{(Lp^2-1/K)^2+R^2p^2}[(Lp^2-1/K)\sin pt+Rp\cos pt].$$

Putting $Lp^2-1/K=r\sin\phi$ and $Rp=r\cos\phi$ [as in Art. 182], this becomes

$$\frac{-Er}{(Lp^2-1/K)^2+R^2p^2}[\sin\phi\sin pt+\cos\phi\cos pt],$$

which, since $r^2=(Lp^2-1/K)^2+R^2p^2$, may be written

$$\frac{-E}{\sqrt{[(Lp^2-1/K)^2+R^2p^2]}}\cos(pt-\phi).$$

Hence, adding the C. F. and the P. I., the charge q in the condenser at the end of time t is given by the equation

$$q = Ae^{-tR/2L}\cos\left[\sqrt{\left(\frac{1}{KL} - \frac{R^2}{4L^2}\right)}\,t - \epsilon\right] - \frac{E\cos(pt-\phi)}{\sqrt{[(Lp^2-1/K)^2+R^2p^2]}};$$

where $\phi = \tan^{-1}\dfrac{Lp^2-1/K}{Rp}$, and the constants A and ϵ are found from the initial conditions.

The first term rapidly decreases as t increases on account of the factor $e^{-tR/2L}$, unless L be very great compared with R, and therefore q approaches the value given by the P. I. Hence the current i, which is equal to $\dot{q}$, approaches the value $\dfrac{pE\sin(pt-\phi)}{\sqrt{[(Lp^2-1/K)^2+R^2p^2]}}$, which may be written

$$\frac{E\sin(pt-\phi)}{\sqrt{[R^2+(pL-1/pK)^2]}}, \text{ where } \phi = \tan^{-1}\left(pL-\frac{1}{pK}\right)\Big/R.$$

Examples XCI.

1. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 12.$

2. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2.$

3. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = -3\sin x.$

$\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{-x}.$

5. $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 12y = 0.$

6. $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 12y = 4.$

7. $\dfrac{d^2y}{dx^2} - 4y = 10.$

8. $\dfrac{d^2y}{dx^2} - 4y = \sin x.$

9. $\dfrac{d^2y}{dx^2} - 4y = e^x.$

10. $\dfrac{d^2y}{dx^2} - 4y = e^{2x}.$

11. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = a.$

12. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 3x - 2.$

13. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 2\cos 2x.$

14. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = \frac{1}{2}\sin x + 1.$

15. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 0.$

16. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = x^2 - 3x - 2.$

17. $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 2.$

18. $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = \cos x.$

19. $\dfrac{d^2y}{dx} - 6\dfrac{dy}{dx} + 9y = 0.$

20. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = e^3$

21. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 10y = 4 - e^{-2x}.$

22. $\dfrac{d^4y}{dx^4} - 4y = 0.$

$\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 7y = 0.$

24. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} -$

25. $\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + k^2x = 0.$

26. $\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + (k^2+n^2)x = 0.$

27. $\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + (k^2+n^2)x = \cos pt.$

28. $\dfrac{d^2x}{dt^2} + a^2x = k\sin(pt+\alpha).$

29. A particle moves in a straight line so that its distance x from the origin at the end of time t satisfies the equation $\ddot{x}+\dot{x}+x=0$; if it starts from the origin with velocity 60 foot-seconds, what will its distance from the origin be after $\pi/\sqrt{3}$ seconds, and what will its velocity and acceleration then be? When will it first come to rest?

30. A particle moves in a straight line under the action of a force to a fixed point O in the line, which varies as the distance from O and is equal to $\frac{1}{4}$ of the weight of the particle at distance 10 feet from O; it starts from rest at distance of 20 feet from O, and moves against a resistance, which varies as the velocity, and is equal to $\frac{1}{2}$ of the weight when the velocity is 50 foot-seconds. Find the distance from O at the end of time t. Find the time taken to reach O for the first time, and the velocity at that instant. Find also the distance to which the particle first goes on the other side of O.

31. In the case of a pendulum making damped oscillations as in Art. 223, and starting from rest at an inclination α to the vertical, prove that

$$\theta=\alpha e^{-\frac{1}{2}kt}[\cos pt+(\tfrac{1}{2}k/p)\sin pt].$$

32. In the preceding example, find the successive angular velocities when the particle is passing through its equilibrium position.

33. A pendulum starts from rest at an inclination 20° to the vertical, and first comes to rest at an inclination 15° on the other side of the vertical after the lapse of one second. Assuming that its displacement follows the law $\theta=Ce^{-\frac{1}{2}kt}\cos(pt-\epsilon)$, find the values of the constants p, k, ϵ, and C. Find the ratio of the successive maximum displacements and of successive angular velocities when passing through the position $\theta=0$.

Find its inclination to the vertical after 10 seconds, and its angular velocity when passing through the equilibrium position for the tenth time. Draw the displacement-time graph of the motion.

34. A point moves in a straight line according to the law

$$x=Ce^{-\frac{1}{2}kt}\cos(pt-\epsilon).$$

It starts at a point 6 inches to the right of a certain point A, moves to a point 5 inches to the left of A and then back to a point 4 inches to the right of A. Find the distance from A of the position of equilibrium (the point from which x is measured). If an interval of 3 seconds was observed to elapse between the first and third of the positions mentioned above, find the values of the constants p, k, ϵ, and C. Find the distance of the point from its equilibrium position and also from A at the end of 15 seconds.

Draw the displacement-time graph of the motion.

35. A particle rests on a rough horizontal table ($\mu=\frac{1}{3}$) and is attached to a fixed point on the table by an elastic string of natural length 20 inches, and modulus equal to the weight of the particle. If the particle is drawn out to a distance of 30 inches from the fixed point and then let go, where will it finally come to rest?

36. The motion of a ballistic galvanometer needle is given by the equation

$$I\ddot{\theta}+K\dot{\theta}+h\theta=0,$$

where I is the moment of inertia of the needle, h the twisting moment per unit angular displacement due to the torsion of the fibre and the magnetic field, and K the retarding moment (per unit angular velocity) of the bath used to damp the motion.

If initially $\theta=\frac{1}{6}\pi$ and $\dot{\theta}=0$, find the value of θ in terms of t when $h=\cdot 2$, $I=\cdot 5$, $K=\cdot 6$. Draw a graph of the motion.

37. Answer the preceding question, if $h = \cdot 4$, $I = \cdot 5$, $K = \cdot 9$. Draw a graph of the motion.

38. A constant E.M.F. of 2000 volts is applied to a circuit of resistance 500 ohms and self-induction ·01 henry, containing a condenser of capacity 2×10^{-6} farads; find the value of the current $\dot{q}$ in terms of the time.

Represent the result graphically.

39. Answer the preceding question when the resistance is 200 ohms, the self-induction ·02 henry, and the capacity 10^{-6} farad.

40. A mass m is supported by a vertical spring which stretches a distance h when supporting 1 lb.; if the resistance of the air be proportional to the velocity, the equation of motion is

$$m\ddot{x} + k\dot{x} + gx/h = 0.$$

Find x in terms of t, if $m = \frac{1}{2}$ lb., $h = 1$ foot, and $k = \cdot 05$ lb. wt.

41. If in the preceding question $m = \frac{1}{2}$ oz., $h = 3$ inches, and $k = \cdot 25$ lb. wt., find x in terms of t.

42. A simple unresisted pendulum is acted upon by a force which is a simple harmonic function of the time represented by $k \cos pt$. Find an expression for θ in terms of t, if the length of the pendulum is 8 feet and its mass unity, (i) when $p = 3$, (ii) when $p = 2$.

225. Solution of linear equation of the second order when a particular solution of the equation with the right-hand side replaced by zero is known.

Taking the equation in the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R,$$

where P, Q, R are functions of x, let u be any solution of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0. \tag{i}$$

In some cases such a solution can be found by inspection.

Substitute $y = uz$ in the given equation;

$$\therefore \quad \frac{dy}{dx} = u\frac{dz}{dx} + z\frac{du}{dx},$$

$$\frac{d^2y}{dx^2} = u\frac{d^2z}{dx^2} + 2\frac{du}{dx}\cdot\frac{dz}{dx} + z\frac{d^2u}{dx^2}.$$

The equation becomes, on substitution,

$$u\frac{d^2z}{dx^2} + 2\frac{du}{dx}\cdot\frac{dz}{dx} + z\frac{d^2u}{dx^2} + P\left(u\frac{dz}{dx} + z\frac{du}{dx}\right) + Quz = R,$$

i.e. $$u\frac{d^2z}{dx^2} + \frac{dz}{dx}\left(2\frac{du}{dx} + Pu\right) + z\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = R. \tag{ii}$$

Since u is a solution of the equation (i), the coefficient of z is 0;

hence we have $$u\frac{d^2z}{dx^2}+\frac{dz}{dx}\left(2\frac{du}{dx}+Pu\right)=R.$$

If dz/dx be replaced by q, this becomes, on dividing by u,

$$\frac{dq}{dx}+q\left(\frac{2}{u}\frac{du}{dx}+P\right)=\frac{R}{u}.$$

This is a linear equation of the first order for q, and can be solved by the method of Art. 214.

Having found q, a further integration with respect to x gives z, and then $y=uz$ will be the solution of the given equation.

Example. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x.$

A particular solution of $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$ is obviously $y=x$.

Therefore substitute $y=xz$ in the given equation.

$$\frac{dy}{dx}=x\frac{dz}{dx}+z;\quad \frac{d^2y}{dx^2}=x\frac{d^2z}{dx^2}+2\frac{dz}{dx}.$$

Hence, on substitution, $x^3\dfrac{d^2z}{dx^2}+2x^2\dfrac{dz}{dx}+x^2\dfrac{dz}{dx}+xz-xz=x,$

i.e. $$x^3\frac{d^2z}{dx^2}+3x^2\frac{dz}{dx}=x.$$

The left-hand side is the d. c. of $x^3\,dz/dx$,

$\therefore$ integrating, $x^3\dfrac{dz}{dx}=\frac{1}{2}x^2+A$, i. e. $\dfrac{dz}{dx}=\dfrac{1}{2x}+\dfrac{A}{x^3}.$

Integrating again, $z=\frac{1}{2}\log x-\frac{1}{2}A/x^2+B,$

and therefore the solution of the given equation is

$$y=xz=\tfrac{1}{2}x\log x-\tfrac{1}{2}A/x+Bx.$$

If u is not a particular solution of (i), the substitution $y=uz$ will still in some cases solve the given equation. For u may be chosen so that the coefficient of dz/dx in equation (ii) shall be zero [i.e. so that $2du/dx+Pu=0$, an equation for u which is at once soluble], and the resulting equation may then admit of solution.

Example: As an illustration of this method, let us take an equation which occurs in various branches of Physics:

$$\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}+k^2\phi=0.$$

Substitute $\phi=uz$; the values of $d\phi/dr$ and $d^2\phi/dr^2$ are given above (with y and x replacing ϕ and r respectively).

The equation becomes

$$u\frac{d^2z}{dr^2}+2\frac{du}{dr}\cdot\frac{dz}{dr}+z\frac{d^2u}{dr^2}+\frac{2}{r}\cdot u\frac{dz}{dr}+\frac{2}{r}\cdot z\frac{du}{dr}+k^2uz=0.$$

The coefficient of $\dfrac{dz}{dr}$ is $2\dfrac{du}{dr}+\dfrac{2u}{r}$, which will be zero

if $\dfrac{1}{u}\dfrac{du}{dr}=-\dfrac{1}{r}$, i.e. if $\log u=-\log r$, i.e. if $u=1/r$.

The equation then becomes

$$\frac{1}{r}\frac{d^2z}{dr^2}+z\frac{2}{r^3}+\frac{2z}{r}\left(-\frac{1}{r^2}\right)+\frac{k^2z}{r}=0,$$

which reduces to
$$\frac{d^2z}{dr^2}+k^2z=0.$$

Therefore
$$z=A\cos kr+B\sin kr,$$
and the solution is
$$\phi=uz=(A\cos kr+B\sin kr)/r.$$

Examples XCII.

Solve the equations:

1. $(1+x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0.$

2. $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=0.$

3. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=0.$

4. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=x^2.$

5. $x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}-k^2xy=0.$

6. $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=0.$

7. $x\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+xy=e^x.$

8. $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+3y=x^2.$

CHAPTER XXII

TAYLOR'S THEOREM

226. Form of the series.

It is impossible in a work like the present to give a full account and a rigorous investigation of this famous and important theorem. It will, however, not be out of place to indicate one way in which the theorem may be arrived at, especially as this method is but an extension of the mean value theorems of Arts. 117 and 119. It was there shown that, provided $f(x)$ and its first and second differential coefficients are continuous throughout the range of the independent variable from $x=a$ to $x=b$, then for any value of x within that range,

(i) $f(x)=f(a)+(x-a)f'(x_1)$,

(ii) $f(x)=f(a)+(x-a)f'(a)+\frac{1}{2}(x-a)^2 f''(x_2)$,

where x_1 and x_2 are between a and x.

By adopting a method of proof similar to that used in obtaining these results, the expression on the right-hand side may be developed to any number of terms.

In the first place, *assuming that* f(x) *can be expanded in an infinite series of ascending powers of* x−a, *and that the successive differential coefficients of* f(x) *are obtained by differentiating this series term by term*,* it is easy to find the form which the series must take. For suppose

$$f(x)=A_0+A_1(x-a)+A_2(x-a)^2+A_3(x-a)^3+A_4(x-a)^4+\dots.$$

Differentiating,

$$f'(x)=A_1+2A_2(x-a)+3A_3(x-a)^2+4A_4(x-a)^3+\dots;$$

differentiating again,

$$f''(x)=2A_2+3.2.A_3(x-a)+4.3.A_4(x-a)^2+\dots;$$

differentiating again,

$$f'''(x)=3.2.A_3+4.3.2.A_4(x-a)+\dots.$$

* We have proved that this is the case for a series with a finite number of terms, but it must not be assumed from this that it is also the case for an infinite series. For a proof in the case of infinite series, the student should consult Lamb's *Infinitesimal Calculus*, Chapter XIII.

Substituting $x = a$ in these results, we have

$$f(a) = A_0;\ f'(a) = A_1;\ f''(a) = 2A_2;\ f'''(a) = 3 \cdot 2 \cdot A_3, \text{ \&c.},$$

i.e. $A_0 = f(a);\ A_1 = f'(a);\ A_2 = \frac{1}{2} f''(a);\ A_3 = \frac{1}{2.3} f'''(a); \ldots$

and generally $$A_n = \frac{1}{n!} f^{(n)}(a),$$

where $f^{(n)}(a)$ stands for the number obtained by differentiating $f(x)$ n times, and substituting a for x in the result.

Hence

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots.$$

This of course is no proof that the expansion is possible, and takes no account of the conditions under which the series is convergent; it is only of value in showing what form the expansion takes if and when it does exist, and it gives the clue to the construction of the series (i) and the function $F(z)$ which occur in the next article.

227. Proof of Taylor's Theorem.

Let $f(x)$ and all its differential coefficients up to the n^{th} be continuous throughout the range extending from $x = a$ to $x = b$, and consider any value of x within the range.

Let the expression

$$f(x) - f(a) - (x-a) f'(a) - \frac{(x-a)^2}{2!} f''(a) - \ldots - \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a)$$

be denoted by $$\frac{(x-a)^n}{n!} \cdot R. \qquad \text{(i)}$$

This expression is the difference between the function $f(x)$ and the sum of the first n terms of the series obtained in the preceding article. We want to find the form of R.

Consider the function

$$F(z) \equiv f(x) - f(z) - (x-z) f'(z) - \frac{(x-z)^2}{2!} f''(z) - \\ \ldots - \frac{(x-z)^{n-1}}{(n-1)!} f^{(n-1)}(z) - \frac{(x-z)^n}{n!} R,$$

where z is between a and x. x, and therefore R, as defined above, are independent of z.

When $z = a$, $F(z) = 0$, since the first $n+1$ terms on the right-hand side then cancel out the last term, from equation (i).

When $z = x$, $F(z) = 0$, since the first two terms on the right cancel out, and all the others vanish owing to the factor $x - z$.

Also, $F(z)$ and $F'(z)$ are continuous within the given range, since every term in the value of $F(z)$ is continuous.

Hence, since $F(z)$ vanishes when $z = a$ and when $z = x$, it follows

(Art. 113) that its differential coefficient $F'(z)$ must vanish for some value of z between a and x.

Now, by differentiation with respect to z (and remembering that x and R are independent of z), we have

$$F'(z) = 0 - f'(z) - [(-1)f'(z) + (x-z)f''(z)]$$
$$- \left[-(x-z)f''(z) + \frac{(x-z)^2}{2!}f'''(z)\right] - \dots$$
$$\dots - \left[-\frac{(x-z)^{n-2}}{(n-2)!}f^{(n-1)}(z) + \frac{(x-z)^{n-1}}{(n-1)!}f^{(n)}(z)\right] - \left[-\frac{(x-z)^{n-1}}{(n-1)!}R\right].$$

In this expression, successive terms cancel in pairs with the exception of the last two;

$$\therefore \quad F'(z) \equiv -\frac{(x-z)^{n-1}}{(n-1)!}f^{(n)}(z) + \frac{(x-z)^{n-1}}{(n-1)!}R = 0$$

for some value of z between a and x.

Any value of z between a and x may be written $a + \theta(x-a)$, where $0 < \theta < 1$.

Therefore, since $x - z \neq 0$, we have

$$-f^{(n)}(z) + R = 0 \quad \text{when} \quad z = a + \theta(x-a),$$

i.e. $$R = f^{(n)}[a + \theta(x-a)].$$

Hence, substituting in (i), and transferring all the terms except $f(x)$ to the right-hand side, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$
$$\dots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{x^n}{n!}f^{(n)}[a + \theta(x-a)].$$

This result is known as Taylor's Theorem.

The last term on the right-hand side is known as Lagrange's form of the remainder after n terms. This 'remainder after n terms' can be put into various other forms. One of the most useful of them is obtained by taking the remainder in (i) in the form $(x-a)R$. $F'(z)$ will then be equal to

$$-\frac{(x-z)^{n-1}}{(n-1)!}f^{(n)}(z) + R, \text{ and } R = \frac{(x-z)^{n-1}}{(n-1)!}f^{(n)}(z)$$

for some value of z between a and x. Taking this value in the form $a + \theta(x-a)$ as before, $x - z = x - a - \theta(x-a) = (x-a)(1-\theta)$, and

$$R = \frac{(x-a)^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}\{a + \theta(x-a)\},$$

so that the remainder after n terms takes the form

$$\frac{(x-a)^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}\{a + \theta(x-a)\}.$$

This is known as Cauchy's form of the remainder.

228. Other forms of the theorem.

A very important particular case is obtained by putting $a = 0$. The theorem then becomes

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x),$$

where $f(0)$, $f'(0)$, $f''(0)$, ... are obtained by substituting 0 for x in the successive differential coefficients of $f(x)$.

This form of the theorem is known as Maclaurin's Theorem.

If, in Taylor's Theorem, we substitute $x+h$ and x for x and a respectively, the theorem takes the following form, which is often convenient,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^{(n)}(x+\theta h).$$

These theorems have been obtained on the supposition that $f(x)$ and all its differential coefficients up to the n^{th} are continuous throughout the range from a to b within which x lies. If in Taylor's Theorem we put $n = 1$ and $n = 2$, we get the mean-value theorem and its extension [Arts. 117 and 119].

If n be increased indefinitely, the series becomes an infinite series and the theorem remains true in general, provided this series be convergent. If the remainder after n terms $\frac{(x-a)^n}{n!} f^{(n)}\{a+\theta(x-a)\}$ tends to the limit zero as $n \longrightarrow \infty$, the series converges to the value $f(x)$.

There are cases in which Taylor's series converges to a value other than $f(x)$. It may happen that the series converges and that the remainder does not tend to zero; in this case the value to which the series converges will not be $f(x)$.* Such cases do not occur in

* This is a fact which the elementary student usually finds difficult to understand. The following example, due to Pringsheim, is a case in point:

If Taylor's Theorem be used for the function

$$\sum \frac{(-1)^n a^n}{n!\,(1+x^2 a^{2n})},$$

it gives the series

$$e^{-a} - x^2 e^{-a^3} + x^4 e^{-a^5} - \ldots,$$

and both series are convergent for real values of x if $a > 1$. Nevertheless, they are not equal (except when $x = 0$), e.g. if we take $x = a = 2$, it is easy to see, by taking a few terms of each series (the terms of each decrease and are alternately + and −), that the sum of the first series $< \cdot 1086$, whereas the sum of the second series $> \cdot 1349$.

ordinary work, and it is beyond the scope of this book to consider them.

By Taylor's Theorem as first given, i.e. as an expansion of $f(x)$ in ascending powers of $x-a$, if we know the value of the function and its derivatives for any value a of x, we can calculate the value of the function for any other value of x within the range throughout which the function and its derivatives are continuous. If this value of x is near a so that $x-a$ is small, a few terms will generally suffice to give an approximate value of the function. For instance, if $y=f(x)$, and if x be increased to $x+h$, δy will be $f(x+h)-f(x)$; expanding the first term by Taylor's Theorem, we have

$$\delta y = hf'(x) + \frac{h^2}{2!} f''(x) + \dots.$$

If we neglect squares and higher powers of h, this gives

$$\delta y = hf'(x) = \delta x \cdot \frac{dy}{dx}. \quad \text{[Cf. Art. 24.]}$$

In calculating numerical values of a function, Maclaurin's Theorem is often extremely useful. If a function of x admits of expansion by Maclaurin's Theorem, it is obtained as a series of positive integral powers of x with constant coefficients, and, by taking a sufficient number of terms, the value of the function for any given value of the variable can be obtained to any required degree of accuracy. It must be remembered, however, that the general form of the n^{th} differential coefficient of a function cannot be obtained as a rule, and therefore, in order that this method of calculating a function may be of value, it is necessary that the series should converge rapidly, and that a sufficient number of the successive differential coefficients of the function should admit of being worked out without excessive labour.

229. Particular cases and examples of Taylor's and Maclaurin's Theorems.

We now proceed to consider a number of important particular cases of these theorems.

It will be seen that many of the expansions with which the student is already familiar are included among them.

To expand a function of $x+h$ in a series of powers of x or of h, and to expand a function of x in powers of $x-a$, Taylor's Theorem is used; to expand a function of x in powers of x, the form known as Maclaurin's Theorem is used. In each case, the remainder after n terms, and the conditions under which it tends to zero as $n \to \infty$, should be examined.

Examples:

(i) *Expand* e^x *in powers of* x *or of* $x-a$.

All the differential coefficients of $f(x)$ in this case are equal to e^x; therefore $f(0), f'(0) \dots f^{(n-1)}(0)$ are all equal to unity.

Hence, by Maclaurin's Theorem, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

The remainder after n terms, $\frac{x^n}{n!} f^{(n)}(\theta x)$, is equal to $\frac{x^n}{n!} e^{\theta x}$.

Whatever be the (finite) value of x, this $\to 0$ as $n \to \infty$, since the first factor then $\to 0$ (Art. 13 (6)) and the second is finite. Hence the series is convergent, the remainder tends to zero, and the expansion holds for all values of x.*

Again, to expand e^x in a series of powers of $x-a$, we have $f(a)$, $f'(a)$, $\dots f^{(n-1)}(a)$ all equal to e^a; therefore, substituting in Taylor's Theorem, we get

$$e^x = e^a + (x-a)e^a + \frac{(x-a)^2}{2!}e^a + \dots + \frac{(x-a)^n}{n!}e^a + \dots,$$

a result which can also be obtained by writing

$$e^x = e^a \times e^{x-a} = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \dots\right]$$

as before.

(ii) *Expand* $\log(1+x)$ *in a series of ascending powers of* x.

If $f(x) = \log(1+x)$, $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$,

$$f'''(x) = \frac{1.2}{(1+x)^3}, \dots, f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n},$$

whence $f(0) = \log 1 = 0$, $f'(0) = 1$, $f''(0) = -1$,

$$f'''(0) = 1.2, \dots f^{(n)}(0) = (-1)^{n-1}(n-1)!.$$

Hence $\quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + \dots.$

The remainder after n terms,

$$\frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!}(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n} = \frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

If x be positive and $<$ or $=1$, $x/(1+\theta x) < 1$, and $1/n \to 0$ as $n \to \infty$; hence the remainder tends to zero, and therefore the expansion holds for values of x from $x=0$ to $x=1$, both inclusive.

If x be negative, $x/(1+\theta x)$ is not necessarily <1 (numerically), and the preceding argument does not hold good.

In this case, taking Cauchy's form of the remainder, we get it in the form

$$(-1)^{n-1}\frac{x^n(1-\theta)^{n-1}}{(n-1)!} \cdot \frac{(n-1)!}{(1+\theta x)^n}, \text{ i.e. } (-1)^{n-1}\frac{x^n}{1+\theta x} \cdot \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.$$

If $|x| < 1$, $1-\theta < 1+\theta x$, therefore the last factor $\to 0$, as also does x^n when $n \to \infty$; the other factor, $1+\theta x$, is finite, hence the remainder $\to 0$,

* It should be noticed that this is not a proof of the exponential theorem, if we have used this theorem in obtaining the differential coefficient of e^x, as in Art. 97.

and the expansion holds if $|x| < 1$. If $x = -1$, the series is divergent and the expansion does not hold.

The function $\log x$ cannot be expanded in a series of powers of x, since all its differential coefficients become infinite for $x = 0$, but, by using Taylor's Theorem in the form first obtained, it can be expanded in a series of powers of $x-a$. For instance, to expand it in powers of $x-1$, we have

$$f(x) = \log x,\ \ f'(x) = \frac{1}{x},\ \ f''(x) = -\frac{1}{x^2}, \ldots,\ \ f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n};$$

$$\therefore \text{ putting } x = 1,\ f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 1.2, \ldots,$$
$$f^{(n)}(1) = (-1)^{n-1}(n-1)!$$

Hence $$\log x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \ldots + (-1)^{n-1}\frac{(x-1)^n}{n} + \ldots.$$

This result can also be deduced from the preceding expansion for $\log(1+x)$; for

$$\log x = \log[1+(x-1)] = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \ldots,$$

by substituting $x-1$ for x in the former expansion, and since that result is true when x is between -1 and $+1$ or equal to $+1$, the latter will be true when $x-1$ is between -1 and $+1$ or equal to $+1$, i.e. when x is between 0 and 2 or equal to 2.

(iii) *Expand* $\sin x$ *in a series of powers of* x.

$$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,$$
$$f''''(x) = \sin x, \text{ \&c.}$$

$$\therefore\ f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f''''(0) = 0, \text{ \&c.}$$

All the coefficients of even powers of x are zero [as follows from the fact that $\sin x$ is an odd function of x (Art. 5)], and the coefficients of the odd powers of x are alternately $+1$ and -1.

Hence, by Maclaurin's Theorem,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$$

The remainder after n terms, $\dfrac{x^n}{n!}f^{(n)}(\theta x) = \pm\dfrac{x^n}{n!} \times$ a factor which is either $\sin\theta x$ or $\cos\theta x$. This factor cannot be numerically greater than 1, and the first factor $\to 0$ as $n \to \infty$ [Art. 13 (6)]. Therefore the series is convergent, the remainder $\to 0$, and the expansion holds for all values of x.

This result may be written $\dfrac{\sin x}{x} = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \ldots$, from which it is obvious that as $x \to 0$, $(\sin x)/x \to 1$, since the right-hand side is equal to $1 - x^2 \times$ [a convergent series]. This is the limit obtained geometrically in Art. 13 (10), and in a number of cases in the preceding chapters we have taken $\sin x$ as approximately equal to x. We can now form some idea as to the amount of error involved in this approximation.

In the first place we notice that, since the terms of the series $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$ are continually diminishing (provided $x>\frac{x^3}{3!}$, i.e. $x^2<6$) and are alternately + and −, the error involved in terminating the series at any term is numerically less than the next term, for if $u_n-u_{n+1}+u_{n+2}-\dots$ represent the rest of the series, this may be written

$$u_n-(u_{n+1}-u_{n+2})-(u_{n+3}-u_{n+4})-\dots,$$

and this is less than u_n, since the contents of the brackets are all +.

Hence the error involved in taking $\sin x$ equal to x is numerically $<\frac{1}{6}x^3$.

Suppose that the angle is 5°; then $x=\pi/36$, and the amount of error $<\pi^3/(6\times 36^3)$, i.e. $<\cdot 00011$. The proportional error is

$$\frac{\cdot 00011}{\sin 5^\circ}=\frac{\cdot 00011}{\cdot 0872}=\cdot 00126, \text{ or about } \tfrac{1}{8} \text{ per cent.}$$

If we want to find for what values of x the substitution of x for $\sin x$ will be correct to 3 places of decimals [i.e. so that the error may be $<\cdot 001$], we put $\frac{1}{6}x^3<\cdot 001$ numerically.

This gives $|x|<\sqrt[3]{\cdot 006}$, i.e. $<\cdot 1817$, which is the circular measure of 10·4°.

Hence, if this degree of approximation is required, it is sufficient to substitute x for $\sin x$, provided the angle is between −10° and +10°. If the angle is larger than this, or if a higher degree of accuracy be required, we can take $x-\frac{1}{6}x^3$ instead of $\sin x$. The error involved in this case is numerically less than $x^5/5!$, i.e. $\frac{1}{120}x^5$. Therefore this will give the result correct to 3 decimal places if $\frac{1}{120}x^5$ is numerically $<\cdot 001$, i.e. if $|x|<\sqrt[5]{\cdot 12}$, i.e. if $|x|<\cdot 6543$, which is true for angles between $-37\frac{1}{2}^\circ$ and $+37\frac{1}{2}^\circ$. It will give the result correct to 4 decimal places if $\frac{1}{120}x^5<\cdot 0001$, i.e. if $|x|<\cdot 4129$, which is true for angles between $-23\frac{1}{2}^\circ$ and $+23\frac{1}{2}^\circ$.

(iv) *Expand* cos x *in a series of powers of* x.

$$f(x)=\cos x,\ f'(x)=-\sin x,\ f''(x)=-\cos x,\ f'''(x)=\sin x,$$
$$f''''(x)=\cos x, \text{ \&c.}$$

$$\therefore\quad f(0)=1,\ f'(0)=0,\ f''(0)=-1,\ f'''(0)=0,\ f''''(0)=1,\dots.$$

All the coefficients of odd powers of x are 0 [as is evident from the fact that $\cos x$ is an even function of x], and the coefficients of the even powers of x are alternately +1 and −1.

Hence $$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+$$

Exactly as in the last case the series is convergent, and the expansion holds for all values of x.

From this also it follows that as $x\to 0$, $\cos x\to 1$, and if a more accurate approximation be required, $1-\frac{1}{2}x^2$ may be substituted for $\cos x$ [Art. 13 (10)]. The range of values for which these substitutions agree with the value of $\cos x$ to any given degree of accuracy may be found as in the similar cases for $\sin x$.

and the expansion holds if $|x|<1$. If $x=-1$, the series is divergent and the expansion does not hold.

The function $\log x$ cannot be expanded in a series of powers of x, since all its differential coefficients become infinite for $x=0$, but, by using Taylor's Theorem in the form first obtained, it can be expanded in a series of powers of $x-a$. For instance, to expand it in powers of $x-1$, we have

$$f(x)=\log x,\quad f'(x)=\frac{1}{x},\quad f''(x)=-\frac{1}{x^2},\ \dots,\quad f^{(n)}(x)=(-1)^{n-1}\frac{(n-1)!}{x^n};$$

$$\therefore \text{ putting } x=1,\ f(1)=0,\ f'(1)=1,\ f''(1)=-1,\ f'''(1)=1.2,\ \dots,$$
$$f^{(n)}(1)=(-1)^{n-1}(n-1)!$$

Hence $\log x=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\dots+(-1)^{n-1}\dfrac{(x-1)^n}{n}+\dots.$

This result can also be deduced from the preceding expansion for $\log(1+x)$; for

$$\log x=\log[1+(x-1)]=(x-1)-\tfrac{1}{2}(x-1)^2+\tfrac{1}{3}(x-1)^3-\dots,$$

by substituting $x-1$ for x in the former expansion, and since that result is true when x is between -1 and $+1$ or equal to $+1$, the latter will be true when $x-1$ is between -1 and $+1$ or equal to $+1$, i.e. when x is between 0 and 2 or equal to 2.

(iii) *Expand* $\sin x$ *in a series of powers of* x.

$$f(x)=\sin x,\ f'(x)=\cos x,\ f''(x)=-\sin x,\ f'''(x)=-\cos x,$$
$$f''''(x)=\sin x,\ \&c.$$

$$\therefore\ f(0)=0,\ f'(0)=1,\ f''(0)=0,\ f'''(0)=-1,\ f''''(0)=0,\ \&c.$$

All the coefficients of even powers of x are zero [as follows from the fact that $\sin x$ is an odd function of x (Art. 5)], and the coefficients of the odd powers of x are alternately $+1$ and -1.

Hence, by Maclaurin's Theorem,

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots.$$

The remainder after n terms, $\dfrac{x^n}{n!}f^{(n)}(\theta x)=\pm\dfrac{x^n}{n!}\times$ a factor which is either $\sin\theta x$ or $\cos\theta x$. This factor cannot be numerically greater than 1, and the first factor $\rightarrow 0$ as $n\rightarrow\infty$ [Art. 13 (6)]. Therefore the series is convergent, the remainder $\rightarrow 0$, and the expansion holds for all values of x.

This result may be written $\dfrac{\sin x}{x}=1-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\dots$, from which it is obvious that as $x\rightarrow 0$, $(\sin x)/x\rightarrow 1$, since the right-hand side is equal to $1-x^2\times$[a convergent series]. This is the limit obtained geometrically in Art. 13 (10), and in a number of cases in the preceding chapters we have taken $\sin x$ as approximately equal to x. We can now form some idea as to the amount of error involved in this approximation.

In the first place we notice that, since the terms of the series $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$ are continually diminishing (provided $x>\frac{x^3}{3!}$, i.e. $x^2<6$) and are alternately + and −, the error involved in terminating the series at any term is numerically less than the next term, for if $u_n-u_{n+1}+u_{n+2}-\ldots$ represent the rest of the series, this may be written

$$u_n-(u_{n+1}-u_{n+2})-(u_{n+3}-u_{n+4})-\ldots,$$

and this is less than u_n, since the contents of the brackets are all +.

Hence the error involved in taking $\sin x$ equal to x is numerically $<\frac{1}{6}x^3$.

Suppose that the angle is 5°; then $x=\pi/36$, and the amount of error $<\pi^3/(6\times 36^3)$, i.e. $<{\cdot}00011$. The proportional error is

$$\frac{{\cdot}00011}{\sin 5^\circ}=\frac{{\cdot}00011}{{\cdot}0872}={\cdot}00126,\text{ or about } \tfrac{1}{8} \text{ per cent.}$$

If we want to find for what values of x the substitution of x for $\sin x$ will be correct to 3 places of decimals [i.e. so that the error may be $<{\cdot}001$], we put $\frac{1}{6}x^3<{\cdot}001$ numerically.

This gives $|x|<\sqrt[3]{{\cdot}006}$, i.e. $<{\cdot}1817$, which is the circular measure of 10·4°.

Hence, if this degree of approximation is required, it is sufficient to substitute x for $\sin x$, provided the angle is between −10° and +10°. If the angle is larger than this, or if a higher degree of accuracy be required, we can take $x-\frac{1}{6}x^3$ instead of $\sin x$. The error involved in this case is numerically less than $x^5/5!$, i.e. $\frac{1}{120}x^5$. Therefore this will give the result correct to 3 decimal places if $\frac{1}{120}x^5$ is numerically $<{\cdot}001$, i.e. if $|x|<\sqrt[5]{{\cdot}12}$, i.e. if $|x|<{\cdot}6543$, which is true for angles between $-37\frac{1}{2}^\circ$ and $+37\frac{1}{2}^\circ$. It will give the result correct to 4 decimal places if $\frac{1}{120}x^5<{\cdot}0001$, i.e. if $|x|<{\cdot}4129$, which is true for angles between $-23\frac{1}{2}^\circ$ and $+23\frac{1}{2}^\circ$.

(iv) *Expand* $\cos x$ *in a series of powers of* x.

$f(x)=\cos x,\ f'(x)=-\sin x,\ f''(x)=-\cos x,\ f'''(x)=\sin x,$
$f''''(x)=\cos x$, &c.

$\therefore\ f(0)=1,\ f'(0)=0,\ f''(0)=-1,\ f'''(0)=0,\ f''''(0)=1,\ldots.$

All the coefficients of odd powers of x are 0 [as is evident from the fact that $\cos x$ is an even function of x], and the coefficients of the even powers of x are alternately +1 and −1.

Hence $$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots.$$

Exactly as in the last case the series is convergent, and the expansion holds for all values of x.

From this also it follows that as $x\to 0$, $\cos x\to 1$, and if a more accurate approximation be required, $1-\frac{1}{2}x^2$ may be substituted for $\cos x$ [Art. 13 (10)]. The range of values for which these substitutions agree with the value of $\cos x$ to any given degree of accuracy may be found as in the similar cases for $\sin x$.

(v) *Expand* $(1+x)^m$ *in ascending powers of* x.

$$f(x)=(1+x)^m,\ f'(x)=m(1+x)^{m-1},\ f''(x)=m(m-1)(1+x)^{m-2},\ \dots$$
$$\dots f^{(n)}(x)=m(m-1)\dots(m-n+1)(1+x)^{m-n}.$$

Hence $f(0)=1,\ f'(0)=m,\ f''(0)=m(m-1),\ \dots$
$$\dots f^{(n)}(0)=m(m-1)\dots(m-n+1).$$

$$\therefore\quad (1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\dots+\frac{m(m-1)\dots(m-n+1)}{n!}x^n+\dots,$$

the well-known binomial series. This series is convergent if $|x|<1$ [Ex. XXXI. 9].

The remainder after n terms is, using Cauchy's form,

$$\frac{x^n}{(n-1)!}m(m-1)\dots(m-n+1)(1+\theta x)^{m-n}(1-\theta)^{n-1}$$
$$=mx\times\frac{(m-1)(m-2)\dots(m-n+1)}{(n-1)!}x^{n-1}\times(1+\theta x)^{m-1}\times\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.$$

Of these factors the first and third are finite, the second $\rightarrow 0$, being the n^{th} term of a series, viz. $(1+x)^{m-1}$, known to be convergent if $|x|<1$, and the fourth cannot be more than 1 since $1-\theta$ cannot be $>1+\theta x$.

Hence, if $|x|<1$, the remainder tends to 0, and the expansion holds.

By the use of Taylor's Theorem we can obtain the expansion of $(x+y)^m$ when m is a positive integer.

This being $f(x+y)$, $f(x)=x^m$, $f'(x)=mx^{m-1}$, $f''(x)=m(m-1)x^{m-2}, \dots$ $f^{(m)}(x)=m!$, and all higher differential coefficients are zero, so that the series terminates at this stage. Hence, using the form of the theorem given in Art. 228, and replacing h by y,

$$(x+y)^m=x^m+y\,.\,mx^{m-1}+\frac{y^2}{2!}m(m-1)x^{m-2}+\dots+\frac{y^m}{m!}\,.\,m!$$
$$=x^m+mx^{m-1}y+\frac{m(m-1)}{2!}x^{m-2}y^2+\dots+y^m.$$

(vi) *Expand* $\cos(x+h)$ *in a series of ascending powers of* h.

Here $f(x+h)=\cos(x+h)$, $f(x)=\cos x$, $f'(x)=-\sin x$,
$$f''(x)=-\cos x,\ f'''(x)=\sin x\dots.$$

Therefore, by Taylor's Theorem,
$$\cos(x+h)=\cos x-h\sin x-\frac{h^2}{2!}\cos x+\frac{h^3}{3!}\sin x+\dots.$$

If we transfer the $\cos x$ to the left-hand side and divide by h, we get
$$\frac{\cos(x+h)-\cos x}{h}=-\sin x-\frac{h}{2!}\cos x+\frac{h^2}{3!}\sin x+\dots.$$

Hence, if h is very small, $\dfrac{\cos(x+h)-\cos x}{h}$ is approximately equal to $-\sin x$ and tends to the limit $-\sin x$ as $h\rightarrow 0$, as in the differentiation of

$\cos x$ from first principles. Hence taking the (small) increase in $\cos x$ equal to $-\sin x \times$ the increase in x, as in Art. 43, Ex. (ii), is equivalent to expanding by Taylor's Theorem and retaining only the first two terms. We have, in Art. 43, evaluated $\cos 135° 1'$ in this way.

We can now find approximately the increase in the function due to a larger increase in the variable. E.g. to find $\cos 136°$. If in the preceding result we put $x = 135°$, $h =$ the radian measure of $1° = \frac{1}{180}\pi$, and retain the first three terms, we get

$$\begin{aligned}\cos 136° &= \cos 135° - \frac{\pi}{180}\sin 135° - \frac{\pi^2}{2\,.\,32400}\cos 135° \\ &= -\cdot 707107 - \cdot 012341 + \cdot 000108 \\ &= -\cdot 719340, \text{ which is correct to 6 decimal places.}\end{aligned}$$

(vii) *To expand* $\tan^{-1} x$ *in a series of powers of* x.

This example is rather more difficult than those hitherto considered.

We have $f(x) = \tan^{-1} x$, $f'(x) = \dfrac{1}{1+x^2}$, $f''(x) = \dfrac{-2x}{(1+x^2)^2}$, whence $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$. The successive differential coefficients calculated in this manner soon become very complicated; but their values for $x = 0$ can be obtained by making use of Leibnitz's Theorem (Art. 111).

We first obtain a differential equation connecting any three consecutive differential coefficients.

We have $\qquad (1+x^2) f'(x) = 1;$

differentiating this n times, the result, by Art. 111, is

$$(1+x^2) f^{(n+1)}(x) + n\,.\,2x\,.\,f^{(n)}(x) + \frac{n(n-1)}{2}\,.\,2\,.\,f^{(n-1)}(x) = 0,$$

which is the differential equation.

Putting $x = 0$ in this, we get $f^{(n+1)}(0) + n(n-1) f^{(n-1)}(0) = 0$,

i.e. $$f^{(n+1)}(0) = -n(n-1) f^{(n-1)}(0).$$

Putting $n = 2, 3, 4, 5, \ldots$ in turn, this gives

$f'''(0) = -2\,.\,1 f'(0) = -2\,!$; $\qquad f^{(4)}(0) = -3\,.\,2 f''(0) = 0$;

$f^{(5)}(0) = -4\,.\,3 f^{(3)}(0) = 4\,!$; $\qquad f^{(6)}(0) = -5\,.\,4 f^{(4)}(0) = 0$;

$f^{(7)}(0) = -6\,.\,5 f^{(5)}(0) = -6\,!$; $\qquad$ and so on.

All the even coefficients are zero; hence $\tan^{-1} x$ consists only of odd powers of x, as is evident also from the fact that it is an odd function of x. Therefore, substituting in Maclaurin's Theorem, we get

$$\tan^{-1} x = x - 2\,!\frac{x^3}{3\,!} + 4\,!\frac{x^5}{5\,!} - \ldots = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \ldots.$$

The expansion holds if x is between -1 and $+1$, and for $\tan^{-1} x$ we must take that value which lies between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$, since we have taken $\tan^{-1} x$ as 0 when $x = 0$.

[This important series can be otherwise obtained as follows:

$$\tan^{-1}x = \int_0^x \frac{1}{1+x^2}\,dx = \int_0^x (1+x^2)^{-1}\,dx = \int_0^x (1-x^2+x^4-\ldots)\,dx \ [\text{if } x^2<1]$$
$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^2 + \ldots,$$

assuming that the conditions under which an infinite series can be integrated term by term are here satisfied.]

We omit the investigation of the remainder after n terms.

As a numerical example, let us find, to 4 places of decimals, the value of $\tan^{-1}(\cdot 3)$. Substituting in the series just obtained, we have

$$\tan^{-1}(\cdot 3) = \cdot 3 - \tfrac{1}{3}\times\cdot 027 + \tfrac{1}{5}\times\cdot 00243 - \tfrac{1}{7}\times\cdot 0002187$$
$$= \cdot 3 - \cdot 009 + \cdot 000486 - \cdot 00003$$
$$= \cdot 2915 \text{ radians.}$$

This is about $16°\,42'$.

A few further examples of less important kind, illustrating uses of Taylor's Theorem, will now be given.

(viii) *Find the first four terms in the expansion of* $\log(1+e^x)$ *in ascending powers of* x.

In this case

$$f(x) = \log(1+e^x); \qquad \therefore\ f(0) = \log 2.$$

$$f'(x) = \frac{e^x}{1+e^x}, \text{ which may be written } 1 - \frac{1}{1+e^x} \qquad \therefore\ f'(0) = \tfrac{1}{2}.$$

$$f''(x) = \frac{1}{(1+e^x)^2}\times e^x \qquad \therefore\ f''(0) = \tfrac{1}{4}.$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - e^x\,2(1+e^x)\,e^x}{(1+e^x)^4} = \frac{e^x - e^{2x}}{(1+e^x)^3}; \qquad \therefore\ f'''(0) = 0.$$

$$f''''(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})\,3(1+e^x)^2 e^x}{(1+e^x)^6}$$
$$= \frac{(1+e^x)(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})}{(1+e^x)^4}; \qquad \therefore\ f''''(0) = -\tfrac{1}{8}.$$

Hence, using Maclaurin's Theorem,

$$\log(1+e^x) = \log 2 + \tfrac{1}{2}x + \tfrac{1}{8}x^2 - \tfrac{1}{192}x^4 + \ldots.$$

(ix) *Expand* $\sin(m\sin^{-1}x)$ *in a series of powers of* x.

In this case, $f(x) = \sin(m\sin^{-1}x)$; $\qquad \therefore\ f(0) = 0$.

$$f'(x) = \cos(m\sin^{-1}x)\times\frac{m}{\sqrt{(1-x^2)}}; \quad \therefore\ f'(0) = \cos 0\times m = m.$$

Again, $f(x)$ is an odd function of x, and therefore consists of odd powers of x only. The n^{th} differential coefficient cannot be obtained, but, as in Ex. (vii), a relation between successive differential coefficients can be obtained by Leibnitz's Theorem, from which their values when $x=0$ can easily be calculated.

We have $\qquad \sqrt{(1-x^2)}\,.\,f'(x) = m\cos(m\sin^{-1}x).$

Differentiating again,

$$\sqrt{(1-x^2)}\,.\,f''(x)+f'(x)\,.\,\frac{-x}{\sqrt{(1-x^2)}} = m\times -\sin(m\sin^{-1}x)\,.\,\frac{m}{\sqrt{(1-x^2)}},$$

$$\therefore\quad (1-x^2)\,f''(x)-xf'(x) = -m^2 f(x).$$

If this be differentiated n times by Leibnitz's Theorem, we get

$$[(1-x^2)f^{(n+2)}(x)+n\,(-2\,x)\,f^{(n+1)}(x)+\frac{n\,(n-1)}{2\,!}\,.\,(-2)\,f^{(n)}(x)]$$
$$-[xf^{(n+1)}(x)+n\,.\,f^{(n)}(x)] = -m^2 f^{(n)}(x).$$

Putting $x=0$, this becomes

$$f^{(n+2)}(0)-n\,(n-1)\,f^{(n)}(0)-nf^{(n)}(0) = -m^2 f^{(n)}(0),$$

i.e. $$f^{(n+2)}(0) = -(m^2-n^2)\,f^{(n)}(0).$$

If $n=1$, then $f'''(0) = -(m^2-1^2)\,f'(0) = -m\,(m^2-1^2)$.

If $n=3$, then $f^{(5)}(0) = -(m^2-3^2)\,f'''(0) = +m\,(m^2-1^2)\,(m^2-3^2)$.

If $n=5$, then $f^{(7)}(0) = -(m^2-5^2)\,f^{(5)}(0) = -m\,(m^2-1^2)\,(m^2-3^2)\,(m^2-5^2)$.

Hence we have

$$\sin(m\sin^{-1}x) = mx-\frac{m\,(m^2-1^2)}{3\,!}x^3+\frac{m\,(m^2-1^2)\,(m^2-3^2)}{5\,!}x^5-\ldots.$$

Taylor's Theorem is often very useful in tabulating the values of a function for a series of values of the variable which are close together. For instance,

(x) *Calculate the values of the function* $y = x^2(16-x^2)$ *from* $x = 1{\cdot}7$ *to* $x = 2{\cdot}3$ *at intervals of* $\cdot 1$.

Here $f(x) = 16\,x^2-x^4$, $f'(x) = 32\,x-4\,x^3$, $f''(x) = 32-12\,x^2$,
$f'''(x) = -24\,x$, $f''''(x) = -24$, and all higher d. c.'s are zero.

Now, by Taylor's Theorem,

$$f(2+h) = f(2)+hf'(2)+\frac{h^2}{2\,!}f''(2)+\frac{h^3}{3\,!}f'''(2)+\frac{h^4}{4\,!}f''''(2)$$
$$= 48+h\,.\,32+\frac{h^2}{2}(-16)+\frac{h^3}{6}(-48)+\frac{h^4}{24}(-24)$$
$$= 48+32h-8h^2-8h^3-h^4.$$

If $h = -{\cdot}3$, $f(1{\cdot}7) = 48-9{\cdot}6-{\cdot}72+{\cdot}216-{\cdot}0081 = 37{\cdot}8879$.

If $h = -{\cdot}2$, $f(1{\cdot}8) = 48-6{\cdot}4-{\cdot}32+{\cdot}064-{\cdot}0016 = 41{\cdot}3424$.

If $h = -{\cdot}1$, $f(1{\cdot}9) = 48-3{\cdot}2-{\cdot}08+{\cdot}008-{\cdot}0001 = 44{\cdot}7279$.

If $h = 0$, $f(2) = 48$.

If $h = +{\cdot}1$, $f(2{\cdot}1) = 48+3{\cdot}2-{\cdot}08-{\cdot}008-{\cdot}0001 = 51{\cdot}1119$.

If $h = +{\cdot}2$, $f(2{\cdot}2) = 48+6{\cdot}4-{\cdot}32-{\cdot}064-{\cdot}0016 = 54{\cdot}0144$.

If $h = +{\cdot}3$, $f(2{\cdot}3) = 48+9{\cdot}6-{\cdot}72-{\cdot}216-{\cdot}0081 = 56{\cdot}6559$.

230. Failure of Taylor's Theorem.

The theorem was proved on the supposition that $f(x)$ and all its differential coefficients up to the n^{th} are continuous throughout the range of the variable considered, and the expansion cannot be effected if any one of the differential coefficients becomes discontinuous for a value within this range. If the n^{th} differential coefficient becomes infinite for a value of x within the range, the function cannot be expanded in an infinite series, but the theorems of Arts. 227 and 228 still hold provided the series be terminated at the $(n+1)^{\text{th}}$ term.

For instance, neither $\log x$ nor $\operatorname{cosec} x$ can be expanded in a series of positive integral powers of x, for both are discontinuous when $x=0$; $f(0)=\infty$, and the series fails at the first term. But, as in Ex. (ii) of the preceding article, $\log x$ can be expanded in a series of powers of $x-a$. Also it can easily be proved that the function $\operatorname{cosec} x - 1/x$ and its differential coefficients are continuous when $x=0$, and therefore $\operatorname{cosec} x - 1/x$ can be expanded in a series of positive integral powers of x.

Again, if we expand $(x+y)^{\frac{5}{2}}$ by Taylor's Theorem, we have

$$f(x)=x^{\frac{5}{2}},\quad f'(x)=\tfrac{5}{2}x^{\frac{3}{2}},\quad f''(x)=\tfrac{15}{4}x^{\frac{1}{2}},\quad f'''(x)=\frac{15}{8}\cdot\frac{1}{x^{\frac{1}{2}}},\ldots$$

$$(x+y)^{\frac{5}{2}}=x^{\frac{5}{2}}+\tfrac{5}{2}x^{\frac{3}{2}}y+\tfrac{15}{4}x^{\frac{1}{2}}\cdot\frac{y^2}{2}+\frac{15}{8}\,\frac{y^3}{x^{\frac{1}{2}}\,3\,!}+\ldots.$$

But this ceases to hold if $x=0$, because then the third differential coefficient $f'''(x)$, and therefore the fourth term of the expansion, become infinite.

In this case the function expanded becomes $y^{\frac{5}{2}}$, and this, being a fractional power of y, cannot be expressed as a series of positive integral powers of y. But the result of Article 228 still holds if we terminate the series at the fourth term, for then

$$(x+y)^{\frac{5}{2}}=x^{\frac{5}{2}}+\tfrac{5}{2}x^{\frac{3}{2}}y+\tfrac{15}{8}x^{\frac{1}{2}}y^2+\frac{15}{48}\cdot\frac{y^3}{(x+\theta y)^{\frac{1}{2}}}.$$

If we put $x=0$ in this, we get

$$y^{\frac{5}{2}}=0+0+0+\frac{5}{16}\,\frac{y^3}{(\theta y)^{\frac{1}{2}}},$$

i. e. $y^{\frac{5}{2}}=\frac{5}{16}\cdot y^{\frac{5}{2}}/\theta^{\frac{1}{2}}$, which is true when $5=16\theta^{\frac{1}{2}}$, i.e. when $\theta=\frac{25}{256}$.

Examples XCIII.

Expand the following functions in series of ascending powers of x, giving the first 4 terms, and state for what values of x they are convergent:

1. e^{ax}. 2. $\sin mx$. 3. $\cos mx$. 4. $\log(a+x)$. 5. $\log(a-x)$.
6. 2^x. 7. a^{mx}. 8. $\sin^2 x$. 9. $\sinh x$.

Verify the following expansions:

10. $\sin(x+\alpha)=\sin\alpha+x\cos\alpha-\dfrac{x^2}{2!}\sin\alpha-\dfrac{x^3}{3!}\cos\alpha+\ldots.$

11. $\tan(x+\alpha) = \tan\alpha + x\sec^2\alpha + x^2\sec^2\alpha\tan\alpha + \ldots.$

12. $e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \ldots.$

13. $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \ldots$

14. $\log(1+\sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \ldots.$

15. $e^x\sin x = x + x^2 + \dfrac{2x^3}{3!} - \dfrac{2^2x^5}{5!} - \dfrac{2^3x^6}{6!} - \dfrac{2^3x^7}{7!} + \ldots$

16. $e^x\cos x = 1 + x - \dfrac{2x^3}{3!} - \dfrac{2^2x^4}{4!} - \dfrac{2^2x^5}{5!} + \dfrac{2^3x^7}{7!} + \ldots$

17. $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots.$

18. $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \ldots.$

19. $x\cot x = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6 - \ldots$

20. $x\operatorname{cosec} x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \ldots.$

21. $x/(e^x-1) = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \ldots$

22. $x/(e^x+1) = \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{48}x^4 - \ldots.$

23. $e^{a\sin^{-1}x} = 1 + ax + \dfrac{a^2x^2}{2!} + \dfrac{a(a^2+1^2)}{3!}x^3 + \dfrac{a^2(a^2+2^2)}{4!}x^4 + \ldots$

24. $\cos(m\sin^{-1}x) = 1 - \dfrac{m^2}{2!}x^2 + \dfrac{m^2(m^2-2^2)}{4!}x^4 - \dfrac{m^2(m^2-2^2)(m^2-4^2)}{6!}x^6 + \ldots.$

25. $\log(1+x+x^2) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \ldots.$

26. $\sin x\cosh x = x + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \ldots.$

27. $\cos x\sinh x = x - \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots.$

28. $\log\sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots.$

29. $\tan(\frac{1}{4}\pi + x) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \ldots.$

30. $\log(1+\cos x) = \log 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 - \ldots.$

31. $\dfrac{1}{x+h} = \dfrac{1}{x} - \dfrac{h}{x^2} + \dfrac{h^2}{x^3} - \dfrac{h^3}{x^4} + \ldots.$

32. $\tan^{-1}(1+x) = \frac{1}{4}\pi + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 - \ldots.$

33. $\sinh^{-1}x = \log\{x + \sqrt{(1+x^2)}\} = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \ldots.$

34. $\frac{1}{2}(\sin^{-1}x)^2 = \dfrac{1}{2!}x^2 + \dfrac{2^2}{4!}x^4 + \dfrac{2^2.4^2}{6!}x^6 + \dfrac{2^2.4^2.6^2}{8!}x^8 + \ldots$

35. $\tan^{-1}(x+h) = \tan^{-1}x + \dfrac{h}{1+x^2} - \dfrac{xh^2}{(1+x^2)^2} + \ldots.$

36. $e^{2x}\sin 3x = 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + \ldots.$

37. $e^{ax}\cos bx = 1 + ax + \frac{1}{2}(a^2-b^2)x^2 + \frac{1}{6}a^3x^3 + \ldots.$

38. Show that $e^{ax}\cos bx$ may be expanded in the form

$$1 + xr\cos\phi + \frac{x^2r^2\cos 2\phi}{2!} + \ldots + \frac{x^nr^n\cos n\phi}{n!} + \ldots,$$

where $\phi = \tan^{-1}(b/a) = \cos^{-1}(a/r)$.

39. For what values of x (in degrees) will the substitution of 1 for $\cos x$ be correct to 2 decimal places?

40. For what values will the substitution of $1 - \frac{1}{2}x^2$ for $\cos x$ be correct to 3 decimal places?

41. For what values will the substitution of x for $\tan^{-1}x$ be correct to 2 decimal places?

42. For what values will the substitution of $x-\frac{1}{3}x^3$ for $\tan^{-1}x$ be correct to (i) 2 decimal places, (ii) 3 decimal places?

43. Given $\cos 60° = \cdot 5$, $\sin 60° = \cdot 86603$, find to 5 places of decimals the values of $\cos 61°$, $\sin 61°$, $\cos 62°$, $\sin 62°$.

44. Calculate from Maclaurin's Theorem the value to 4 places of decimals of $\sin 30°$, $\sin 60°$, $\cos 30°$, $\cos 60°$.

45. Find to 4 places of decimals the values of $\tan 15°$ and $\tan 55°$ (see Ex. 17 and 29).

46. Calculate, by aid of Art. 229 (ix) and Ex. 24 above, the value of $\sin(3\sin^{-1}\frac{1}{10})$ and $\cos(2\sin^{-1}\frac{1}{5})$.

47. Prove that the d. c. of $e^{x\cos\alpha}\cos(x\sin\alpha) = e^{x\cos\alpha}\cos(x\sin\alpha+\alpha)$. Hence find the n^{th} differential coefficient.
Deduce the expansion of $e^{x\cos\alpha}\cos(x\sin\alpha)$ in a series of ascending powers of x.

48. Draw on the same diagram the graphs of x, $x-\dfrac{x^3}{3!}$, $x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}$, and compare with the graph of $\sin x$.

49. Draw on the same diagram the graphs of x, $x-\frac{1}{3}x$, $x-\frac{1}{3}x^3+\frac{1}{5}x^5$, from $x=-1$ to $+1$, and compare with the graph of $\tan x$.

50. Deduce, by expanding $f(a+h)$ and $f(a-h)$ in powers of h, the conditions obtained for a maximum and minimum in Art. 58, that $f(x)$ is a maximum if $f'(a)=0$ and $f''(a)$ is $-$, and a minimum if $f'(a)=0$ and $f''(a)$ is $+$.

51. Prove, generally, that if the first of the quantities $f'(a), f''(a), f'''(a), \ldots$, which does not vanish is of odd order, $f(a)$ is neither a maximum nor a minimum value of $f(x)$; and that if the first that does not vanish is of even order, $f(a)$ is a maximum or minimum according as the first non-vanishing function is $-$ or $+$.

52. By putting $n=\frac{1}{2}(1/x-1)$ [whence $x=1/(2n+1)$], and using Art. 229 (ii), deduce an expansion for $\log(1+1/n)$ in a series of negative powers of $2n+1$.
Hence calculate to 4 places of decimals the logarithms to base e of all integers from 1 to 20.

53. An arc of a circle subtends an angle of $2x$ radians at the centre. If a and b be the lengths of the chords of the whole arc and half the arc respectively, $a=2r\sin x$ and $b=2r\sin\frac{1}{2}x$. By expanding the sines by Maclaurin's Theorem, find the difference between $\frac{1}{3}(8b-a)$ and the length of the arc.
This is known as Huyghen's approximation to the length of a circular arc. Show that, if this approximation be used to find the length of the arc which subtends an angle of 30° at the centre of a circle of radius 100,000 feet, the error is only about 2 inches.

54. Show, by taking the remainder in (i), Art. 227, in the form $(x-a)^pR$, that the remainder after n terms may be expressed in the form

$$\frac{(x-a)^n(1-\theta)^{n-p}}{p(n-1)!}f^{(n)}\{a+\theta(x-a)\}.$$

This is known as the 'Schlömilch-Roche form of the remainder'. Lagrange's and Cauchy's forms are obtained by taking $p=n$ and $p=1$ respectively.

CHAPTER XXIII

PARTIAL DIFFERENTIATION

231. Functions of more than one variable. Partial differential coefficients.

Hitherto we have dealt exclusively with functions of only a single variable such as x or t, but functions of more than one variable frequently occur. For example, the area of a rectangle is a function of two variables, the length and the breadth; the volume of a rectangular parallelepiped is a function of three variables, the length, breadth, and thickness; the pressure of a given mass of gas depends upon its density and its temperature, and so on.

If z be a function of two variables x and y, a fact which is indicated by the notation $z = f(x, y)$, either x alone or y alone or both x and y simultaneously may be varied, and in each case a change in the value of z will result. Generally the change in the value of z will be different in each of these three cases, e.g. the area of a rectangle whose sides are 6 and 10 inches is 60 square inches; an increase of 1 inch in the length alone will increase the area by 6 square inches, an increase of 1 inch in the breadth alone will increase the area by 10 square inches, and an increase of 1 inch in both simultaneously will increase the area by 17 square inches.

If x and y be changed to $x+\delta x$, $y+\delta y$ respectively, the new value of z will be denoted by $f(x+\delta x,\ y+\delta y)$.

The function z may be defined as *continuous* for any particular values of x and y if, when x and y have these values,

$$\text{Lt}\ [f(x+\delta x,\ y+\delta y)-f(x, y)] = 0,$$

when δx and $\delta y \longrightarrow 0$ in any manner whatever.

Briefly, z is a continuous function of x and y if indefinitely small changes in either x or y separately, or in both together, produce only an indefinitely small change in z.

Suppose that, when x is changed to $x+\delta x$ *and y remains constant*, z becomes $z+\delta z$. The ratio $\delta z/\delta x$ will tend to a finite limit as $\delta x \longrightarrow 0$, if z is continuous for these values of x and y.

This limit is called the *partial differential coefficient* of z with respect to x, and is denoted by the symbol $\partial z/\partial x$ [or sometimes $D_x z$, or $\partial f/\partial x$ or f_x, if z be written as $f(x, y)$].

Similarly, when y is changed to $y+\delta y$ *and x remains constant*, let z change to $z+\delta' z$.* The limit of $\delta' z/\delta y$ as $\delta y \to 0$ is called the partial differential coefficient of z with respect to y, and is denoted by the symbol $\partial z/\partial y$ [or sometimes $D_y z$, or $\partial f/\partial y$ or f_y, if z be written in the form $f(x, y)$]. A similar notation is used if z be a function of more than two variables.

If z be written in the form $f(x, y)$, we may define $\partial z/\partial x$ as the limit, when $\delta x \to 0$, of $\dfrac{f(x+\delta x, y)-f(x, y)}{\delta x}$, and $\partial z/\partial y$ as the limit, when $\delta y \to 0$, of $\dfrac{f(x, y+\delta y)-f(x, y)}{\delta y}$.

Hence, to find $\partial z/\partial x$, differentiate z with respect to x, regarding y as constant.

To find $\partial z/\partial y$, differentiate z with respect to y, regarding x as constant.

Examples:

(i) If $z = x^3+2axy+y^3$, $\quad \partial z/\partial x \quad 3x^2+2ay, \quad \partial z/\partial y = 2ax+3y^2$.

(ii) If $z=\tan^{-1}\dfrac{x}{y}$,

$$\frac{\partial z}{\partial x} = \frac{1}{1+x^2/y^2} \times \frac{1}{y} = \frac{y}{y^2+x^2},$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+x^2/y^2} \times -\frac{x}{y^2} \quad \frac{-}{y^2+x^2}.$$

(iii) The volume V of a cylinder of radius r and height h is $\pi r^2 h$.

$$\therefore \quad \partial V/\partial h = \pi r^2; \quad \partial V/\partial r = 2\pi rh,$$

i.e. the rate of increase of the volume per unit increase of the height, the radius remaining constant, is πr^2; the rate of increase of the volume per unit increase of the radius, the height remaining constant, is $2\pi rh$. These results can be verified geometrically, for, when the radius remains constant and the height is increased by a small amount δh, the volume is increased by a thin circular slice added to one end, of volume $\pi r^2 \delta h$,

i.e. $$\delta V = \pi r^2 . \delta h \quad \text{and} \quad \frac{\partial V}{\partial h} = \mathrm{Lt}\frac{\delta V}{\delta h} = \pi r^2.$$

Similarly, if the radius is increased by a small amount δr, while the height remains constant, the volume is increased by a thin coating all over the curved surface, whose inner superficial area is $2\pi rh$ and outer superficial area $2\pi(r+\delta r)h$, and hence its volume

$$\delta V > 2\pi rh\delta r \quad \text{and} \quad < 2\pi(r+\delta r)h\delta r.$$

Therefore $$\delta V/\delta r > 2\pi rh \quad \text{and} \quad < 2\pi(r+\delta r)h,$$

and when $$\delta r \to 0, \quad \partial V/\partial r = 2\pi rh.$$

* The change in z in this case will generally be different from the change in z in the preceding case.

232. Geometrical representation of partial differential coefficients.

If values of x and y be taken as coordinates of a point in a plane XOY, to each pair of simultaneous values of x and y corresponds a point Q in the plane (Fig. 176). At Q erect a perpendicular QP to the plane XOY to represent the corresponding value of z; then, if x and y vary continuously, and z is a continuous function of x and y, P traces out a surface.

For instance, if $z=\sqrt{(a^2-x^2-y^2)}$, $z=QP$, and $x^2+y^2=OQ^2$.

$$\therefore \quad QP=\sqrt{(a^2-OQ^2)}, \quad \text{i.e.} \quad a^2=OQ^2+QP^2=OP^2.$$

Hence $OP=a$, and the locus of P is a sphere with centre O and radius a.

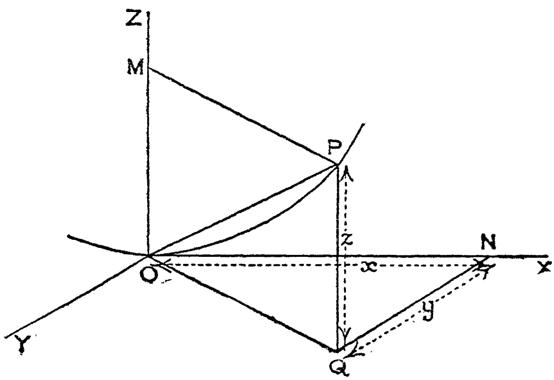


Fig. 176.

Again, if $az=x^2+y^2$, the coordinates of all the points Q, at which the height of the perpendicular QP is b, satisfy the equation $ab=x^2+y^2$, and this is the equation of a circle, centre O and radius $\sqrt{(ab)}$. Hence the locus of P is a circle of radius $\sqrt{(ab)}$ whose centre is on OZ at height b above O; therefore the section of the surface by a plane parallel to the plane XOY is a circle. Moreover, since $QP=z=(x^2+y^2)/a=OQ^2/a$, i.e. $MP^2=aOM$ [cf. $y^2=ax$], it follows that if the plane QOM (Fig. 176) be fixed, all positions of P in that plane are on a parabola, vertex O and axis OM; hence the section by the plane MOQ, and similarly by any other plane through OM, is a parabola. Therefore the equation $az=x^2+y^2$ represents the paraboloid of revolution formed by the rotation of this parabola about its axis OZ.

In the general case (Fig. 177), by taking y constant and varying x and therefore z, we get a section of the surface by a plane parallel to

the plane XOZ, the curve EPF in the figure. Then, exactly as in Art. 23, $\partial z/\partial x$ is the slope of this curve at P, i.e. the tangent of the angle ψ, which the tangent to the curve EPF at P makes with the line MN in which the plane of the section cuts the plane XOY.

Similarly, by taking x constant and varying y and therefore z, we get a section of the surface by a plane parallel to the plane YOZ, the curve HPK in the figure; and $\partial z/\partial y$ is the slope of this curve at P, i.e. the tangent of the angle ψ' which the tangent to the curve HPK at P makes with the line DG in which the plane of the section cuts the plane XOY.

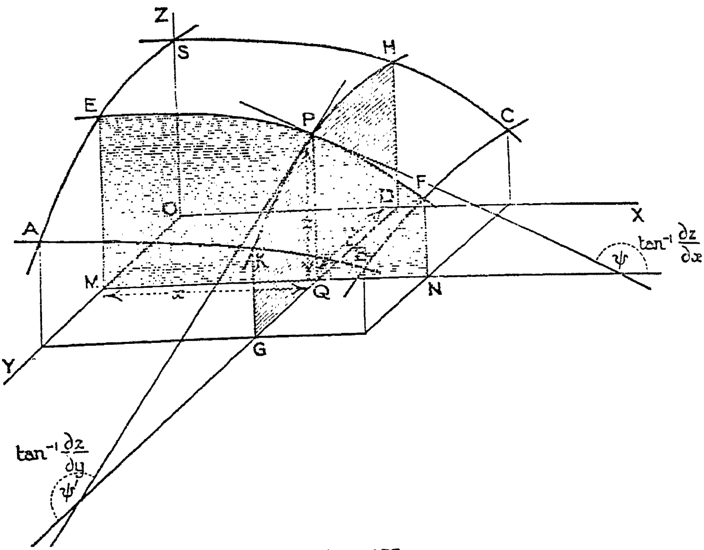


Fig. 177.

For example, in the first case mentioned above, where $z^2 = a^2 - x^2 - y^2$, and P moves on the surface of a sphere, centre O and radius a, the partial differential coefficient of z with respect to x is given by

$$2z \,.\, \partial z/\partial x = -2x, \quad \text{i.e.} \quad \partial z/\partial x = -x/z.$$

This is easily verified geometrically, for the section EPF will in this case be a circle, centre M, and the angle ψ which the tangent at P makes with MN is $90° + PMQ$, since the tangent is now perpendicular to MP.

Hence $\tan\psi = -\cot QMP = -MQ/QP = -x/z$.

Similarly $\partial z/\partial y = -y/z$, which can be verified geometrically in exactly similar manner.

Examples XCIV.

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in the following cases:

1. $z = \tan(ax+by)$.
2. $z = (x-y)/(x+y)$.
3. $z = e^{px+qy}$.
4. $z = ax^2+2hxy+by^2+2gx+2fy+c$.
5. $z = (ax^2+by^2)^r$.
6. $\sin^{-1}(x/y)$.
7. $z = xy/(x+y)$.
8. $z^2 = x^2-y^2$.
9. $z^n = x^n+y^n$.
10. $z^2 = (x^2-y^2)/(x^2+y^2)$.
11. $ax^2+by^2+cz^2 = 1$.
12. $xy+yz = zx$.

Find $\partial V/\partial x$, $\partial V/\partial y$, $\partial V/\partial z$ in the following cases:

13. $V = x^2+y^2+z^2$.
14. $V = \tan^{-1}\{(x+y)/z\}$.
15. $V = 1/\sqrt{(x^2+y^2+z^2)}$.
16. $V = ax^2+by^2+cz^2+2hxy+2fyz+2gzx$.
17. Prove that, if $z = x^3-3x^2y-2y^3$, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = 3z$.
18. Prove that, if $z = \dfrac{x^2+y^2}{\sqrt{(x+y)}}$, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = \frac{3}{2}z$.
19. Prove that, if $z = \sin^{-1}\dfrac{x-y}{x+y}$, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = 0$.
20. Prove that, if $z = xf\left(\dfrac{x}{y}\right)$, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}$
21. The last four examples are particular cases of 'Euler's Theorem of Homogeneous Functions', viz.: If z be a homogeneous function of x and y of degree n, then $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = nz$. By writing such a function in the form $x^n f(y/x)$, prove this theorem.
22. Find the rate of increase of the volume of a right circular cone (i) when the radius of the base is constant and the height increases at the rate of 1 inch per second, (ii) when the height is constant and the radius of the base increases at the rate of 1 inch per second.
23. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^2+y^2$. Verify the result geometrically.
24. The radius of a cylinder of volume V and height h is equal to $\sqrt{(V/\pi h)}$. Find the rate of increase of the radius at the instant when r is 4 inches and h is 1 foot (i) if the height is constant and the volume increases at the rate of 10 cubic inches per second, (ii) if the volume is constant and the height decreases at the rate of 1 inch per second.
25. The area of the curved surface of a right circular cone, height h and radius of base r, is $\pi r\sqrt{(r^2+h^2)}$; find the rate of increase of the area at the instant when r is 6 inches and h is 8 inches (i) if the radius is constant and the height is increasing at the rate of 1 inch per second, (ii) if the height is constant and the radius is increasing at the rate of $\frac{1}{2}$ inch per second.
26. Find $\partial z/\partial x$ and $\partial z/\partial y$ when $x^2/a^2+y^2/b^2+z^2/c^2 = 1$. Explain the result geometrically.

27. If v be the volume, T the absolute temperature, and p the intensity of pressure of a given mass of a perfect gas, p, v, T are connected by the relation $pv = kT$, where k is a constant. Find (i) the rate of increase of the intensity of pressure per unit increase of temperature, supposing the volume to remain constant; (ii) the rate of increase of the intensity of pressure per unit increase of volume, the temperature being supposed to remain constant; (iii) the rate of increase of the volume per unit increase of temperature, the pressure being supposed to remain constant.

233. Total differential of a function of two variables.

If z be a continuous function of x and y, and if x and y receive small increments δx and δy (which are usually quite independent of one another), z will receive a small increment δz; to find the relation between δz, δx, and δy.

If x alone varies and y remains constant, we know (Art. 24) that the resulting increment of z is $\frac{\partial z}{\partial x} . \delta x$ approximately, to the first order of small quantities; and if y alone varies and x remains constant, the resulting increment of z is $\frac{\partial z}{\partial y} . \delta y$ approximately. We shall now show that, when x and y vary simultaneously, the total resulting increment δz is, to the first order of small quantities, equal to the sum of these two partial increments, i.e. the ratio of the total increment δz to the sum of these two partial increments $\to 1$, when δx and δy each $\to 0$.

If $z = f(x, y)$, we have the total increment

$$\begin{aligned}\delta z &= f(x+\delta x,\ y+\delta y) - f(x, y)\\ &= [f(x+\delta x,\ y+\delta y) - f(x,\ y+\delta y)] + [f(x,\ y+\delta y) - f(x, y)].\end{aligned}$$

By the Mean-Value Theorem (Art. 117) the expression in the first square brackets

$$= \delta x . f_x(x+\theta\delta x,\ y+\delta y), \text{ where } 0 < \theta < 1,$$

and f_x denotes the partial d. c. with respect to x.

Similarly, the expression in the second square brackets

$$= \delta y . f_y(x,\ y+\theta'\delta y), \text{ where } 0 < \theta' < 1,$$

and f_y denotes the partial d. c. with respect to y.

$$\therefore \quad \delta z = \delta x . f_x(x+\theta\delta x,\ y+\delta y) + \delta y . f_y(x,\ y+\theta'\delta y).$$

Since z and its differential coefficients are supposed continuous, $f_x(x+\theta\delta x,\ y+\delta y)$ tends to the limit $f_x(x, y)$, i.e. $\partial f/\partial x$, as δx and $\delta y \to 0$, and therefore may be written $\partial f/\partial x + \epsilon$. Similarly, $f_y(x,\ y+\theta'\delta y) \to$ the limit $\partial f/\partial y$, and may be written $\partial f/\partial y + \epsilon'$, where ϵ and $\epsilon' \to 0$ when δx and $\delta y \to 0$;

$$\therefore \quad \delta z = \delta x\left(\frac{\partial f}{\partial x} + \epsilon\right) + \delta y\left(\frac{\partial f}{\partial y} + \epsilon'\right). \qquad \text{(i)}$$

Hence, since the terms $\epsilon\delta x$ and $\epsilon'\delta y$ are of the second order of small quantities, and the other three terms of the first order, δz tends to equality with $\dfrac{\partial f}{\partial x}\delta x + \dfrac{\partial f}{\partial y}\delta y$ as δx and $\delta y \longrightarrow 0$,

i.e. $$dz = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y \text{ approximately.} \qquad \text{(ii)}$$

This is called the *total differential* of z.

If x and y, and therefore also z, are continuous functions of some other variable t, then, if δx, δy, and δz be the increments of x, y, and z due to an increment δt of t, we have, by dividing (i) by δt,

$$\frac{\delta z}{\delta t} = \left(\frac{\partial f}{\partial x} + \epsilon\right)\frac{\delta x}{\delta t} + \left(\frac{\partial f}{\partial y} + \epsilon'\right)\frac{\delta y}{\delta t};$$

hence, taking the limits, when δt and therefore also δx, δy, δz and therefore also ϵ and $\epsilon' \longrightarrow 0$,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

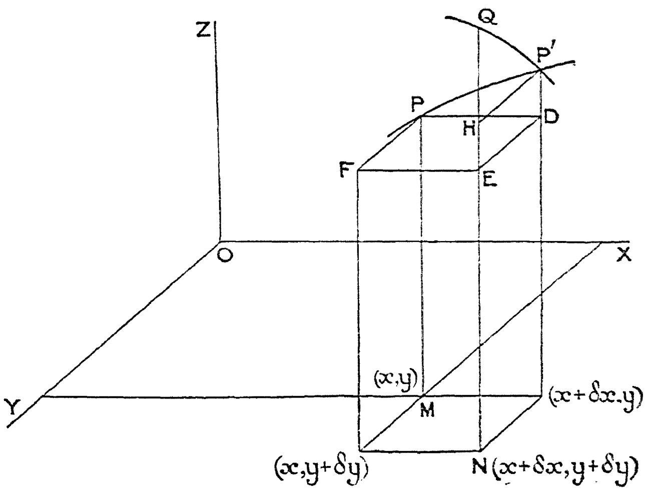


Fig. 178.

234. Geometrical illustrations.

The relation (ii) of the preceding article may be obtained geometrically by the method of Art. 232.

Let M, N (Fig. 178) be the points (x, y) and $(x+\delta x, y+\delta y)$ in the plane XOY, and let MP, NQ be the corresponding perpendiculars z and $z+\delta z$

The curve PP' represents the path of P as x increases to $x+\delta x$, y remaining constant; $P'Q$ represents the path of P as y increases to $y+\delta y$, the abscissa remaining constant and equal to its increased value $x+\delta x$; then EQ represents δz, the total increase in z.

$$\delta z = EQ = DP' + HQ = \delta x \, . \tan P'PD + \delta y \, . \tan QP'H.$$

Taking the limits, when δx and $\delta y \longrightarrow 0$, $\tan P'PD$ becomes the slope of the section $P'P$, i.e. $\partial f/\partial x$, and $\tan QP'H$ becomes the slope of the section QP' which ultimately approaches coincidence with the section through P parallel to YOZ, whose slope is $\partial f/\partial y$.

Therefore we have, approximately,

$$\delta z = \frac{\partial f}{\partial x} \, . \, \delta x + \frac{\partial f}{\partial y} \, . \, \delta y.$$

Examples:

(i) Let A be the area of a rectangle whose sides are x and y (Fig. 179); then $A = xy$.

$$\frac{\partial A}{\partial x} = \mathop{\mathrm{Lt}} \frac{\text{increase in area}}{\text{increase in length } x}, \text{ the breadth } y \text{ remaining constant,}$$

$$= \mathop{\mathrm{Lt}} \frac{\text{area } FK}{EK} = \mathop{\mathrm{Lt}} \frac{EF \, . \, EK}{EK} = EF = y.$$

$$\frac{\partial A}{\partial y} = \mathop{\mathrm{Lt}} \frac{\text{increase in area}}{\text{increase in breadth } y}, \text{ the length } x \text{ remaining constant,}$$

$$= \mathop{\mathrm{Lt}} \frac{\text{area } HF}{GH} = \mathop{\mathrm{Lt}} \frac{GF \, . \, GH}{GH} = GF = x.$$

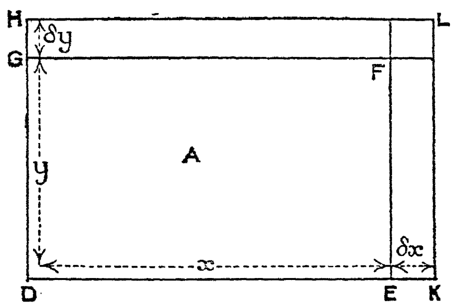


Fig. 179.

If x and y are simultaneously increased to DK and DH respectively, δA, the resulting increase in area, $= HF + FK + FL = x\delta y + y\delta x + \delta x \, . \, \delta y$.

The last of these terms is ultimately indefinitely small compared with the others, being of the second order of small quantities; hence to the first order of small quantities,

$$\delta A = y\delta x + x\delta y = \frac{\partial A}{\partial x} \, . \, \delta x + \frac{\partial A}{\partial y} \, . \, \delta y.$$

(ii) Let (x, y) be the rectangular coordinates of a point in a plane, and (r, θ) its polar coordinates; then x and y may each be regarded as functions of r and θ.

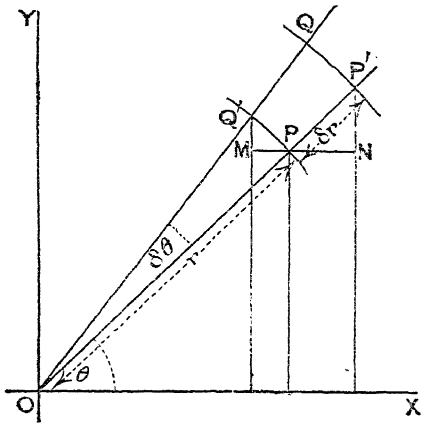


Fig. 180.

In Fig. 180, P is the point whose polar coordinates are (r, θ). An increase $\delta\theta$ in θ, r remaining constant, would move P to Q'; an increase δr in r, θ remaining constant, would move P to P'; the combined effect of both increases is to move P to Q.

$$\frac{\partial x}{\partial \theta} = \mathrm{Lt} - \frac{MP}{\delta\theta} = -\mathrm{Lt}\frac{PQ' \sin MQ'P}{\delta\theta}$$

$$= -\mathrm{Lt}\frac{\text{chord } PQ'}{\text{arc } PQ'} \cdot \frac{\text{arc } PQ'}{\delta\theta} \cdot \sin MQ'P;$$

as $\delta\theta \to 0$, (chord PQ')/(arc PQ') $\to 1$, the arc $PQ' = r\delta\theta$, $MQ'P \to$ the angle which the tangent at P makes with the ordinate of P, i.e. θ.

$$\therefore \quad \partial x/\partial\theta = -r\sin\theta.$$

Similarly,

$$\frac{\partial y}{\partial \theta} = \mathrm{Lt}\frac{MQ'}{\delta\theta} = \mathrm{Lt}\frac{PQ'\cos MQ'P}{\delta\theta}, \text{ as in the preceding case, } r\cos\theta.$$

Again, $\quad \partial x/\partial r = \mathrm{Lt}\,(PN/PP') = \cos P'PN = \cos\theta;$

$\quad \partial y/\partial r = \mathrm{Lt}\,(NP'/PP') = \sin P'PN = \sin\theta.$

[All these results follow immediately by differentiation from the relations $x = r\cos\theta$, $y = r\sin\theta$.]

δx, the total increment of x, due to increments δr and $\delta\theta$ in r and θ,

$=$ the projection of PQ on the axis of x

$= PP'\cos\theta - QP'\sin(\theta + \tfrac{1}{2}\delta\theta) = \delta r\cos\theta - (r+\delta r)\,\delta\theta\sin(\theta+\tfrac{1}{2}\delta\theta)$

$= \delta r\cos\theta - r\sin\theta\,\delta\theta$, to the first order (the other terms are infinitesimals of higher order)

$= \dfrac{\partial x}{\partial r}\delta r + \dfrac{\partial x}{\partial\theta}\delta\theta$ from the results just obtained.

Similarly for δy.

The student must be cautious when applying the theorems of Arts. 29–35 to partial differential coefficients, e.g. $\partial x/\partial r$ and $\partial r/\partial x$ are reciprocals *provided the same coordinate is constant in both cases*, but not otherwise.

In fact, it is easily seen that $\partial x/\partial r$ (θ constant) and $\partial r/\partial x$ (y constant), instead of being reciprocals, are equal to one another, for, from the preceding, we have $\partial x/\partial r$ (θ constant) $= \cos\theta$. To find $\partial r/\partial x$ (y constant), we take Fig. 181.

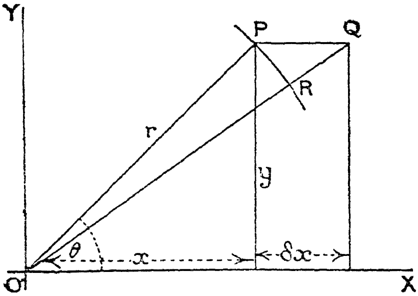


Fig. 181.

An increase δx in x, y remaining constant, will move P to Q, through a distance δx parallel to the axis of x; the new radius vector is OQ.

$$\therefore \quad \delta r = OQ - OP, \text{ and } \frac{\partial r}{\partial x} = \mathrm{Lt}\,\frac{(\quad)}{PQ} = \mathrm{Lt}\,\frac{\quad}{PQ},$$

if a circle, centre O and radius OP, cuts OQ in R. In the limit, $PRQ \to 90^\circ$ and therefore $RQ/PQ \to \cos RQP$, i.e. $\cos\theta$.

$$\therefore \quad \partial r/\partial x = \cos\theta.$$

Hence $\partial x/\partial r$ (θ constant) and $\partial r/\partial x$ (y constant) are equal.

This also follows analytically, for, since $x = r\cos\theta$,

$$\partial x/\partial r \ (\theta \text{ constant}) = \cos\theta;$$

and since $r^2 = x^2 + y^2$, $2r\dfrac{\partial r}{\partial x} = 2x$, i.e. $\dfrac{\partial r}{\partial x}$ (y constant) $= \dfrac{x}{r} = \cos\theta$.

235. Total differential coefficient.

It has been proved that, if z be a function of x and y when both are functions of t, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

If we take t to be x, i.e. if z be a function of x and y where y is also a function of x, then, since dx/dt is now unity, this relation becomes

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}. \qquad \text{(i)}$$

dz/dx is called the total differential coefficient of z with respect to x.

Similarly dz/dy, the total differential coefficient of z with respect to y, $= \dfrac{\partial z}{\partial y} + \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dy}$.

The quantities dz/dx and $\partial z/\partial x$ which occur in equation (i) are quite distinct.

$\partial z/\partial x$ is the limit of $\delta z/\delta x$, where δz is the increase in z due to a variation in x only where it occurs explicitly in the equation, i.e. on the supposition that y is independent of x; dz/dx is the limiting value of $\delta z/\delta x$, where δz is the total increment of z, due partly to the increment of x and partly to the increment of y which is itself due to that of x, since y is a function of x.

Geometrically, in Fig. 178, $\partial z/\partial x$ is the limiting value of DP'/PD; dz/dx is the limiting value of EQ/PD, i.e. of EQ/FE, and these two are usually quite different.

For instance, let $z = a^2 - x^2 - xy - y^2$, and let y be a function of x;

then $\qquad \partial z/\partial x = -2x - y$, and $\partial z/\partial y = -x - 2y$,

$$\therefore \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = -2x - y - (x + 2y)\frac{dy}{dx},$$

and the value of dy/dx will depend upon the relation between y and x.

Suppose, for instance, that $x^2 + y^2 = r^2$ (r constant); then

$$2x + 2y\frac{dy}{dx} = 0, \quad \text{and} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Hence in this case, $dz/dx = -2x - y + (x + 2y)\, x/y = (x^2 - y^2)/y$.

[This might have been obtained by first eliminating y from the given equations and thereby obtaining z as a function of x alone; but generally by this process, when it is feasible, the differentiation is rendered more complicated; and in many cases the actual elimination cannot be carried out. In the example under consideration, we should get

$$z = a^2 - x^2 - y^2 - xy = a^2 - r^2 - x\sqrt{(r^2 - x^2)};$$

$$\frac{dz}{dx} = -x \cdot \frac{-2x}{2\sqrt{(r^2 - x^2)}} - \sqrt{(r^2 - x^2)} = \frac{x^2 - (r^2 - x^2)}{\sqrt{(r^2 - x^2)}} = \frac{x^2 - y^2}{y}, \text{ as before.]}$$

The preceding results can easily be extended to a function of any number of variables.

236. Adiabatic expansion of a gas.

To illustrate the foregoing principles, let us consider the adiabatic expansion of a gas. We will prove the well-known theorem that if a given mass of gas expands adiabatically (i.e. so that heat neither enters nor leaves it),

$$pv^{\gamma} = \text{constant},$$

where p is the intensity of pressure, v the volume, and γ a numerical constant. If T be the absolute temperature, i.e. the temperature measured from $-273°$ C. or $-469°$ F., then, in the case of a 'perfect gas', p, v, T are connected by the relation $pv = kT$, where k is a constant.

If, *when the volume is kept constant*, a small quantity δQ of heat supplied to the gas raises the temperature by an amount δT, then as δQ and therefore also $\delta T \to 0$, $\delta Q/\delta T \to$ a limiting value, which is called the 'specific heat at constant volume' and is denoted by K_v.

If, *when the pressure is kept constant*, a small quantity δQ of heat raises the temperature by an amount δT, then $\delta Q/\delta T \to$ a limiting value, which is called the 'specific heat at constant pressure' and is denoted by K_p.

It can be shown that, for a perfect gas, the ratio K_p/K_v is a constant γ. The value of γ in the case of air (regarded as a perfect gas) is 1·404.

Since $pv = kT$, only two of the three variables p, v, T are independent. The third can be calculated when two of them are known.

Taking p and v as the independent variables, we have, if a small quantity δQ of heat be supplied,

$$\delta Q = \frac{\partial Q}{\partial v}\,\delta v + \frac{\partial Q}{\partial p}\,\delta p. \tag{i}$$

$\partial Q/\partial v$ is the d. c. of Q with respect to v, p being regarded as constant.

Now $\dfrac{\partial Q}{\partial v} = \dfrac{\partial Q}{\partial T}\cdot\dfrac{\partial T}{\partial v}$ [p constant]; also $\dfrac{\partial Q}{\partial T}$ (p constant) $= K_p$, and

$$\frac{\partial T}{\partial v}\ [p \text{ constant}] = \frac{p}{k}, \text{ since } pv = kT. \qquad \frac{\partial Q}{\partial v} = K_p\,.\,\frac{p}{k}.$$

Similarly, $\dfrac{\partial Q}{\partial p}$ (v constant) $= \dfrac{\partial Q}{\partial T}\cdot\dfrac{\partial T}{\partial p}$ (v constant) $= K_v\,.\,\dfrac{v}{k}$.

Hence, substituting in (i), $\delta Q = K_p\dfrac{p}{k}\,\delta v + K_v\dfrac{v}{k}\,\delta p$.

If the gas expands adiabatically, the amount of heat it contains is constant, i.e. $\delta Q = 0$.

Therefore $$K_p\frac{p}{k}\,\delta v + K_v\frac{v}{k}\,\delta p = 0.$$

Dividing by K_v/k and putting $K_p/K_v = \gamma$, we have

$$\gamma p\,\delta v + v\,\delta p = 0,$$

whence, in the limit, $$\frac{\gamma}{v}\frac{dv}{dp} + \frac{1}{p} = 0.$$

Integrating, $$\gamma \log v + \log p = \log C,$$

i.e. $$pv^{\gamma} = C,$$

which is the relation required.

237. Application to implicit functions.

If the relation between x and y be given in the form

$$f(x, y) = \text{constant},$$

then df/dx, the total d. c. with respect to x, $= 0$, since the d. c. of a constant is zero; hence, by Art. 235,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0, \quad \text{and therefore } \frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}.$$

This gives an alternative method to that of Art. 36 of finding the d. c. of y with respect to x, when y is given as an implicit function of x.

E.g. if $x^3 + 3axy + y^3 = a^3$, $\dfrac{\partial f}{\partial x} = 3x^2 + 3ay$, $\dfrac{\partial f}{\partial y} = 3ax + 3y^2$;

$$\frac{dy}{dx} = -\frac{3x^2 + 3ay}{3ax + 3y^2} = -\frac{x^2 + ay}{ax + y^2}, \text{ as in Art. 36, Ex. (ii).}$$

238. Applications to analytical geometry.

(i) *Equation of tangent to a curve.* This result can be used to obtain a convenient form of equation of the tangent to a curve $f(x, y) = 0$ at a given point.

The equation of the tangent at (x, y) was obtained in Art. 46 in the form

$$Y - y = (X - x)\frac{dy}{dx}.$$

Substituting $-\dfrac{\partial f}{\partial x} \Big/ \dfrac{\partial f}{\partial y}$ for $\dfrac{dy}{dx}$ and rearranging, the equation becomes

$$(X - x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} = 0.$$

(ii) *Centre of a curve.* At any point on the curve whose equation is $f(x, y) = 0$, the direction of the tangent is found from the equation

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}.$$

If $f(x, y) = 0$ be of the second degree, the curve will be a conic. $\partial f/\partial x = 0$ will then be an equation of the first degree, and therefore will represent a straight line; moreover, when $\partial f/\partial x = 0$, $dy/dx = 0$, i.e. the tangent to the curve is parallel to the axis of x. Hence $\partial f/\partial x = 0$ is the equation of the straight line joining the points on the curve where the tangent is parallel to the axis of x (Fig. 182). Similarly, $\partial f/\partial y = 0$ is the equation of a straight line, and when $\partial f/\partial y = 0$, dy/dx is infinite, and the tangent is parallel to the axis of y. Hence $\partial f/\partial y = 0$ is the equation of the straight line joining the points on the curve where the tangent is parallel to the axis of y. These two straight lines are diameters of the conic and intersect at

its centre. Hence the coordinates of the centre of the conic whose equation is $f(x, y) = 0$ are obtained by solving the equations

$$\partial f/\partial x = 0, \qquad \partial f/\partial y = 0.$$

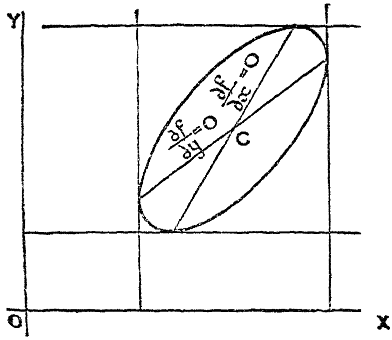


Fig. 182.

Example:

Find the centre of the ellipse $36x^2-24xy+29y^2-168x+106y+21=0$, *and the equation of the tangent to it at the point* $(1, 1)$.

Here $\partial f/\partial x = 72x-24y-168 = 0$; $\partial f/\partial y = -24x+58y+106 = 0$, and we have to solve these equations.

Dividing the first by 3 and adding to the second, we have $50y+50=0$, i.e. $y=-1$, and thence $x=2$. Hence the centre is the point $(2, -1)$.

Also, at the point $(1, 1)$,

$$\partial f/\partial x = -120, \quad \partial f/\partial y = 140.$$

Therefore the equation of the tangent is

$$-120(x-1)+140(y-1)=0,$$

i.e. $$6x-7y+1=0.$$

239. Applications to errors of measurement.

The result of Art. 233 is of importance in that it enables us, when calculating the value of a quantity from the values of several variables upon which it depends, to find the total effect of small errors in the observed values of the several variables. The theorem is equivalent to the statement that, to the first order of small quantities, the total error due to errors in the measurements of several variables is equal to the sum of the errors due to each separately.

Examples:

(i) *The length of the hypotenuse of a right-angled triangle is calculated from the lengths of its sides. If these are measured as* 8·5 *and* 11·5 *feet respectively, with a possible error of* $\frac{1}{5}$ *of an inch in each, find the possible error in the calculated length of the hypotenuse.*

In this case, $c^2 = a^2+b^2$, $\qquad 2c\dfrac{\partial c}{\partial a} = 2a$ and $2c\dfrac{\partial c}{\partial b} = 2b$;

$$\therefore\ \delta c = \frac{\partial c}{\partial a}\cdot\delta a + \frac{\partial c}{\partial b}\cdot\delta b = \frac{a}{c}\cdot\delta a + \frac{b}{c}\cdot\delta b = \frac{a+b}{c}\times\tfrac{1}{5}\,\text{inch} = \frac{20}{\sqrt{(8{\cdot}5^2+11{\cdot}5^2)}}\times\tfrac{1}{5}$$

$$= {\cdot}28 \text{ inch approximately.}$$

(ii) *The height and the radius of the base of a cylinder are at a given instant 10 and 4 inches respectively; if they are increasing at the rate of 2 inches and 1 inch per second respectively, at what rate is the volume of the cylinder increasing at that instant?*

$V=\pi r^2 h$, and by Art. 233, $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\cdot\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\cdot\dfrac{dh}{dt}$

$$= 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$$

$$= 80\pi\times 1 + 16\pi\times 2$$

$$= 112\pi \text{ cubic inches per second.}$$

This is the rate of increase of the volume at the given instant.

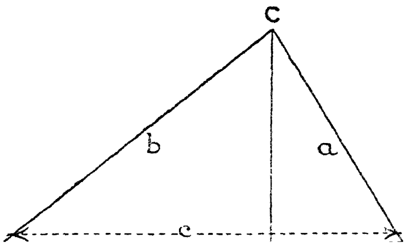


Fig. 183.

(iii) *The area of a triangle is calculated from the length of one of its sides and the magnitudes of the adjacent angles; if the measurements made are* c = 40 *feet,* A = 35°, B = 71°, *find the error in the area due to an error of* $\frac{1}{4}$° *in each angle.*

The area (Fig. 183)

$$S = \tfrac{1}{2}AB\,.\,CD = \tfrac{1}{2}c\,.\,b\sin A = \tfrac{1}{2}c\,.\,c\frac{\sin B}{\sin C}\sin A = \frac{c^2}{2}\,\frac{\sin A\sin B}{\sin(A+B)}.$$

This gives S in terms of the quantities whose measurements are taken c remains constant, therefore $\delta S = \dfrac{\partial S}{\partial A}\cdot\delta A + \dfrac{\partial S}{\partial B}\cdot\delta B.$

$$\frac{\partial S}{\partial A} = \frac{c^2}{2}\sin B\,.\,\frac{\sin(A+B)\cos A - \sin A\cos(A+B)}{\sin^2(A+B)}$$

$$= \frac{c^2}{2}\sin B\,.\,\frac{\sin B}{\sin^2(A+B)} = 800\,\frac{\sin^2 71^\circ}{\sin^2 106^\circ}.$$

Similarly $\dfrac{\partial S}{\partial B} = \dfrac{c^2}{2}\sin A \, . \dfrac{\sin A}{\sin^2(A+B)} = 800\dfrac{\sin^2 35^\circ}{\sin^2 106^\circ}$.

Also $\delta A = \delta B =$ the circular measure of $\frac{1}{4}^\circ = \frac{1}{720}\pi$.

$$\therefore \quad \delta S = \frac{800(\sin^2 71^\circ + \sin^2 35^\circ)}{\sin^2 74^\circ} \times \tfrac{1}{720}\pi = 4{\cdot}623 \text{ square feet.}$$

The *proportional error*, i. e. the ratio of the possible error to the estimated value, and this is what is usually wanted in such cases, is obtained more easily by taking logarithms and differentiating, thus:

$$\log S = 2\log c - \log 2 + \log \sin A + \log \sin B - \log \sin(A+B).$$

Now $$\delta(\log S) = \frac{d(\log S)}{dS} \, . \, \delta S = \frac{1}{S} \, . \, \delta S;$$

$$\delta(\log \sin A) = \frac{\cos A}{\sin A} \, . \, \delta A = \cot A \, . \, \delta A, \quad \&c.$$

Hence, c being constant, we have

$\delta S/S = \cot A \, . \, \delta A + \cot B \, . \, \delta B - \cot(A+B) \, . \, (\delta A + \delta B)$

$= (\cot A + \cot C)\,\delta A + (\cot B + \cot C)\,\delta B$ [since $\cot(A+B) = -\cot C$]

$= \frac{1}{720}\pi(\cot 35^\circ + \cot 71^\circ + 2\cot 74^\circ)$

$= {\cdot}0102.$

Hence the proportional error is about 1 per cent.

(iv) *Given that the volume of a quantity of a gas whose temperature is* 47° C. *and pressure* 15 *lb. weight per square inch is* 6 *cubic feet, find its volume when the pressure is increased to* 15·1 *lb. weight per square inch, and the temperature raised to* 48° C.

In books on Hydrostatics it is proved that, if p be the intensity of pressure of a gas whose volume is v and absolute temperature T, then $pv = kT$, where k is constant.

Regarding T as a function of p and v, we have

$$\delta T = \frac{\partial T}{\partial p} \, . \, \delta p + \frac{\partial T}{\partial v} \, . \, \delta v.$$

Now $$\frac{\partial T}{\partial p} = \frac{v}{k} = \frac{T}{p} = \frac{273+47}{15 \times 144} = \frac{4}{27},$$

and $$\frac{\partial T}{\partial v} = \frac{p}{k} = \frac{T}{v} = \frac{273+47}{6} = \frac{160}{3}.$$

Also $\delta T = 1^\circ$; $\delta p = {\cdot}1 \times 144$. Therefore, substituting in the first equation, $1 = \frac{4}{27} \times {\cdot}1 \times 144 + \frac{160}{3} \times \delta v$, whence $160\,\delta v = -3{\cdot}4$, and $\delta v = -{\cdot}021$.

Hence the volume is diminished by about ·021 cubic foot.

Examples XCV.

1. If (x, y), (r, θ) be rectangular and polar coordinates of a point, find
   (i) the total increment in y, due to small increments δr, $\delta\theta$;
   (ii) the total increment in r, due to small increments δx, δy;
   (iii) the total increment in θ, due to small increments δx, δy.

2. Prove, both geometrically and analytically, that $\partial y/\partial r$ (θ constant) and $\partial r/\partial y$ (x constant) are equal.

Find dz/dt in the following cases:

3. $z = x^n y^n$, where $x = \cos at$, $y = \sin bt$.
4. $z = \log(x^2+y^2)$, where $x = a(1-\cos t)$, $y = a\sin t$.
5. $z = (ax-by)/(cx+dy)$, where $x = e^t \sin t$, $y = e^t \cos t$.

Find du/dt in the following cases:

6. $u = x^2+y^2+z^2$, where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$.
7. $u = \log(x+y+z)$, where $x = \sin^2 t$, $y = \cos^2 t$, $z = \sin 2t$.
8. $u = xyz$, where $x = e^{-t}\cos t$, $y = \sin^2 t$, $z = e^{-t}\sin t$.

Find dz/dx in the following cases:

9. $z = x^2+y^2$, where $y = (1-x)/x$.
10. $z = x^3+y^3+a^3$, where $x^2+y^2 = a^2$.
11. $z = x^2 y^3$, where $x^2-xy+y^2 = a^2$.
12. $z = \sin^{-1}(x/y)$, where $y^2 = a^2+x^2$.
13. $z = \tan^{-1}(y/x)$, where $y = \sin^2 x$.
14. $z = x^3+3axy+y^3$, where $x^2+y^2 = xy$.

Find, by the method of Art. 237, the value of dy/dx in the following cases:

15. $x^3+5x^2y-4xy^2-2y^3 = 0$.
16. $\sin^2 x+\sin^2 y-2\cos x\cos y = 0$.
17. $(x^2+y^2)^2 = a^2(x^2-y^2)$.
18. $x^n y^m + x^m y^n = a^{m+n}$.
19. $\sin(x+y)+\cos(x-y) = 1$.
20. $(bx-ay)^2 = 1+(ax+by)^2$.

Find the relation between the differentials of the variables in the following cases:

21. $V = \frac{1}{3}\pi r^2 h$.
22. $pv = kT$ [k constant].
23. $xyz = a^3$ [a constant].
24. $x^2+y^2 = z^2$.
25. $f = mv^2/r$ [m constant].
26. $Fs = \frac{1}{2}mv^2$ [m constant].
27. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial u}{\partial r}$ and $-\frac{\partial u}{\partial \theta}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, u being a function of x and y.
28. Find the equation of the normal to the curve $f(x, y) = 0$ at any point on the curve.
29. Find the coordinates of the centre of the conic
$$y^2-5xy+6x^2-14x+5y = 0,$$
and the equations of the tangent and normal at the origin.
30. Find the centre of the conic $3x^2+2xy+3y^2 = 4ax+4ay$, and the equations of the tangents at the points where it meets the axes.
31. If $K/K_0 = (T/T_0)^n \times \frac{1}{760}p$, find the change in K due to small variations δp and δT in p and T.
32. If $\mu-1 = \frac{\mu_0-1}{1+\alpha\theta}\cdot\frac{p}{760}$, find the change in μ due to small variations δp, $\delta\theta$ in p and θ.
33. The hypotenuse and one side of a right-angled triangle are measured as 143 and 93 feet; find the error in the third side due to an error of 1 inch in each measurement.

34. The length of a side of a right-angled triangle is calculated from the length of the hypotenuse and the angle between them, which are found to be 140 inches and 43°: find the error in the length of the side due to the measurement of the hypotenuse being $\frac{1}{2}$ inch too small and the size of the angle (i) a quarter of a degree too small, (ii) a quarter of a degree too large.

35. The area of a triangle is calculated from the formula $S = \frac{1}{2}bc \sin A$, and the measurements taken are $b = 72$ feet, $c = 55$ feet, $A = 56°$. Find the possible error in the area due to (i) errors of 2 inches in each side; (ii) errors of 2 inches in each side and half a degree in the angle. Find the proportional error in each case.

36. If p be the intensity of pressure of a gas of volume v and absolute temperature T, $pv = kT$ where k is constant. Given that p is 20 lb. weight per square inch when $v = 10$ cubic feet, and the temperature 40° C., find approximately

(i) the change in the pressure when v is increased to 10·2 cubic feet and the temperature to 40·5° C.;

(ii) the change of volume when p is increased to 20·1 lb. weight per sq. inch and the temperature reduced to 39·7° C.;

(iii) the change of volume when p is reduced to 19·7 lb. weight per sq. inch and the temperature raised to 40·3° C.;

(iv) the change of temperature required to raise p to 20·2 lb. weight per sq. inch when the volume is increased to 10·2 cubic feet;

(v) the change of temperature required to lower the pressure to 19·6 lb. weight per sq. inch when the volume is increased to 10·1 cubic feet.

37. The side b of a triangle is calculated from the formula $b = a \sin B/\sin A$, and the observed values are $a = 125$, $B = 73°$, $A = 42°$. Find the error in the calculated value if the true values of A and B are 41·8° and 72·7°.

38. The side c of a triangle is calculated from the following observations: $a = 175$ feet, $A = 60°$, $C = 38·5°$. Find the error in the calculated value of c (i) if the true values are 175·5, 60°, and 38·8°; (ii) if the true values are 175·5, 59·6°, and 38·8° respectively.

39. The side c of a triangle is calculated from the formula

$$c^2 = a^2 + b^2 - 2ab \cos C;$$

find the relation between the differentials δc, δa, δb, δC.
Find the error in c if the observed values of a, b, C are 120, 180, and 32°, and the real values 121, 179, and $32\frac{1}{4}°$.
Find also the proportional error in this case.

40. The area of an ellipse whose semi-axes are a and b is πab; find the possible error in the area due to possible errors of $\frac{1}{2}$ inch in each measurement, the observed values being 3 feet and 2 feet.

41. Find the proportional error in the area of an ellipse due to small errors δa, δb in the lengths of the semi-axes.

42. Find the proportional error in the area of a triangle calculated from the lengths of its sides, due to small errors δa, δb, δc in the measurements of the lengths of the sides.

43. The angle A of a triangle is calculated from the formula

$$\cos A = (b^2 + c^2 - a^2)/2bc;$$

find the error in the angle due to small errors δa, δb, δc in the sides.

44. The height of a building is calculated from the observed elevation (α) at a measured distance (a) from its base; find the error in the height due to small errors $\delta\alpha$, δa in the observations.

If the observed values of α and a be 24° and 120 feet, and the true values 24·2° and 119·8 feet, find the error in the height.

45. Find the proportional error in the volume of a cone due to small errors δh, δr in the height and radius of the base.

46. The time of oscillation T of a simple pendulum is $2\pi\sqrt{(l/g)}$; find the error in the time due to small errors in measuring l and g.

Find also the proportional error.

47. If the value of g be calculated from the preceding formula, find the percentage error in the value of g due to positive errors of ·5 per cent. in the measurement of both l and T.

48. If (x, y) and (r, θ) be the rectangular and polar coordinates of a point in a plane, prove that the differentials of x, y, θ are connected by the relation $x \,.\, \delta y - y \,.\, \delta x = r^2 \delta\theta$.

Prove also that $(\delta x)^2 + (\delta y)^2 = (\delta r)^2 + r^2 (\delta\theta)^2$.

49. Supposing x, y, r, θ functions of the time t, deduce from the last relation that $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$. What is the significance of this result in Mechanics?

50. The rectangular coordinates of a moving point in a plane are at a given instant (10, 6), and the velocities of the point at that instant, parallel to the axes of x and y, are respectively 3 and 2 foot-seconds respectively; find the angular velocity of the point about the origin at that instant.

51. The specific gravity of a solid heavier than water is $W/(W - W')$, where W and W' are its weights in air and water respectively; if W and W' are observed to be 20·7 and 11·2, find the maximum error in the calculated value of the specific gravity due to errors of ·05 in each observation.

Find also the percentage error.

52. If the H. P. required to propel a steamer vary as the cube of the velocity and the square of the length, prove that a 2 per cent. increase in velocity and a 3 per cent. increase in length will require approximately a 12 per cent. increase in H. P.

53. The specific gravity of a liquid is $(W - W_2)/(W - W_1)$, where W, W_1, W_2 are the weights of a solid in air, water, and the liquid respectively; find the proportional error due to small errors δW, δW_1, δW_2 in the weighings.

54. Find the rate of increase of (i) the volume, (ii) the area of the curved surface of a right circular cone, at the instant when the height and the radius of the base are 12 inches and 4 inches respectively, and each is increasing at the rate of $\frac{1}{2}$ inch per second.

55. Find the rate of increase of (i) the volume, (ii) the superficial area of a rectangular parallelepiped, at the instant when its sides are 20, 15, 10 inches, and are increasing at the rate of ·8, ·6, ·4 inch per second respectively.

240. Partial derivatives of higher orders.

If z is a function of two variables x and y, denoted by $f(x, y)$, then $\partial z/\partial x$ and $\partial z/\partial y$ will generally be functions of x and y, and therefore they can be differentiated again partially with respect to x and y.

The partial d. c. of $\dfrac{\partial z}{\partial x}$ or f_x with respect to x, i. e. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$, is denoted by $\dfrac{\partial^2 z}{\partial x^2}$ or f_{xx}.

The partial d. c. of $\dfrac{\partial z}{\partial x}$ or f_x with respect to y, i. e. $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$, is denoted by $\dfrac{\partial^2 z}{\partial y\,\partial x}$ or f_{yx}.

The partial d. c. of $\dfrac{\partial z}{\partial y}$ or f_y with respect to x, i. e. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$, is denoted by $\dfrac{\partial^2 z}{\partial x\,\partial y}$ or f_{xy}.

The partial d. c. of $\dfrac{\partial z}{\partial y}$ or f_y with respect to y, i. e. $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)$, is denoted by $\dfrac{\partial^2 z}{\partial y^2}$ or f_{yy};

and so on for derivatives of higher order.

Similarly for functions of more than two variables.

Examples:

(i) If $z \equiv f(x, y) = x \sin y + y \sin x$,

$$\frac{\partial z}{\partial x} \text{ or } f_x = \sin y + y \cos x; \qquad \frac{\partial z}{\partial y} \text{ or } f_y = x \cos y + \sin x;$$

$$\frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx} = -y \sin x; \qquad \frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy} = -x \sin y;$$

$$\frac{\partial^2 z}{\partial y\,\partial x} \text{ or } f_{yx} = \cos y + \cos x; \qquad \frac{\partial^2 z}{\partial x\,\partial y} \text{ or } f_{xy} = \cos y + \cos x.$$

(ii) As an example of a function of three variables x, y, z, if

$$V = \frac{1}{\sqrt{\{(x-a)^2 + (y-b)^2 + (z-c)^2\}}}$$

(where $x-a$, $y-b$, $z-c$ are supposed not to be simultaneously zero),

prove that
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Denoting the expression under the radical sign by u, for convenience, we have
$$V = u^{-\frac{1}{2}}.$$

$$\therefore \quad \frac{\partial V}{\partial x} = -\tfrac{1}{2} u^{-3/2} \cdot \frac{\partial u}{\partial x} = -\tfrac{1}{2} u^{-3/2} \times 2(x-a) = -\frac{x-a}{u^{3/2}};$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{u^{3/2} \times 1 - (x-a)\,\frac{3}{2} u^{1/2} \cdot 2(x-a)}{u^3} = -\frac{u - 3(x-a)^2}{u^{5/2}} = \frac{3(x-a)^2 - u}{u^{5/2}}.$$

Similarly $\quad \dfrac{\partial^2 V}{\partial y^2} = \dfrac{3(y-b)^2 - u}{u^{5/2}}$ and $\dfrac{\partial^2 V}{\partial z^2} = \dfrac{3(z-c)^2 - u}{u^{5/2}}.$

Adding, $\quad \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = \dfrac{3(x-a)^2 + 3(y-b)^2 + 3(z-c)^2 - 3u}{u^{5/2}} = 0,$

since $\quad u = (x-a)^2 + (y-b)^2 + (z-c)^2.$

This equation is a very important one in the theory of electricity and attractions, and is known as *Laplace's Equation.*

If r be the distance between two points whose coordinates in space are (a, b, c) and (x, y, z), then $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, and the potential at $P(x, y, z)$ of mass or charge m at (a, b, c)

$$= m/r \text{ (Art. 178)} = m/\sqrt{\{(x-a)^2 + (y-b)^2 + (z-c)^2\}}$$

$$= m \times \text{the function } V \text{ of the preceding example.}$$

Hence the potential satisfies Laplace's Equation, and this result will be true for any number of masses or charges at points not coincident with P.

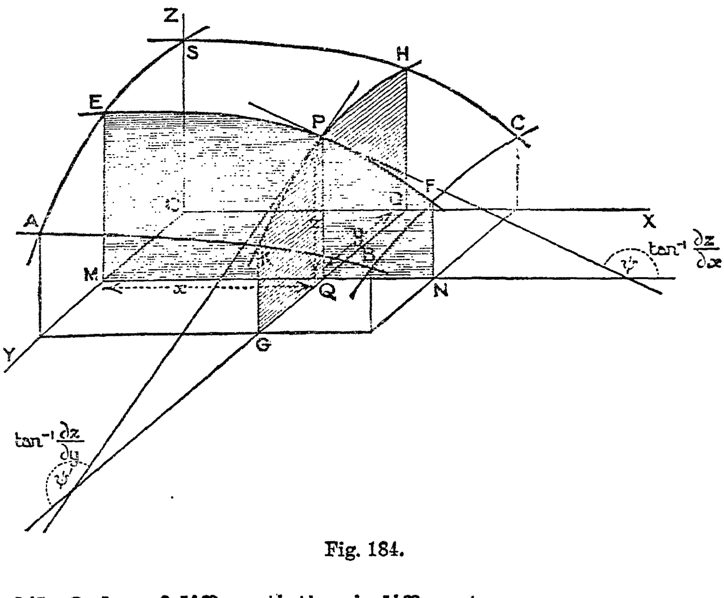


Fig. 184.

241. Order of differentiation indifferent.

In the example at the beginning of the previous article, it will be noticed that $\dfrac{\partial^2 z}{\partial y\, \partial x}$ and $\dfrac{\partial^2 z}{\partial x\, \partial y}$ are equal, i.e. if the function be differentiated partially with respect to the variables x and y in succession, the order of differentiation is immaterial. This is always the case if the function and the differential coefficients involved are continuous. A geometrical proof of this property readily follows from Fig. 184.

Let V be the volume enclosed by the surface $ESHP$, the coordinate planes, and the planes PM, PD.

Then, as in Arts. 81, 159,

$$\partial V/\partial x = \text{the area } HDQP,$$

$$\text{and } \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \text{the d. c. of the area } HDQP \text{ with respect to } y$$
$$= \text{the ordinate } QP \text{ (Art. 79).}$$

Similarly $\partial V/\partial y =$ the area $EMQP$,

$$\text{and } \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = \text{the d. c. of the area } EMQP \text{ with respect to } x$$
$$= \text{the ordinate } QP \text{ (Art. 79).}$$

Hence $$\frac{\partial^2 V}{\partial y\, \partial x} = \frac{\partial^2 V}{\partial x\, \partial y}.$$

Analytical proof.

An analytical proof of the foregoing important theorem can be obtained by the use of the Mean-Value Theorem (Art. 116).

We have

$$\frac{\partial}{\partial y} f_x = \operatorname*{Lt}_{k \to 0} \frac{1}{k} \{f_x(x, y+k) - f_x(x, y)\}$$
$$= \operatorname*{Lt}_{k \to 0} \frac{1}{k} \left\{ \operatorname*{Lt}_{h \to 0} \frac{1}{h} [\{f(x+h, y+k) - f(x, y+k)\} - \{f(x+h, y) - f(x,y)\}] \right\}.$$

Similarly,

$$\frac{\partial}{\partial x} f_y = \operatorname*{Lt}_{h \to 0} \frac{1}{h} \left\{ \operatorname*{Lt}_{k \to 0} \frac{1}{k} [\{f(x+h,y+k) - f(x+h,y)\} - \{f(x,y+k) - f(x,y)\}] \right\}.$$

It must not be assumed that these two expressions are identical although they consist of the same terms, for the assumption that the limits are the same, whether $h \to 0$ before k or whether $k \to 0$ before h, is equivalent to assuming the theorem which is being proved.*

By the Mean-Value Theorem,

$$F(x+h) - F(x) = hF_x(x+\theta h), \quad \text{where} \quad 0 < \theta < 1.$$

In this equation, take $F(x)$ to be $f(x, y+k) - f(x, y)$.

* That this assumption is unjustifiable is easily seen from the following example: Consider $\dfrac{a \sin x + b \sin y}{cx + dy}$, and find its limit when first x, and afterwards y, $\to 0$. The limit when $x \to 0$ is $\dfrac{b \sin y}{dy}$, i.e. $\dfrac{b}{d} \times \dfrac{\sin y}{y}$, and the limit of this as $y \to 0$ is b/d. But, if $y \to 0$ first, the limit is $\dfrac{a \sin x}{cx}$, i. e. $\dfrac{a}{c} \times \dfrac{\sin x}{x}$, and the limit of this as $x \to 0$ is a/c.

Then $[f(x+h,\ y+k)-f(x+h,\ y)]-[f(x,\ y+k)-f(x,\ y)]$

$$= h\,[f_x(x+\theta_1 h,\ y+k)-f_x(x+\theta_1 h,\ y)]$$

$$= {}^*h\,[k\frac{\partial}{\partial y}f_x(x+\theta_1 h,\ y+\theta_2 k)]$$

$$= hkf_{yx}(x+\theta_1 h,\ y+\theta_2 k).$$

Similarly $\quad F(y+k)-F(y)=kF_y(y+\theta' k)$, where $0<\theta'<1$.

In this equation, take $F(y)$ to be $f(x+h,\ y)-f(x,\ y)$.

Then $\quad [f(x+h,\ y+k)-f(x,\ y+k)]-[f(x+h,\ y)-f(x,\ y)]$

$$= k\,[f_y(x+h,\ y+\theta_3 k)-f(x,\ y+\theta_3 k)]$$

$$= {}^\dagger k\,[h\frac{\partial}{\partial x}f_y(x+\theta_4 h,\ y+\theta_3 k)]$$

$$= khf_{xy}(x+\theta_4 h,\ y+\theta_3 k).$$

Hence, since the expressions on the left-hand sides in these two equations are identical, we have, after dividing out the factor hk (which is not zero),

$$f_{yx}(x+\theta_1 h,\ y+\theta_2 k)=f_{xy}(x+\theta_4 h,\ y+\theta_3 k),$$

where all the θ's are between 0 and 1.

Hence, in the limit when h and k both $\to 0$, since the functions are continuous,

$$f_{yx}(x,\ y)=f_{xy}(x,\ y).$$

242. Exact differential equations.

To find the condition that $P\delta x+Q\delta y$, where P and Q are functions of x and y, may be a perfect differential.

If the given expression is the total differential of a function u of x and y,

$$P\delta x+Q\delta y=\delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y\,;$$

hence $\quad P=\partial u/\partial x$ and $Q=\partial u/\partial y$;

$$\therefore\quad \frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial Q}{\partial x}$$

This is a necessary condition. Conversely, if this condition is satisfied, it follows that P and Q are partial differential coefficients with respect to x and y respectively of some function u of x and y. For let $P=\partial z/\partial x$;

then $$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right).$$

Integrating with respect to x,

$$Q=\frac{\partial z}{\partial y}+\text{ a function of } y=\frac{\partial}{\partial y}(z+\text{ a function of } y).$$

* Using the Mean-Value Theorem for the expression in the brackets regarded as a function of y.

† Using the Mean-Value Theorem for the expression in the brackets regarded as a function of x.

If the expression in the brackets be denoted by u,

$$Q = \partial u/\partial y, \text{ and } P = \partial z/\partial x = \partial u/\partial x,$$

since the partial d. c. of the function of y with respect to x is zero.

Therefore P and Q are equal to $\partial u/\partial x$ and $\partial u/\partial y$ respectively,

and $P\delta x + Q\delta y = \dfrac{\partial u}{\partial x}\delta x + \dfrac{\partial u}{\partial y}\delta y = \delta u$, a perfect differential.

From this result it follows that, if the condition $\partial P/\partial y = \partial Q/\partial x$ be satisfied, the differential equation $P + Q\dfrac{dy}{dx} = 0$ may be put into the form

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0, \quad \text{i.e. } \frac{du}{dx} = 0 \text{ [Art. 235]},$$

the integral of which is $u = C$.

The differential equation is then said to be exact [Art. 216].

Example. $$x^2 + ay + (y^2 + ax)\frac{dy}{dx} = 0.$$

The condition is satisfied, and the equation is exact, since

$$\partial P/\partial y = \partial Q/\partial x = a.$$

In this case the integral can be written down at once, since the equation may be put in the form

$$x^2 + a\left(y + x\frac{dy}{dx}\right) + y^2\frac{dy}{dx} = 0,$$

which gives on integration $\qquad \frac{1}{3}x^3 + axy + \frac{1}{3}y^3 = C.$

In the general case, since, in finding $\partial u/\partial x$, y is regarded as constant, and in finding $\partial u/\partial y$, x is regarded as constant, it follows that the terms of u which contain x only are represented only in $\partial u/\partial x$, i.e. P, and those which contain y only are represented only in $\partial u/\partial y$, i.e. Q, whereas those which contain both x and y are represented in both P and Q. Hence we have the following working rule for integrating an exact equation: Integrate P with respect to x and Q with respect to y; add the integrals together, but only insert once the terms common to both the integrals, and equate the sum to a constant.

Example: $x^2 + 2ay + y^2 - 2xy + (2ax - x^2 + 2xy - y^2)\dfrac{dy}{dx} = 0.$

$$\partial P/\partial y = 2a + 2y - 2x; \; \partial Q/\partial x = 2a - 2x + 2y = \partial P/\partial y;$$

hence the equation is exact.

$$\int P\,dx = \int(x^2 + 2ay + y^2 - 2xy)\,dx = \tfrac{1}{3}x^3 + 2axy + xy^2 - x^2y,$$

$$\int Q\,dy = \int(2ax - x^2 + 2xy - y^2)\,dy = 2axy - x^2y + xy^2 - \tfrac{1}{3}y^3.$$

The terms $2axy+xy^2-x^2y$ are common to both, hence the integral is

$$\tfrac{1}{3}x^3+2axy+xy^2-x^2y-\tfrac{1}{3}y^3 = C.$$

Or we may proceed as follows:

$$\partial u/\partial x = P = x^2+2ay+y^2-2xy.$$

Therefore, integrating, and remembering that the 'constant of integration' will involve y,

$$u = \tfrac{1}{3}x^3+2axy+xy^2-x^2y+f(y);$$

$$\therefore \quad \partial u/\partial y = 2ax+2xy-x^2+f'(y).$$

But $$\partial u/\partial y = Q = 2ax-x^2+2xy-y^2;$$

$$\therefore \quad f'(y) = -y^2, \text{ and } f(y) = -\tfrac{1}{3}y^3.$$

Hence $$u = \tfrac{1}{3}x^3+2axy+xy^2-x^2y-\tfrac{1}{3}y^3,$$

and the integral is $$\tfrac{1}{3}x^3+2axy+xy^2-x^2y-\tfrac{1}{3}y^3 = C, \text{ as before.}$$

Examples XCVI.

1. If $z = ax^3+3bx^2y+3cxy^2+dy^3$, find $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}, \dfrac{\partial^2 z}{\partial x\,\partial y}, \dfrac{\partial^2 z}{\partial y\,\partial x}$.

2. If $z = x^2\sin y+y^2\sin x$, find the values of the same functions.

3. If $z = x^m/y^n$, find the values of the same functions; find also
$$\frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial y^3}, \frac{\partial^3 z}{\partial y\,\partial x^2}, \frac{\partial^3 z}{\partial x\,\partial y^2}, \frac{\partial^3 z}{\partial x\,\partial y\,\partial x}, \frac{\partial^3 z}{\partial y\,\partial x\,\partial y}.$$

4. If $z = \log r$, where $r^2 = (x-a)^2+(y-b)^2$, prove that $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2} = 0$, provided $x-a$ and $y-b$ are not simultaneously zero.

5. Prove that the equation $\partial^2 z/\partial t^2 = a^2\,\partial^2 z/\partial x^2$ is satisfied by each of the functions $z = A\sin(x+at)$, $z = A\sin(x+at)+B\cos(x-at)$.

6. If $z = f(x+ay)$, prove that $\partial^2 z/\partial y^2 = a^2\,\partial^2 z/\partial x^2$.

7. Prove that the same differential equation is satisfied by
$$z = f(x+ay)+F(x-ay).$$

8. If $u = e^{xyz}$, find the value of $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$.

9. If $u = \tan^{-1}(y/x)$, verify that $\dfrac{\partial^3 u}{\partial y^2\,\partial x} = \dfrac{\partial^3 u}{\partial x\,\partial y^2} = \dfrac{\partial^3 u}{\partial y\,\partial x\,\partial y}$.

10. If $u = \dfrac{x^3-y^3}{x^3+y^3}$, verify that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$.

11. If $u = x^n f(y/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$.

12. If $u = \dfrac{x^2+y^2}{x+y}$, verify that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

13. If $\dfrac{\partial^2 z}{\partial x\,\partial y} = 0$, prove that $z = f(x)+F(y)$.

14. If $r^2 = (x-a)^2+(y-b)^2+(z-c)^2$, prove that $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}+\dfrac{\partial^2 r}{\partial z^2} = \dfrac{2}{r}$.

15. Find d^2z/dt^2 in terms of partial d. c.'s, when $z = f(x, y)$, where x and y are both functions of t.

16. Find d^2y/dx^2 in terms of partial d. c.'s, when y is an implicit function of x given by the equation $f(x, y) = 0$.

17. If $u = f(x, y)$, and $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$ in terms of partial d. c.'s of u with respect to r and θ.

Prove that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

18. If x, y, r, θ are functions of t, prove that
$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r} - r\dot{\theta}^2; \quad \ddot{y}\cos\theta - \ddot{x}\sin\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$
What is the meaning of these equations in Mechanics?

19. If u is a function of x and y, and if
$$x = X\cos\alpha - Y\sin\alpha, \quad y = X\sin\alpha + Y\cos\alpha,$$
show that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 y}{\partial Y^2}.$$

20. By expanding $f(x+h, y+k)$ in powers of h, then expanding each of the resulting terms in powers of k (by Taylor's Theorem), and neglecting small quantities of the third order, h and k being taken as of the first order, obtain the value of $f(x+h, y+k) - f(x, y)$.

21. Prove that the radius of curvature at any point of the curve $f(x, y) = 0$ is
$$\frac{(f_x^2 + f_y^2)^{3/2}}{f_x^2 f_{yy} - 2f_x f_y f_{xy} + f_y^2 f_{xx}}.$$

22. Prove that the equation $(ax+by)\dfrac{dy}{dx} + ay + cx = 0$ is exact, and solve it.

23. Prove that the equation $2xy - y^2 + ay + (x^2 - 2xy + ax)\dfrac{dy}{dx} = 0$ is exact, and solve it.

24. Show that the equation $x + ky + (y - kx)\dfrac{dy}{dx}$ is not exact, but that it is made exact by dividing both sides by $x^2 + y^2$. Hence integrate it.

25. Prove the same fact in the case of the equation
$$(x^2+y^2)\left(x + y\frac{dy}{dx}\right) + \frac{y}{x}\left(x\frac{dy}{dx} - y\right) = 0,$$
and integrate it.

26. If $y = a\sin(px/b)\sin(pt+\epsilon)$, where a, p, b, ϵ are constants, prove that
$$\partial^2 y/\partial t^2 = b^2\partial^2 y/\partial x^2.$$

27. If $u = f(x + \alpha t, y + \beta t)$, where x and y are independent of t, prove that
$$\frac{du}{dt} = \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y};$$
$$\frac{d^2 u}{dt^2} = \alpha^2\frac{\partial^2 u}{\partial x^2} + 2\alpha\beta\frac{\partial^2 u}{\partial x\,\partial y} + \beta^2\frac{\partial^2 u}{\partial y^2}.$$
Find d^3u/dt^3, and state a general rule for finding $d^n u/dt^n$.
[Put $x + \alpha t = X$, $y + \beta t = Y$.]

NUMERICAL TABLES

NUMERICAL TABLES

TABLE I

MISCELLANEOUS FORMULAE, EQUIVALENTS, ETC.

$\pi = 3{\cdot}1416$, $\log\pi = {\cdot}4971$. [See Table II for powers of π.]

$e = 2{\cdot}7183$, $\log_{10} e = {\cdot}4343$, $\log_e 10 = 2{\cdot}3026$. [See Tables II and X for powers of e.]

$\log_{10} x = \log_e x \times {\cdot}4343$; $\log_e x = \log_{10} x \times 2{\cdot}3026$.

1 radian = 57·30 degrees; 1 minute = ·0002909 radian.

1 metre = 39·37 inches = 1·094 yards = ·0006214 miles.

1 inch = 2·540 cm.

1 gallon = ·1604 c. ft. = volume of 10 lb. of water = ·4545 litre.

1 c. ft. of water contains 62·28 lb.

1 lb. wt. = g (= 32·18) poundals = 453·6 gm. wt. = 445,300 dynes.

1 kilogram = 2·2046 lb.

Value of g (in London) = 32·18 ft. secs. per sec. = 980·8 cm. secs. per sec.

60 miles per hour = 88 ft. secs.

Circle of radius r. Equation referred to centre, $x^2 + y^2 = r^2$. Area $= \pi r^2$; circumference $= 2\pi r$. Length of arc which subtends θ radians at the centre $= r\theta$; distance of C. G. of arc from centre $= (r \sin \frac{1}{2}\theta)/\frac{1}{2}\theta$; area of sector on this arc $= \frac{1}{2}r^2\theta$; distance of C. G. of sector from centre $= \frac{2}{3}(r \sin \frac{1}{2}\theta)/\frac{1}{2}\theta$.

Parabola. $y^2 = 4ax$. Latus rectum $= 4a$; focus $(a, 0)$; equation of directrix, $x + a = 0$.

Ellipse, semi-axes a and b. Equation referred to principal axes, $x^2/a^2 + y^2/b^2 = 1$. Eccentricity $e = \sqrt{(1 - b^2/a^2)}$; semi-latus rectum $= b^2/a$; foci $(\pm ae, 0)$; equations of directrices, $x = \pm a/e$; area $= \pi ab$.

Hyperbola, semi-axes a and b. Equation referred to principal axes,

$$x^2/a^2 - y^2/b^2 = 1.$$

Eccentricity $e = \sqrt{(1 + b^2/a^2)}$; semi-latus rectum $= b^2/a$; foci $(\pm ae, 0)$; equations of directrices, $x = \pm a/e$; equations of asymptotes, $x/a = \pm y/b$.

Rectangular Hyperbola, eccentricity $= \sqrt{2}$; equation referred to principal axes, $x^2 - y^2 = a^2$; equation referred to asymptotes, $xy = \frac{1}{2}a^2$.

Sphere of radius r. Volume $= \frac{4}{3}\pi r^3$; surface $= 4\pi r^2$.

Distance of C. G. of hemisphere from centre $= \frac{3}{8}r$, if solid; $\frac{1}{2}r$, if a thin shell.

Cylinder of height h and radius r. Volume $= \pi r^2 h$; curved surface $= 2\pi rh$.

Cone of height h and radius of base r. Volume $= \frac{1}{3}\pi r^2 h$; curved surface $= \pi r\sqrt{(r^2 + h^2)}$. Height of C. G. above base $= \frac{1}{4}h$, if solid; $\frac{1}{3}h$, if a thin shell (open at base).

Moment of Inertia of rod or rectangle, length $2l$, about perpendicular axis through centre $= \frac{1}{3}Ml^2$; of circular disc about a diameter, $\frac{1}{4}Mr^2$; of circular disc about a perpendicular to its plane through the centre, $\frac{1}{2}Mr^2$; of sphere about a diameter, $\frac{2}{5}Mr^2$.

TABLE II

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS OF INTEGERS FROM 1 TO 100, AND OF e AND π

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1	1	1	51	2601	132651	7·141	3·708	·01961
2	4	8	1·414	1·260	·50000	52	2704	140608	7·211	3·733	·01923
3	9	27	1·732	1·442	·33333	53	2809	148877	7·280	3·756	·01887
4	16	64	2·000	1·587	·25000	54	2916	157464	7·348	3·780	·01852
5	25	125	2·236	1·710	·20000	55	3025	166375	7·416	3·803	·01818
6	36	216	2·449	1·817	·16667	56	3136	175616	7·483	3·826	·01786
7	49	343	2·646	1·913	·14286	57	3249	185193	7·550	3·849	·01754
8	64	512	2·828	2·000	·12500	58	3364	195112	7·616	3·871	·01724
9	81	729	3·000	2·080	·11111	59	3481	205379	7·681	3·893	·01695
10	100	1000	3·162	2·154	·10000	60	3600	216000	7·746	3·915	·01667
11	121	1331	3·317	2·224	·09091	61	3721	226981	7·810	3·936	·01639
12	144	1728	3·464	2·289	·08333	62	3844	238328	7·874	3·958	·01613
13	169	2197	3·606	2·351	·07692	63	3969	250047	7·937	3·979	·01587
14	196	2744	3·742	2·410	·07143	64	4096	262144	8·000	4·000	·01563
15	225	3375	3·873	2·466	·06667	65	4225	274625	8·062	4·021	·01538
16	256	4096	4·000	2·520	·06250	66	4356	287496	8·124	4·041	·01515
17	289	4913	4·123	2·571	·05882	67	4489	300763	8·185	4·062	·01493
18	324	5832	4·243	2·621	·05556	68	4624	314432	8·246	4·082	·01471
19	361	6859	4·359	2·668	·05263	69	4761	328509	8·307	4·102	·01449
20	400	8000	4·472	2·714	·05000	70	4900	343000	8·367	4·121	·01429
21	441	9261	4·583	2·759	·04762	71	5041	357911	8·426	4·141	·01408
22	484	10648	4·690	2·802	·04545	72	5184	373248	8·485	4·160	·01389
23	529	12167	4·796	2·844	·04348	73	5329	389017	8·544	4·179	·01370
24	576	13824	4·899	2·884	·04167	74	5476	405224	8·602	4·198	·01351
25	625	15625	5·000	2·924	·04000	75	5625	421875	8·660	4·217	·01333
26	676	17576	5·099	2·962	·03846	76	5776	438976	8·718	4·236	·01316
27	729	19683	5·196	3·000	·03704	77	5929	456533	8·775	4·254	·01299
28	784	21952	5·291	3·037	·03571	78	6084	474552	8·832	4·273	·01282
29	841	24389	5·385	3·072	·03448	79	6241	493039	8·888	4·291	·01266
30	900	27000	5·477	3·107	·03333	80	6400	512000	8·944	4·309	·01250
31	961	29791	5·568	3·141	·03226	81	6561	531441	9·000	4·327	·01235
32	1024	32768	5·657	3·175	·03125	82	6724	551368	9·055	4·344	·01220
33	1089	35937	5·745	3·208	·03030	83	6889	571787	9·110	4·362	·01205
34	1156	39304	5·831	3·240	·02941	84	7056	592704	9·165	4·380	·01191
35	1225	42875	5·916	3·271	·02857	85	7225	614125	9·220	4·397	·01177
36	1296	46656	6·000	3·302	·02778	86	7396	636056	9·274	4·414	·01163
37	1369	50653	6·083	3·332	·02703	87	7569	658503	9·327	4·431	·01149
38	1444	54872	6·164	3·362	·02632	88	7744	681472	9·381	4·448	·01136
39	1521	59319	6·245	3·391	·02564	89	7921	704969	9·434	4·465	·01124
40	1600	64000	6·325	3·420	·02500	90	8100	729000	9·487	4·481	·01111
41	1681	68921	6·403	3·448	·02439	91	8281	753571	9·539	4·498	·01099
42	1764	74088	6·481	3·476	·02381	92	8464	778688	9·592	4·514	·01087
43	1849	79507	6·557	3·503	·02326	93	8649	804357	9·644	4·531	·01075
44	1936	85184	6·633	3·530	·02273	94	8836	830584	9·695	4·547	·01064
45	2025	91125	6·708	3·557	·02222	95	9025	857375	9·747	4·563	·01053
46	2116	97336	6·782	3·583	·02174	96	9216	884736	9·798	4·579	·01042
47	2209	103823	6·856	3·609	·02128	97	9409	912673	9·849	4·595	·01031
48	2304	110592	6·928	3·634	·02083	98	9604	941192	9·899	4·610	·01020
49	2401	117649	7·000	3·659	·02041	99	9801	970299	9·950	4·626	·01010
50	2500	125000	7·071	3·684	·02000	100	10000	1000000	10·000	4·642	·01000
e	7·389	20·086	1·649	1·396	·36788	π	9·8696	31·006	1·7725	1·465	·31831

The squares, cubes, and reciprocals of numbers from 0 to 10 at intervals of ·1 may be written down at once from the above table by inserting the decimal point in its proper position, e.g. $3{\cdot}7^2 = 13{\cdot}69$, $3{\cdot}7^3 = 50{\cdot}653$, $\frac{1}{3{\cdot}7} = {\cdot}2703$.

TABLE III

SQUARE ROOTS AND CUBE ROOTS OF NUMBERS FROM 1 TO 10 AT INTERVALS OF ·1

n	$\sqrt{n}$	$\sqrt[3]{n}$	n	$\sqrt{n}$	$\sqrt[3]{n}$	n	$\sqrt{n}$	$\sqrt[3]{n}$	n	$\sqrt{n}$	$\sqrt[3]{n}$	n	$\sqrt{n}$	$\sqrt[3]{n}$
·1	·316	·464	2·1	1·449	1·281	4·1	2·025	1·601	6·1	2·470	1·827	8·1	2·846	2·00[illegible]
·2	·447	·585	2·2	1·483	1·301	4·2	2·049	1·613	6·2	2·490	1·837	8·2	2·864	2·017
·3	·548	·669	2·3	1·517	1·320	4·3	2·074	1·626	6·3	2·510	1·847	8·3	2·881	2·025
·4	·632	·737	2·4	1·549	1·339	4·4	2·098	1·639	6·4	2·530	1·857	8·4	2·898	2·03[illegible]
·5	·707	·794	2·5	1·581	1·357	4·5	2·121	1·651	6·5	2·550	1·866	8·5	2·915	2·04[illegible]
·6	·775	·843	2·6	1·612	1·375	4·6	2·145	1·663	6·6	2·569	1·876	8·6	2·933	2·049
·7	·837	·888	2·7	1·643	1·392	4·7	2·168	1·675	6·7	2[illegible]	1[illegible]	8·7	2·950	2·057
·8	·894	·928	2·8	1·673	1·409	4·8	2·191	1·687	6·8	2[illegible]	1·895	8·8	2·966	2·06[illegible]
·9	·949	·965	2·9	1·703	1·426	4·9	2·214	1·698	6·9	2·627	1·904	8·9	2·983	2·072
1·0	1·000	1·000	3·0	1·732	1·442	5·0	2·236	1·710	7·0	2·646	1·913	9·0	3·000	2·08[illegible]
1·1	1·049	1·032	3·1	1·761	1·458	5·1	2·258	1·721	7·1	2·665	1·922	9·1	3·017	2·08[illegible]
1·2	1·095	1·063	3·2	1·789	1·474	5·2	2·280	1·732	7·2	2·683	1·931	9·2	3·033	2·09[illegible]
1·3	1·140	1·091	3·3	1·817	1·489	5·3	2·302	1·744	7·3	2·702	1·940	9·3	3·050	2·10[illegible]
1·4	1·183	1·119	3·4	1·844	1·504	5·4	2·324	1·754	7·4	2·720	1·94[illegible]	9·4	3·066	2·11[illegible]
1·5	1·225	1·145	3·5	1·871	1·518	5·5	2·345	1·765	7·5	2·739	1·95[illegible]	9·5	3·082	2·11[illegible]
1·6	1·265	1·170	3·6	1·897	1·533	5·6	2·366	1·776	7·6	2·757	1·96[illegible]	9·6	3·098	2·12[illegible]
1·7	1·304	1·193	3·7	1·924	1·547	5·7	2·387	1·786	7·7	2·775	1·97[illegible]	9·7	3·114	2·13[illegible]
1·8	1·342	1·216	3·8	1·949	1·560	5·8	2·408	1·797	7·8	2·793	1·98[illegible]	9·8	3·130	2·14[illegible]
1·9	1·378	1·239	3·9	1·975	1·574	5·9	2·429	1·807	7·9	2·811	1·99[illegible]	9·9	3·146	2·14[illegible]
2·0	1·414	1·260	4·0	2·000	1·587	6·0	2·449	1·817	8·0	2·828	2·00[illegible]	10·0	3·162	2·15[illegible]

To find the square root of an integer between 100 and 1000, e.g. 347, we have from the above table

$$\sqrt{340} = 10\sqrt{3{\cdot}4} = 18{\cdot}44,$$
$$\sqrt{350} = 10\sqrt{3{\cdot}5} = 18{\cdot}71.$$

Using the principle of proportional parts, the difference for $10 = {\cdot}27$; hence the difference for $7 = {\cdot}27 \times \frac{7}{10} = {\cdot}19$.

Therefore $\sqrt{347} = 18{\cdot}44 + {\cdot}19 = 18{\cdot}63$.

The root may be found more readily by looking out from Table VI the antilog. of $\frac{1}{2}\log 347$ obtained from Table V.

Similarly, the cube root of any number x is the antilog. of $\frac{1}{3}\log x$.

TABLE IV

TRIGONOMETRICAL RATIOS AND RADIAN MEASURE OF ANGLES FROM 0° TO 90° AT INTERVALS OF 1°

Radians.	Degrees.	Sines.	Tangents.	Secants.	Cosecants.	Cotangents.	Cosines.		
0	0	0	0	1	∞	∞	1	90	1·5708
·0175	1	·0175	·0175	1·0002	57·2987	57·2900	·9998	89	1·5533
·0349	2	·0349	·0349	1·0006	28·6537	28·6363	·9994	88	1·5359
·0524	3	·0523	·0524	1·0014	19·1073	19·0811	·9986	87	1·5184
·0698	4	·0698	·0699	1·0024	14·3356	14·3007	·9976	86	1·5010
·0873	5	·0872	·0875	1·0038	11·4737	11·4301	·9962	85	1·4835
·1047	6	·1045	·1051	1·0055	9·5668	9·5144	·9945	84	1·4661
·1222	7	·1219	·1228	1·0075	8·2055	8·1443	·9925	83	1·4486
·1396	8	·1392	·1405	1·0098	7·1853	7·1154	·9903	82	1·4312
·1571	9	·1564	·1584	1·0125	6·3925	6·3138	·9877	81	1·4137
·1745	10	·1736	·1763	1·0154	5·7588	5·6713	·9848	80	1·3963
·1920	11	·1908	·1944	1·0187	5·2408	5·1446	·9816	79	1·3788
·2094	12	·2079	·2126	1·0223	4·8097	4·7046	·9781	78	1·3614
·2269	13	·2250	·2309	1·0263	4·4454	4·3315	·9744	77	1·3439
·2443	14	·2419	·2493	1·0306	4·1336	4·0108	·9703	76	1·3265
·2618	15	·2588	·2679	1·0353	3·8637	3·7321	·9659	75	1·3090
·2793	16	·2756	·2867	1·0403	3·6280	3·4874	·9613	74	1·2915
·2967	17	·2924	·3057	1·0457	3·4203	3·2709	·9563	73	1·2741
·3142	18	·3090	·3249	1·0515	3·2361	3·0777	·9511	72	1·2566
·3316	19	·3256	·3443	1·0576	3·0716	2·9042	·9455	71	1·2392
·3491	20	·3420	·3640	1·0642	2·9238	2·7475	·9397	70	1·2217
·3665	21	·3584	·3839	1·0711	2·7904	2·6051	·9336	69	1·2043
·3840	22	·3746	·4040	1·0785	2·6695	2·4751	·9272	68	1·1868
·4014	23	·3907	·4245	1·0864	2·5593	2·3559	·9205	67	1·1694
·4189	24	·4067	·4452	1·0946	2·4586	2·2460	·9135	66	1·1519
·4363	25	·4226	·4663	1·1034	2·3662	2·1445	·9063	65	1·1345
·4538	26	·4384	·4877	1·1126	2·2812	2·0503	·8988	64	1·1170
·4712	27	·4540	·5095	1·1223	2·2027	1·9626	·8910	63	1·0996
·4887	28	·4695	·5317	1·1326	2·1301	1·8807	·8829	62	1·0821
·5061	29	·4848	·5543	1·1434	2·0627	1·8040	·8746	61	1·0647
·5236	30	·5000	·5774	1·1547	2·0000	1·7321	·8660	60	1·0472
·5411	31	·5150	·6009	1·1666	1·9416	1·6643	·8572	59	1·0297
·5585	32	·5299	·6249	1·1792	1·8871	1·6003	·8480	58	1·0123
·5760	33	·5446	·6494	1·1924	1·8361	1·5399	·8387	57	·9948
·5934	34	·5592	·6745	1·2062	1·7883	1·4826	·8290	56	·9774
·6109	35	·5736	·7002	1·2208	1·7434	1·4281	·8192	55	·9599
·6283	36	·5878	·7265	1·2361	1·7013	1·3764	·8090	54	·9425
·6458	37	·6018	·7536	1·2521	1·6616	1·3270	·7986	53	·9250
·6632	38	·6157	·7813	1·2690	1·6243	1·2799	·7880	52	·9076
·6807	39	·6293	·8098	1·2868	1·5890	1·2349	·7771	51	·8901
·6981	40	·6428	·8391	1·3054	1·5557	1·1918	·7660	50	·8727
·7156	41	·6561	·8693	1·3250	1·5243	1·1504	·7547	49	·8552
·7330	42	·6691	·9004	1·3456	1·4945	1·1106	·7431	48	·8378
·7505	43	·6820	·9325	1·3673	1·4663	1·0724	·7314	47	·8203
·7679	44	·6947	·9657	1·3902	1·4396	1·0355	·7193	46	·8029
·7854	45	·7071	1·0000	1·4142	1·4142	1·0000	·7071	45	·7854
		Cosines.	Cotangents.	Cosecants.	Secants.	Tangents.	Sines.	Degrees.	Radians.

The Radian Measure of any other angle can be obtained by Proportional Parts [10′ = ·0029 radian].

TABLE V

COMMON LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Difference-Columns. 1	2	3	4	5	6	7	8	9
10	·0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	·0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	·0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	·1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	·4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	·5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	
42	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	
43	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	
44	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	
45	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	
46	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	
47	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	
48	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	
49	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	
50	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	
51	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	
52	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	
53	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	
54	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	

	0	1	2	3	4	5	6	7	8	9	Difference-Columns.								
											1	2	3	4	5	6	7	8	9
55	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
88	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE VI

COMMON ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											Difference-Columns.								
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	Difference-Columns.								
											1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	21

TABLE VIII

NATURAL TANGENTS OF ANGLES FROM 0° TO 90° AT INTERVALS OF 1′

Natural Cotangents may be obtained from this table, by using the fact that $\cot A = \tan(90° - A)$.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes.				
											1′	2′	3′	4′	5′
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	6	10	13	17
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	10	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	10	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	2
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	2
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	2
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	2
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	2

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes. 1′	2′	3′	4′	5′
45°	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1.0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1.1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	26	33
49	1.1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1.2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1.2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	23	31	39
53	1.3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1.3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1.4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1.5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1.6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1.7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1.8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1.8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1.9626	9711	9797	9883	9970	$\bar{0}057$	$\bar{0}145$	$\bar{0}233$	$\bar{0}323$	$\bar{0}413$	15	29	44	58	73
64	2.0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2.1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2.2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	74	92
67	2.3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2.4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2.6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	118
70	2.7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	130
71	2.9042	9208	9375	9544	9714	9887	$\bar{0}061$	$\bar{0}237$	$\bar{0}415$	$\bar{0}595$	29	58	87	115	144
72	3.0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3.2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3.4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	203
75	3.7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	Use Proportional Parts.				
76	4.0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4.3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4.7046	7453	7867	8288	8716	9152	9594	$\bar{0}045$	$\bar{0}504$	$\bar{0}970$					
79	5.1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5.6713	7297	7894	8502	9124	9758	$\bar{0}405$	$\bar{1}066$	$\bar{1}742$	$\bar{2}432$					
81	6.3138	3859	4596	5350	6122	6912	7920	8548	9395	$\bar{0}264$					
82	7.1154	2066	3002	3962	4947	5958	6996	8062	9158	$\bar{0}285$					
83	8.1443	2636	3863	5126	6427	7769	9152	$\bar{0}579$	$\bar{2}052$	$\bar{3}572$					
84	9.5144	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20					
85	11.430	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.301	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.081	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.636	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.290	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					

TABLE IX

NAPIERIAN OR HYPERBOLIC LOGARITHMS OF NUMBERS FROM 1 TO 10 AT INTERVALS OF ·001

From this table, the hyperbolic logarithm of any four-digit number up to 10000 may be obtained,

e. g. log 27·9 = log 2·79 + log 10 = 1·0260 + 2·3026 = 3·3286.
log 5137 = log 5·137 + log 1000 = 1·6365 + 6·9078 = 8·5443.
log ·0279 = log 2·79 − log 100 = 1·0260 − 4·6052 = −3·5792.

	0	1	2	3	4	5	6	7	8	9	Difference-Columns. 1	2	3	4	5	6	7	8	9
1·0	·0000	0100	0198	0296	0392	0488	0583	0677	0770	0862	10	19	28	38	48	57	67	76	86
1·1	·0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	9	17	26	35	44	52	61	70	78
1·2	·1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	8	16	24	32	40	48	56	64	72
1·3	·2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	7	15	22	30	37	45	52	59	67
1·4	·3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	7	14	20	27	34	41	48	55	62
1·5	·4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	6	13	19	25	32	38	45	51	58
1·6	·4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	6	12	18	24	30	36	42	48	54
1·7	·5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	6	11	17	23	28	34	40	45	51
1·8	·5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	5	11	16	21	27	32	37	43	48
1·9	·6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	5	10	15	20	26	31	36	41	46
2·0	·6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	5	10	15	19	24	29	34	39	44
2·1	·7419	7467	7514	7561	7608	7655	7701	7747	7793	7839	5	10	14	19	23	28	33	37	42
2·2	·7885	7930	7975	8020	8065	8109	8154	8198	8242	8286	4	9	13	18	22	27	31	36	40
2·3	·8329	8372	8416	8459	8502	8544	8587	8629	8671	8713	4	9	13	17	21	26	30	34	38
2·4	·8755	8796	8838	8879	8920	8961	9002	9042	9083	9123	4	8	12	16	20	24	29	33	37
2·5	·9163	9203	9243	9282	9322	9361	9400	9439	9478	9517	4	8	12	16	20	24	27	31	35
2·6	·9555	9594	9632	9670	9708	9746	9783	9821	9858	9895	4	8	11	15	19	23	26	30	34
2·7	·9933	9969	$\overline{0}006$	$\overline{0}043$	$\overline{0}080$	$\overline{0}116$	$\overline{0}152$	$\overline{0}188$	$\overline{0}225$	$\overline{0}260$	4	7	11	15	18	22	25	29	33
2·8	1·0296	0332	0367	0403	0438	0473	0508	0543	0578	0613	4	7	11	14	18	21	25	28	32
2·9	1·0647	0682	0716	0750	0784	0815	0852	0886	0919	0953	3	7	10	14	17	20	24	27	31
3·0	1·0986	1019	1053	1086	1119	1151	1184	1217	1249	1282	3	7	10	13	16	20	23	26	29
3·1	1·1314	1346	1378	1410	1442	1474	1506	1537	1569	1600	3	6	10	13	16	19	22	25	28
3·2	1·1632	1663	1694	1725	1756	1787	1817	1848	1878	1909	3	6	9	12	15	18	22	25	28
3·3	1·1939	1969	2000	2030	2060	2090	2119	2149	2179	2208	3	6	9	12	15	18	21	24	27
3·4	1·2238	2267	2296	2326	2355	2384	2413	2442	2470	2499	3	6	9	12	15	17	20	23	26
3·5	1·2528	2556	2585	2613	2641	2669	2698	2726	2754	2782	3	6	9	11	14	17	20	23	25
3·6	1·2809	2837	2865	2892	2920	2947	2975	3002	3029	3056	3	5	8	11	14	16	19	22	25
3·7	1·3083	3110	3137	3164	3191	3218	3244	3271	3297	3324	3	5	8	11	13	16	19	21	24
3·8	1·3350	3376	3403	3429	3455	3481	3507	3533	3558	3584	3	5	8	10	13	16	18	21	23
3·9	1·3610	3635	3661	3686	3712	3737	3762	3788	3813	3838	3	5	8	10	13	15	18	20	23
4·0	1·3863	3888	3913	3938	3962	3987	4012	4036	4061	4085	2	5	7	10	12	15	17	20	22
4·1	1·4110	4134	4159	4183	4207	4231	4255	4279	4303	4327	2	5	7	10	12	14	17	19	22
4·2	1·4351	4375	4398	4422	4446	4469	4493	4516	4540	4563	2	5	7	9	12	14	17	19	21
4·3	1·4586	4609	4633	4656	4679	4702	4725	4748	4770	4793	2	5	7	9	12	14	16	18	[illegible]
4·4	1·4816	4839	4861	4884	4907	4929	4951	4974	4996	5019	2	5	7	9	11	13	16	18	[illegible]
4·5	1·5041	5063	5085	5107	5129	5151	5173	5195	5217	5239	2	4	7	9	11	13	15	18	[illegible]
4·6	1·5261	5282	5304	5326	5347	5369	5390	5412	5433	5454	2	4	6	9	11	13	15	17	[illegible]
4·7	1·5476	5497	5518	5539	5560	5581	5602	5623	5644	5665	2	4	6	8	11	13	15	17	[illegible]
4·8	1·5686	5707	5728	5748	5769	5790	5810	5831	5851	5872	2	4	6	8	10	12	14	16	
4·9	1·5892	5913	5933	5953	5974	5994	6014	6034	6054	6074	2	4	6	8	10	12	14	16	

	0	1	2	3	4	5	6	7	8	9	Difference-Columns.								
											1	2	3	4	5	6	7	8	
5·0	1·6094	6114	6134	6154	6174	6194	6214	6233	6253	6273	2	4	6	8	10	12	14	16	18
5·1	1·6292	6312	6332	6351	6371	6390	6409	6429	6448	6467	2	4	6	8	10	12	14	16	18
5·2	1·6487	6506	6525	6544	6563	6582	6601	6620	6639	6658	2	4	6	8	10	11	13	15	
5·3	1·6677	6696	6715	6734	6752	6771	6790	6808	6827	6845	2	4	6	7	9	11	13	15	
5·4	1·6864	6882	6901	6919	6938	6956	6974	6993	7011	7029	2	4	6	7	9	11	13	15	
5·5	1·7047	7066	7084	7102	7120	7138	7156	7174	7192	7210	2	4	5	7	9	11	13	14	
5·6	1·7228	7246	7263	7281	7299	7317	7334	7352	7370	7387	2	4	5	7	9	11	12	14	
5·7	1·7405	7422	7440	7457	7475	7492	7509	7527	7544	7561	2	3	5	7	9	10	12	14	
5·8	1·7579	7596	7613	7630	7647	7664	7681	7699	7716	7733	2	3	5	7	9	10	12	14	
5·9	1·7750	7766	7783	7800	7817	7834	7851	7867	7884	7901	2	3	5	7	8	10	12	13	
6·0	1·7918	7934	7951	7967	7984	8001	8017	8034	8050	8066	2	3	5	7	8	10	12	13	
6·1	1·8083	8099	8116	8132	8148	8165	8181	8197	8213	8229	2	3	5	6	8	10	11	13	
6·2	1·8245	8262	8278	8294	8310	8326	8342	8358	8374	8390	2	3	5	6	8	10	11	13	
6·3	1·8405	8421	8437	8453	8469	8485	8500	8516	8532	8547	2	3	5	6	8	9	11	13	14
6·4	1·8563	8579	8594	8610	8625	8641	8656	8672	8687	8703	2	3	5	6	8	9	11	12	
6·5	1·8718	8733	8749	8764	8779	8795	8810	8825	8840	8856	2	3	5	6	8	9	11	12	
6·6	1·8871	8886	8901	8916	8931	8946	8961	8976	8991	9006	2	3	5	6	8	9	11	12	
6·7	1·9021	9036	9051	9066	9081	9095	9110	9125	9140	9155	1	3	4	6	7	9	10	12	
6·8	1·9169	9184	9199	9213	9228	9242	9257	9272	9286	9301	1	3	4	6	7	9	10	11	
6·9	1·9315	9330	9344	9359	9373	9387	9402	9416	9430	9445	1	3	4	6	7	9	10	11	
7·0	1·9459	9473	9488	9502	9516	9530	9544	9559	9573	9587	1	3	4	6	7	9	10	11	
7·1	1·9601	9615	9629	9643	9657	9671	9685	9699	9713	9727	1	3	4	6	7	8	10	11	
7·2	1·9741	9755	9769	9782	9796	9810	9824	9838	9851	9865	1	3	4	6	7	8	10	11	
7·3	1·9879	9892	9906	9920	9933	9947	9961	9974	9988	0001	1	3	4	5	7	8	9	11	
7·4	2·0015	0028	0042	0055	0069	0082	0096	0109	0122	0136	1	3	4	5	7	8	9	11	
7·5	2·0149	0162	0176	0189	0202	0215	0229	0242	0255	0268	1	3	4	5	7	8	9	11	
7·6	2·0281	0295	0308	0321	0334	0347	0360	0373	0386	0399	1	3	4	5	7	8	9	10	
7·7	2·0412	0425	0438	0451	0464	0477	0490	0503	0516	0528	1	3	4	5	6	8	9	10	
7·8	2·0541	0554	0567	0580	0592	0605	0618	0631	0643	0656	1	3	4	5	6	8	9	10	
7·9	2·0669	0681	0694	0707	0719	0732	0744	0757	0769	0782	1	3	4	5	6	8	9	10	
8·0	2·0794	0807	0819	0832	0844	0857	0869	0882	0894	0906	1	2	4	5	6	7	9	10	
8·1	2·0919	0931	0943	0956	0968	0980	0992	1005	1017	1029	1	2	4	5	6	7	9	10	
8·2	2·1041	1054	1066	1078	1090	1102	1114	1126	1138	1150	1	2	4	5	6	7	8	10	
8·3	2·1163	1175	1187	1199	1211	1223	1235	1247	1258	1270	1	2	4	5	6	7	8	10	
8·4	2·1282	1294	1306	1318	1330	1342	1353	1365	1377	1389	1	2	4	5	6	7	8	9	
8·5	2·1401	1412	1424	1436	1448	1459	1471	1483	1494	1506	1	2	4	5	6	7	8	9	
8·6	2·1518	1529	1541	1552	1564	1576	1587	1599	1610	1622	1	2	3	5	6	7	8	9	
8·7	2·1633	1645	1656	1668	1679	1691	1702	1713	1725	1736	1	2	3	5	6	7	8	9	
8·8	2·1748	1759	1770	1782	1793	1804	1815	1827	1838	1849	1	2	3	5	6	7	8	9	
8·9	2·1861	1872	1883	1894	1905	1917	1928	1939	1950	1961	1	2	3	4	6	7	8	9	
9·0	2·1972	1983	1994	2006	2017	2028	2039	2050	2061	2072	1	2	3	4	6	7	8	9	
9·1	2·2083	2094	2105	2116	2127	2138	2148	2159	2170	2181	1	2	3	4	5	7	8	9	
9·2	2·2192	2203	2214	2225	2235	2246	2257	2268	2279	2289	1	2	3	4	5	6	8	9	
9·3	2·2300	2311	2322	2332	2343	2354	2364	2375	2386	2396	1	2	3	4	5	6	7	9	10
9·4	2·2407	2418	2428	2439	2450	2460	2471	2481	2492	2502	1	2	3	4	5	6	7	8	10
9·5	2·2513	2523	2534	2544	2555	2565	2576	2586	2597	2607	1	2	3	4	5	6	7	8	9
9·6	2·2618	2628	2638	2649	2659	2670	2680	2690	2701	2711	1	2	3	4	5	6	7	8	9
9·7	2·2721	2732	2742	2752	2762	2773	2783	2793	2803	2814	1	2	3	4	5	6	7	8	9
9·8	2·2824	2834	2844	2854	2865	2875	2885	2895	2905	2915	1	2	3	4	5	6	7	8	9
9·9	2·2925	2935	2946	2956	2966	2976	2986	2996	3006	3016	1	2	3	4	5	6	7	8	9
10·0	2·3026	log 100 = 4·6052			log 1000 = 6·9078			log 10000 = 9·2103			log 100,000 = 11·5129								

e^x	e^{-x}	x	$\sinh x$	$\cosh x$	e^x	e^{-x}	x	$\sinh x$	$\cosh x$	e^x	e^{-x}	x	$\sinh x$	$\cosh x$
	1	·00	0	1	1·5683	·6376	·45	·4653	1·1030	2·4596	·4066	·90	1·0265	1·433
·0101	·9900	·01	·0100	1·0001	1·5841	·6313	·46	·4764	1·1077	2·4843	·4025	·91	1·0409	1·443
·0202	·9802	·02	·0200	1·0002	1·6000	·6250	·47	·4875	1·1125	2·5093	·3985	·92	1·0554	1·45
1·0305	·9704	·03	·0300	1·0005	1·6161	·6188	·48	·4986	1·1174	2·5345	·3946	·93	1·0700	1·46
1·0408	·9608	·04	·0400	1·0008	1·6323	·6126	·49	·5098	1·1225	2·5600	·3906	·94	1·0847	1·47
1·0513	·9512	·05	·0500	1·0013	1·6487	·6065	·50	·5211	1·1276	2·5857	·3867	·95	1·0995	1·48
1·0618	·9418	·06	·0600	1·0018	1·6653	·6005	·51	·5324	1·1329	2·6117	·3829	·96	1·1144	1·49
1·0725	·9324	·07	·0701	1·0025	1·6820	·5945	·52	·5438	1·1383	2·6379	·3791	·97	1·1294	1·50
1·0833	·9231	·08	·0801	1·0032	1·6989	·5886	·53	·5552	1·1438	2·6645	·3753	·98	1·1446	1·51
1·0942	·9139	·09	·0901	1·0041	1·7160	·5827	·54	·5666	1·1494	2·6912	·3716	·99	1·1598	1·53
052	·9048	·10	·1002	1·0050	1·7333	·5769	·55	·5782	1·1551	2·7183	·3679	1·00	1·1752	1·543
63	·8958	·11	·1102	1·0061	1·7507	·5712	·56	·5897	1·1609	2·7456	·3642	1·01	1·1907	1·554
75	·8869	·12	·1203	1·0072	1·7683	·5655	·57	·6014	1·1669	2·7732	·3606	1·02	1·2063	1·5
88	·8781	·13	·1304	1·0085	1·7860	·5599	·58	·6131	1·1730	2·8011	·3570	1·03	1·2220	1·57
503	·8694	·14	·1405	1·0098	1·8040	·5543	·59	·6248	1·1792	2·8292	·3535	1·04	1·2379	1·59
1618	·8607	·15	·1506	1·0113	1·8221	·5488	·60	·6367	1·1855	2·8577	·3499	1·05	1·2539	1·60
1·1735	·8521	·16	·1607	1·0128	1·8404	·5434	·61	·6485	1·1919	2·8864	·3465	1·06	1·2700	1·61
1·1853	·8437	·17	·1708	1·0145	1·8589	·5379	·62	·6605	1·1984	2·9154	·3430	1·07	1·2862	1·62
1·1972	·8353	·18	·1810	1·0162	1·8776	·5326	·63	·6725	1·2051	2·9447	·3396	1·08	1·3025	1·64
1·2092	·8270	·19	·1911	1·0181	1·8965	·5273	·64	·6846	1·2119	2·9743	·3362	1·09	1·3190	1·65
1·2214	·8187	·20	·2013	1·0201	1·9155	·5220	·65	·6967	1·2188	3·0042	·3329	1·10	1·3356	1·6
1·2337	·8106	·21	·2115	1·0221	1·9348	·5169	·66	·7090	1·2258	3·0344	·3296	1·11	1·3524	1·6
1·2461	·8025	·22	·2218	1·0243	1·9542	·5117	·67	·7213	1·2330	3·0649	·3263	1·12	1·3693	1·6
1·2586	·7945	·23	·2320	1·0266	1·9739	·5066	·68	·7336	1·2402	3·0957	·3230	1·13	1·3863	1·7
1·2712	·7866	·24	·2423	1·0289	1·9937	·5016	·69	·7461	1·2476	3·1268	·3198	1·14	1·4035	1·7
1·2840	·7788	·25	·2526	1·0314	2·0138	·4966	·70	·7586	1·2552	3·1582	·3166	1·15	1·4208	1·7
1·2969	·7711	·26	·2629	1·0340	2·0340	·4916	·71	·7712	1·2628	3·1899	·3135	1·16	1·4382	1·7
1·3100	·7634	·27	·2733	1·0367	2·0544	·4868	·72	·7838	1·2706	3·2220	·3104	1·17	1·4558	1·7
1·3231	·7558	·28	·2837	1·0395	2·0751	·4819	·73	·7966	1·2785	3·2544	·3073	1·18	1·4735	1·7
1·3364	·7483	·29	·2941	1·0423	2·0959	·4771	·74	·8094	1·2865	3·2871	·3042	1·19	1·4914	1·7
1·3499	·7408	·30	·3045	1·0453	2·1170	·4724	·75	·8223	1·2947	3·3201	·3012	1·20	1·5095	1·
1·3634	·7334	·31	·3150	1·0484	2·1383	·4677	·76	·8353	1·3030	3·3535	·2982	1·21	1·5276	1·
1·3771	·7261	·32	·3255	1·0516	2·1598	·4630	·77	·8484	1·3114	3·3872	·2952	1·22	1·5460	1·
1·3910	·7189	·33	·3360	1·0549	2·1815	·4584	·78	·8615	1·3199	3·4212	·2923	1·23	1·5645	1·
1·4049	·7118	·34	·3466	1·0584	2·2034	·4538	·79	·8748	1·3286	3·4556	·2894	1·24	1·5831	1·
1·4191	·7047	·35	·3572	1·0619	2·2255	·4493	·80	·8881	1·3374	3·4903	·2865	1·25	1·6019	1
1·4333	·6977	·36	·3678	1·0655	2·2479	·4449	·81	·9015	1·3464	3·5254	·2837	1·26	1·6209	1
1·4477	·6907	·37	·3785	1·0692	2·2705	·4404	·82	·9150	1·3555	3·5609	·2808	1·27	1·6400	1
1·4623	·6839	·38	·3892	1·0731	2·2933	·4360	·83	·9286	1·3646	3·5966	·2780	1·28	1·6593	1
1·4770	·6771	·39	·4000	1·0770	2·3164	·4317	·84	·9423	1·3740	3·6328	·2753	1·29	1·6788	1
1·4918	·6703	·40	·4108	1·0811	2·3396	·4274	·85	·9561	1·3835	3·6693	·2725	1·30	1·6984	1
1·5068	·6637	·41	·4216	1·0852	2·3632	·4232	·86	·9700	1·3932	3·7062	·2698	1·31	1·7182	1
1·5220	·6570	·42	·4325	1·0895	2·3869	·4190	·87	·9840	1·4029	3·7434	·2671	1·32	1·7381	[illegible]
1·5373	·6505	·43	·4434	1·0939	2·4109	·4148	·88	·9981	1·4128	3·7810	·2645	1·33	1·7583	[illegible]
[illegible]	·6440	·44	·4543	1·0984	2·4351	·4107	·89	1·0122	1·4229	3·8190	·2618	1·34	1·7786	[illegible]

e^x	e^{-x}	x	$\sinh x$	$\cosh x$	e^x	e^{-x}	x	$\sinh x$	$\cosh x$	e^x	e^{-x}	x	$\sinh x$	$\cosh x$
3·8574	·2592	**1·35**	1·7991	2·0583	6·6859	·1496	**1·90**	3·2682	3·4177	11·588	·0863	**2·45**	5·7510	5·8373
3·8962	·2567	**1·36**	1·8198	2·0764	6·7531	·1481	**1·91**	3·3025	3·4506	11·705	·0854	**2·46**	5·8097	5·8951
3·9354	·2541	**1·37**	1·8406	2·0947	6·8210	·1466	**1·92**	3·3372	3·4838	11·822	·0846	**2·47**	5·8689	5·9535
3·9749	·2516	**1·38**	1·8617	2·1132	6·8895	·1451	**1·93**	3·3722	3·5173	11·941	·0837	**2·48**	5·9288	6·0125
4·0148	·2491	**1·39**	1·8829	2·1320	6·9588	·1437	**1·94**	3·4075	3·5512	12·061	·0829	**2·49**	5·9892	6·0721
4·0552	·2466	**1·40**	1·9043	2·1509	7·0287	·1423	**1·95**	3·4432	3·5855	12·182	·0821	**2·50**	6·0502	6·1323
4·0960	·2441	**1·41**	1·9259	2·1700	7·0993	·1409	**1·96**	3·4792	3·6201	12·305	·0813	**2·51**	6·1118	6·1931
4·1371	·2417	**1·42**	1·9477	2·1894	7·1707	·1395	**1·97**	3·5156	3·6551	12·429	·0805	**2·52**	6·1741	6·2546
4·1787	·2393	**1·43**	1·9697	2·2090	7·2427	·1381	**1·98**	3·5523	3·6904	12·554	·0797	**2·53**	6·2369	6·3166
4·2207	·2369	**1·44**	1·9919	2·2288	7·3155	·1367	**1·99**	3·5894	3·7261	12·680	·0789	**2·54**	6·3004	6·3793
4·2631	·2346	**1·45**	2·0143	2·2488	7·3891	·1353	**2·00**	3·6269	3·7622	12·807	·0781	**2·55**	6·3645	6·4426
4·3060	·2322	**1·46**	2·0369	2·2691	7·4633	·1340	**2·01**	3·6647	3·7987	12·936	·0773	**2·56**	6·4293	6·5066
4·3492	·2299	**1·47**	2·0597	2·2896	7·5383	·1327	**2·02**	3·7028	3·8355	13·066	·0765	**2·57**	6·4946	6·5712
4·3929	·2276	**1·48**	2·0827	2·3103	7·6141	·1313	**2·03**	3·7413	3·8727	13·197	·0758	**2·58**	6·5607	6·6365
4·4371	·2254	**1·49**	2·1059	2·3312	7·6906	·1300	**2·04**	3·7803	3·9103	13·330	·0750	**2·59**	6·6274	6·7024
4·4817	·2231	**1·50**	2·1293	2·3524	7·7679	·1287	**2·05**	3·8196	3·9483	13·464	·0743	**2·60**	6·6947	6·7690
4·5267	·2209	**1·51**	2·1529	2·3738	7·8460	·1275	**2·06**	3·8593	3·9867	13·599	·0735	**2·61**	6·7628	6·8363
4·5722	·2187	**1·52**	2·1768	2·3955	7·9248	·1262	**2·07**	3·8993	4·0255	13·736	·0728	**2·62**	6·8315	6·9043
4·6182	·2165	**1·53**	2·2008	2·4174	8·0045	·1249	**2·08**	3·9398	4·0647	13·874	·0721	**2·63**	6·9008	6·9729
4·6646	·2144	**1·54**	2·2251	2·4395	8·0849	·1237	**2·09**	3·9806	4·1043	14·013	·0714	**2·64**	6·9709	7·0423
4·7115	·2122	**1·55**	2·2496	2·4619	8·1662	·1225	**2·10**	4·0219	4·1443	14·154	·0707	**2·65**	7·0417	7·1123
4·7588	·2101	**1·56**	2·2743	2·4845	8·2482	·1212	**2·11**	4·0635	4·1847	14·296	·0699	**2·66**	7·1132	7·1831
4·8066	·2080	**1·57**	2·2993	2·5073	8·3311	·1200	**2·12**	4·1056	4·2256	14·440	·0693	**2·67**	7·1854	7·2546
4·8550	·2060	**1·58**	2·3245	2·5305	8·4149	·1188	**2·13**	4·1480	4·2669	14·585	·0686	**2·68**	7·2583	7·3268
4·9037	·2039	**1·59**	2·3499	2·5538	8·4994	·1177	**2·14**	4·1909	4·3085	14·732	·0679	**2·69**	7·3319	7·3998
4·9530	·2019	**1·60**	2·3756	2·5775	8·5849	·1165	**2·15**	4·2342	4·3507	14·880	·0672	**2·70**	7·4063	7·4735
5·0028	·1999	**1·61**	2·4015	2·6014	8·6711	·1153	**2·16**	4·2779	4·3932	15·029	·0665	**2·71**	7·4814	7·5479
5·0531	·1979	**1·62**	2·4276	2·6255	8·7583	·1142	**2·17**	4·3221	4·4362	15·180	·0659	**2·72**	7·5572	7·6231
5·1039	·1959	**1·63**	2·4540	2·6499	8·8463	·1130	**2·18**	4·3666	4·4797	15·333	·0652	**2·73**	7·6338	7·6991
5·1552	·1940	**1·64**	2·4806	2·6746	8·9352	·1119	**2·19**	4·4116	4·5236	15·487	·0646	**2·74**	7·7112	7·7758
5·2070	·1920	**1·65**	2·5075	2·6995	9·0250	·1108	**2·20**	4·4571	4·5679	15·643	·0639	**2·75**	7·7894	7·8533
5·2593	·1901	**1·66**	2·5346	2·7247	9·1157	·1097	**2·21**	4·5030	4·6127	15·800	·0633	**2·76**	7·8683	7·9316
5·3122	·1882	**1·67**	2·5620	2·7502	9·2073	·1086	**2·22**	4·5494	4·6580	15·959	·0627	**2·77**	7·9480	8·0106
5·3656	·1864	**1·68**	2·5896	2·7760	9·2999	·1075	**2·23**	4·5962	4·7037	16·119	·0620	**2·78**	8·0285	8·0905
5·4195	·1845	**1·69**	2·6175	2·8020	9·3933	·1065	**2·24**	4·6434	4·7499	16·281	·0614	**2·79**	8·1095	8·1712
5·4739	·1827	**1·70**	2·6456	2·8283	9·4877	·1054	**2·25**	4·6913	4·7966	16·445	·0608	**2·80**	8·1919	8·2527
5·5290	·1809	**1·71**	2·6740	2·8549	9·5831	·1044	**2·26**	4·7394	4·8437	16·610	·0602	**2·81**	8·2749	8·3351
5·5845	·1791	**1·72**	2·7027	2·8818	9·6794	·1033	**2·27**	4·7880	4·8914	16·777	·0596	**2·82**	8·3586	8·4182
5·6407	·1773	**1·73**	2·7317	2·9090	9·7768	·1023	**2·28**	4·8372	4·9395	16·945	·0590	**2·83**	8·4432	8·5022
5·6973	·1755	**1·74**	2·7609	2·9364	9·8749	·1013	**2·29**	4·8868	4·9881	17·116	·0584	**2·84**	8·5287	8·5871
5·7546	·1738	**1·75**	2·7904	2·9642	9·9742	·1003	**2·30**	4·9370	5·0372	17·288	·0578	**2·85**	8·6150	8·6728
5·8124	·1720	**1·76**	2·8202	2·9922	10·074	·0993	**2·31**	4·9876	5·0868	17·462	·0573	**2·86**	8·7021	8·7594
5·8709	·1703	**1·77**	2·8503	3·0206	10·176	·0983	**2·32**	5·0387	5·1370	17·637	·0567	**2·87**	8·7902	8·8469
5·9299	·1686	**1·78**	2·8806	3·0492	10·278	·0973	**2·33**	5·0903	5·1876	17·814	·0561	**2·88**	8·8791	8·9352
5·9895	·1670	**1·79**	2·9112	3·0782	10·381	·0963	**2·34**	5·1425	5·2388	17·993	·0556	**2·89**	8·9689	9·0244
6·0496	·1653	**1·80**	2·9422	3·1075	10·486	·0954	**2·35**	5·1951	5·2905	18·174	·0550	**2·90**	9·0596	9·1146
6·1104	·1637	**1·81**	2·9734	3·1371	10·591	·0944	**2·36**	5·2483	5·3427	18·357	·0545	**2·91**	9·1512	9·2056
6·1719	·1620	**1·82**	3·0049	3·1669	10·697	·0935	**2·37**	5·3020	5·3954	18·541	·0539	**2·92**	9·2437	9·2976
6·2339	·1604	**1·83**	3·0367	3·1971	10·805	·0926	**2·38**	5·3562	5·4487	18·728	·0534	**2·93**	9·3371	9·3905
6·2965	·1588	**1·84**	3·0689	3·2277	10·913	·0916	**2·39**	5·4109	5·5026	18·916	·0529	**2·94**	9·4315	9·4844
6·3598	·1572	**1·85**	3·1013	3·2585	11·023	·0907	**2·40**	5·4662	5·5569	19·106	·0523	**2·95**	9·5268	9·5791
6·4237	·1557	**1·86**	3·1340	3·2897	11·134	·0898	**2·41**	5·5221	5·6119	19·298	·0518	**2·96**	9·6231	9·6749
6·4883	·1541	**1·87**	3·1671	3·3212	11·246	·0889	**2·42**	5·5785	5·6674	19·492	·0513	**2·97**	9·7203	9·7716
6·5535	·1526	**1·88**	3·2005	3·3530	11·359	·0880	**2·43**	5·6354	5·7235	19·688	·0508	**2·98**	9·8185	9·8693
6·6194	·1511	**1·89**	3·2341	3·3852	11·473	·0872	**2·44**	5·6929	5·7801	19·886	·0503	**2·99**	9·9177	9·9680

(b) Values of e^x, e^{-x}, $\sinh x$ and $\cosh x$ from $x = 3$ to $x = 6$ at intervals of ·05.

e^x	e^{-x}	x	$\sinh x$	$\cosh x$	e^x	e^{-x}	x	$\sinh x$	$\cosh x$	e^x	e^{-x}	x	$\sinh x$	$\cosh x$
20·086	·0498	3·0	10·018	10·068	54·598	·0183	4·0	27·290	27·308	148·41	·0067	5·0	74·203	[illegible]
21·115	·0474	3·05	10·534	10·581	57·397	·0174	4·05	28·690	28·707	156·02	·0064	5·05	78·008	78·014
22·198	·0450	3·1	11·076	11·122	60·340	·0166	4·1	30·162	30·178	164·02	·0061	5·1	82·008	82·014
23·336	·0429	3·15	11·647	11·689	63·434	·0158	4·15	31·709	31·725	172·43	·0058	5·15	86·213	[illegible]
24·533	·0408	3·2	12·246	12·287	66·686	·0150	4·2	33·336	33·351	181·27	·0055	5·2	90·633	[illegible]
25·790	·0388	3·25	12·876	12·915	70·105	·0143	4·25	35·046	35·060	190·57	·0052	5·25	95·280	[illegible]
27·113	·0369	3·3	13·538	13·575	73·700	·0136	4·3	36·843	36·857	200·34	·0050	5·3	100·17	[illegible]
28·503	·0351	3·35	14·234	14·269	77·478	·0129	4·35	38·733	38·746	210·61	·0047	5·35	105·30	[illegible]
29·964	·0334	3·4	14·965	14·999	81·451	·0123	4·4	40·719	40·732	221·41	·0045	5·4	110·70	[illegible]
31·500	·0317	3·45	15·734	15·766	85·627	·0117	4·45	42·808	42·819	232·76	·0043	5·45	116·38	[illegible]
33·115	·0302	3·5	16·543	16·573	90·017	·0111	4·5	45·003	45·014	244·69	·0041	5·5	122·34	[illegible]
34·813	·0287	3·55	17·392	17·421	94·632	·0106	4·55	47·311	47·322	257·24	·0039	5·55	128·62	[illegible]
36·598	·0273	3·6	18·285	18·313	99·484	·0101	4·6	49·737	49·747	270·43	·0037	5·6	135·21	[illegible]
38·475	·0260	3·65	19·224	19·250	104·59	·0096	4·65	52·288	52·297	284·29	·0035	5·65	142·14	[illegible]
40·447	·0247	3·7	20·211	20·236	109·95	·0091	4·7	54·969	54·978	298·87	·0033	5·7	149·43	149·44
42·521	·0235	3·75	21·249	21·272	115·58	·0087	4·75	57·788	57·796	314·19	·0032	5·75	157·09	[illegible]
44·701	·0224	3·8	22·339	22·362	121·51	·0082	4·8	60·751	60·759	330·30	·0030	5·8	165·15	[illegible]
46·993	·0213	3·85	23·486	23·507	127·74	·0078	4·85	63·866	63·874	347·23	·0029	5·85	173·62	173·62
49·402	·0202	3·9	24·691	24·711	134·29	·0074	4·9	67·141	67·149	365·04	·0027	5·9	182·52	182·52
[illegible]	[illegible]	3·95	25·958	25·977	141·17	·0071	4·95	70·584	70·591	383·75	·0026	5·95	191·88	[illegible]
											·0025	6·0		

For intermediate values of x, the values of the functions may be found by using the first three terms of their expansions by Taylor's Theorem, viz.:

$$e^{\pm x+h} = e^{\pm x} + he^{\pm x} + \tfrac{1}{2} h^2 e^{\pm x}.$$
$$\sinh (x+h) = \sinh x + h \cosh x + \tfrac{1}{2} h^2 \sinh x.$$
$$\cosh (x+h) = \cosh x + h \sinh x + \tfrac{1}{2} h^2 \cosh x.$$
$$e^{3\cdot 63} = e^{3\cdot 65 - \cdot 02} = e^{3\cdot 65} - \cdot 02\, e^{3\cdot 65} + \cdot 0002\, e^{3\cdot 65}$$
$$= 38\cdot 475 - \cdot 7695 + \cdot 0077 = 37\cdot 713.$$
$$\sinh 3\cdot 31 = \sinh (3\cdot 3 + \cdot 01) = \sinh 3\cdot 3 + \cdot 01 \cosh 3\cdot 3 + \cdot 00005 \sinh 3\cdot 3$$
$$= 13\cdot 538 + \cdot 1358 + \cdot 0007 = 13\cdot 675.$$

For higher values of x, the values of the functions e^x and e^{-x} may be worked out by the aid of a table of common logarithms [$\log_{10} e = \cdot 4342945$], and the values of both $\sinh x$ and $\cosh x$ may be taken equal to $\frac{1}{2} e^x$.

e.g. $\log_{10} e^9 = 3\cdot 9086505$, whence $e^9 = 8103\cdot 1$,

and $\sinh x = \cosh x = 4051\cdot 5$.

$\log_{10} e^{-9} = -\log_{10} e^9 = \bar{4}\cdot 0913495$, whence $e^{-9} = \cdot 0001234$.

For negative values of x, since $\sinh x$ is an odd function of x and $\cosh x$ an even function, it follows that

$$\sinh(-2) = -\sinh 2 = -3\cdot 6269, \quad \cosh(-2) = \cosh 2 = 3\cdot 7622, \ \&c.$$

The values of the other hyperbolic functions may, if required, be obtained from their definitions in Art. 92:

$$\tanh x = \sinh x/\cosh x; \quad \coth x = \cosh x/\sinh x;$$
$$\operatorname{sech} x = 1/\cosh x; \quad \operatorname{cosech} x = 1/\sinh x;$$

by using the present table and a table of logarithms.

e.g. $\tanh 1\cdot 5 = \sinh 1\cdot 5/\cosh 1\cdot 5 = 2\cdot 1293/2\cdot 3524 = \cdot 905$ nearly.

The values of the inverse hyperbolic functions can be obtained from Table IX by the aid of the formulae of Art. 94:

$$\cosh^{-1} x = \log \{x \pm \sqrt{(x^2 - 1)}\},$$
$$\sinh^{-1} x = \log \{x + \sqrt{(x^2 + 1)}\},$$
$$\tanh^{-1} x = \tfrac{1}{2} \log \{(1+x)/(1-x)\},$$
$$\coth^{-1} x = \tfrac{1}{2} \log \{(x+1)/(x-1)\}.$$

$$\cosh^{-1} 4 = \log (4 \pm \sqrt{15}) = \pm \log 7\cdot 873 = \pm 2\cdot 0635.$$
$$\tanh^{-1} \cdot 35 = \tfrac{1}{2} \log \tfrac{1\cdot 35}{\cdot 65} = \tfrac{1}{2} \log \tfrac{27}{13} = \tfrac{1}{2} \log 2\cdot 077 = \cdot 3654.$$

ANSWERS TO THE EXAMPLES

Examples I, p. 5.

1. -2; -2; 4; 4; -296.
2. $\frac{1}{3}$; 0; -5; $13/147$; $(x-2)(6-x)/x^2$; -21.
4. $ax^2+(b+2a)x+a+b+c$; $ax^2+(b-2a)x+a-b+c$; $ah^2+(2ax+b)h$.
8. odd, even, odd, even, odd, odd, odd, even, even, odd, even, even, odd.
9. (i) $y=\sqrt[3]{(a^3-x^3)}$. (ii) $y=\pm\sqrt{(a^4+b^4)}/x$.
   (iii) $y=a/x$. (iv) $y=-a\pm\sqrt{(x^2+a^2)}$.
   (v) $y=\sin^{-1}\{(cx-b)/a\}$. (vi) $y=-(bx+d)/(ax+c)$.
10. (i) $2xy-3x-y+2=0$. (ii) $x^n+y^n=a^n$.
    (iii) $xy^2=(a+x)^2$. (iv) $(1+x^2)a^{2y}=x^2$.
    (v) $y^2-2xy+2x^2=1$. (vi) $x=a\sin y$.
11. (i) $x=\sqrt[4]{y}$. (ii) $x=(y-1)^2$. (iii) $x=\frac{1}{2}\cos y$.
    (iv) $x=\log_a y$. (v) $x=\tan^{-1}\sqrt{(y/a)}$. (vi) $x=\sqrt[n]{(a^n-y^n)}$.
    (vii) $x=\sqrt{(5-y^2)}$. (viii) $x=1\pm\sqrt{(1-y^2)}$. (ix) $x=y/(y-4)$.
    (x) $x=a^{4y}-1$.

	Number of values of y.	Values of x for which y is defined.	Number of values of x.	Values of y for which x is defined.
12. (i)	1.	All.	4 (2 real).	y positive.
(ii)	2.	x positive.	1.	All.
(iii)	∞.	$\lvert x\rvert<\frac{1}{2}$.	1.	All.
(iv)	q, if $x=p/q$ where p and q are integral.	x rational.	1 (real).	y positive.
(v)	1.	All.	∞.	y/a positive.
(vi)	n*	$\lvert x\rvert<a$, if n be even. All, if n be odd.		$\lvert y\rvert<a$, if n be even. All, if n be odd.
(vii)	2.	$\lvert x\rvert<\sqrt{5}$.	2.	$\lvert y\rvert<\sqrt{5}$.
(viii)	2.	From 0 to 2.	2	$\lvert y\rvert<1$.
(ix)	1.	All except $x=1$.	1.	All except $y=4$.

Examples III, p. 46.

1. 4. 2. 3. 3. 3. 4. $\frac{1}{2}$.
5. $\frac{1}{3}$. 6. b/d; a/c. 7. $\frac{1}{3}$; $\frac{3}{5}$. 8. b/q; ∞.
9. a^2/b^3; 0. 10. 0; -3. 11. ∞; 8. 12. 0; 0.
13. 2. 14. 1. 15. $-1/4\sqrt{(2a)}$. 16. 7; $1/n$.

* Two only are real if n be even, and one only if n be odd.

17. $10a^9$; $1/2\sqrt{a}$; $\frac{10}{7}a^3$. 18. $\frac{3}{2}a^{1/6}$. 19. 0.

20. $\frac{1}{2}a$. 21. $\frac{1}{2}$. 22. 0. 23. 2.

24. p/q. 25. p/q. 26. m. 27. m/n.

28. $\frac{1}{2}p^2$; $\frac{1}{2}(b^2-a^2)$.

30. πrl, if r be the radius of the base, and l the slant height.

31. $2\pi rh$; $\pi r^2 h$, if r be the radius, and h the height.

32. $\frac{4}{3}\pi r^3$. 33. m^2/n^2. 34. 0. 35. m/n.

36. $2\pi a^3$. 37. $\frac{1}{3}$. 38. $1/(1-x)^2$.

Examples IV, p. 54.

3. When $x =$ [illegible] -1; $n\pi$; $\frac{1}{2}(2n+1)\pi$; $\frac{1}{2}n\pi$; ± 2, ± 3; $\frac{1}{2}(2n+1)\pi$; $\frac{1}{6}(2n+1)\pi$; $(2n+1)\pi$; 0, respectively. 4. 27.

Examples V, p. 60.

1. (i) (a) 6, (b) 12, (c) 60 sq. ft. per min. (ii) (a) (b) (c) 4 ft. per min. (iii) (a) (b) (c) $\sqrt{2}$ ft. per min. 2. 71°34′, 82°52′, 99°28′.

3. (i) $x = \cdot 18$. (ii) $x = \cdot 87$. (iii) $x = -\frac{1}{2}$.

4. (i) 3. (ii) -6. At the points $(\frac{2}{3}, \frac{1}{3})$, (1·83, 2·51), $(-\frac{2}{3}\sqrt{3}, 1)$ respectively.

5. 331, 303·01, 300·3001, $300 + 30h + h^2$; 300. 6. $3x^2$.

7. 36°52′, 71°34′, 85°14′. 8. $(\pm\frac{1}{3}\sqrt{3}, \pm\frac{1}{9}\sqrt{3})$; $\frac{3}{2}$.

9. They touch at (0, 0) and intersect at 8°8′ at (1, 1).

10. (i) $\frac{1}{4}$ c. ft., (ii) 1 sq. ft., (iii) 1·73 in. per sec.

11. (i) 36π c. ft., (ii) 24π sq. ft. per min.

12. 1080π c. in. per min. 13. At (1, 1). 14. $1\frac{1}{2}$ c. in. per sec.

15. (i) 12, (ii) ± 16, (iii) -8 ft.-secs., i.e. 8 ft.-secs. downwards.

16. (i) 4, (ii) 1 ft.-sec. per sec. 17. (i) $\frac{1}{10}$ in., (ii) 30 c. in. per sec.

18. (i) $5/(144\pi)$ in., (ii) $\frac{5}{3}$ sq. in. per sec. 19. $\frac{1}{2}$ in. per sec.

Examples VI, p. 70.

1. $4x^3$. 2. $-2(1-x)$. 3. $-1/x^2$.

4. $-2/x^3$. 5. $q/(p-qx)^2$. 6. $7/(3x-2)^2$.

7. $(bc-ad)/(c+dx)^2$. 8. $4x-7$. 9. $2ax+b$.

10. $(1-x^2)/(x^2+1)^2$. 11. $-1/(2x^{3/2})$. 12. $b/\{2(a-bx)^{3/2}\}$.

13. $-x/\sqrt{(a^2-x^2)}$. 14. $3x\sqrt{(x^2+a^2)}$. 15. $x/(1-x^2)^{3/2}$.

16. ·0499375. 17. ·00986. 18. $-\frac{1}{16}$.

19. $-\frac{1}{500}$. 20. $-\frac{3}{25}$. 21. $6x-7$; 6·075.

22. $90/(10+5x)^2$; 1·22027. 23. $-2x/(x^2-1)^2$; ·0126125.

26. $2ax+b$; $x = -b/2a$. 27. $x = \pm 1$. $\pm\frac{1}{2}$.

28. y increases or decreases according as $|x| >$ or < 1.

29. (i) $dx/dt = n\,dz/dt$. (ii) $dy/dx = n\,dy/dz$. (iii) $dy/dx = du/dx + dv/dx$.

30. (i) $dx/dt = n\,dy/dt$. (ii) $dv/dt = kv$. (iii) $dv/dt = -kt$.

31. (i) $dA/dr = kr$. (ii) $dV/dr = kr$. (iii) $dV/dx = k(dA/dx)^2$. (iv) $dV/dt = k\,dA/dt$.

32. (i) $dy/dx = kx$. (ii) $y = k(dy/dx)^2$. (iii) $dy/dx = \frac{1}{2}y/x$.

Examples VII, p. 72.

$5x^4,\ 9x^8,\ 30x^{29},\ 75x^{74}$.

$\frac{1}{5}/\sqrt[5]{x^4},\ \frac{1}{10}/\sqrt[10]{x^9},\ \frac{2}{3}/\sqrt[3]{x},\ \frac{7}{4}\sqrt[4]{x^3},\ \frac{4}{7}/\sqrt[7]{x^3},\ 1/(n\sqrt[n]{x^{n-1}})$.

$-3/x^4,\ -7/x^8,\ -10/x^{11},\ -50/x^{51},\ -n/x^{n+1}$.

$-\frac{2}{3}/\sqrt[3]{x^5},\ -\frac{1}{5}/\sqrt[5]{x^6},\ -\frac{3}{4}/\sqrt[4]{x^7},\ -1/(n\sqrt[n]{x^{n+1}}),\ -p/(q\sqrt[q]{x^{p+q}})$.

5·0267; 4·996; 2·00117.

6. ·1005; ·100167; ·33278.

1·2; 1·02; ·88.

Examples VIII, p. 74.

1. $2x-7$.
2. $6x-8$.
3. $2px+q$.
4. $6x^2-9$.
5. $3ax^2+2bx+c$.
6. $30x^5-30x^4+$
7. $4x^3-4a^2x$.
8. $2nx^{n-1}(x^n-a^n)$.
9. $2(x-5)$.
10. $1+\sqrt{(a/x)}$.
11. $-3(1-x)^2$.
12. $3a(ax-b)^2$.
13. $4/x^2-6/x^3$.
14. $1-1/x^2$.
15. $2x-2/x^3$.
16. $(1-3a/x)/2\sqrt{x}$.
17. $2/x^2-2/x^3$.
18. $-6(2a^6+5a^4x^2+4a^2x^4+x^6)/x^{13}$.
19. $\frac{3}{2}\sqrt{x}-6+6/\sqrt{x}$.
20. $(\sqrt{x}-1)/x^2$.
21. $3b(ax-b)^2/x^4$.

Examples IX, p. 77.

1. $2x(3x^4-8x^2+3)$.
2. $\frac{1}{2}x^{n-\frac{1}{2}}+(1+\sqrt{x})nx^{n-1}$.
3. $(m+n)x^{m+n-1}+ma^n x^{m-1}+na^m x^{n-1}$.
4. $\frac{1}{2}x^{3/2}(7x+15)$.
5. $3ax^2+2(ac+b)x+bc+ac^2$.
6. $-(ax+3b)/2x^{5/2}$.
7. $5x^4+9x^2-4$.
8. $6(3x+2)$.
9. $9(3x+2)^2$.
10. $3n(3x+2)^{n-1}$.
11. $(5x^2-9x+2)/2\sqrt{x}$.
12. $2(a-bx+cx^2)(2cx-b)$.
13. $3(a-bx+cx^2)^2(2cx-b)$.
14. $n(2cx-b)(a-bx+cx^2)^{n-1}$.

15. $2xy\dfrac{dy}{dx}+y^2;\quad 2xy^3\left(2x\dfrac{dy}{dx}+y\right);\quad x^2y\left(2x\dfrac{dy}{dx}+3y\right)$

16. $x^{n-1}\left(x\dfrac{dy}{dx}+ny\right);\quad y^{n-1}\left(nx\dfrac{dy}{dx}+y\right)$.

17. $x^{n-1}y^2\left(3x\dfrac{dy}{dx}+ny\right);\quad x^2y^{n-1}\left(nx\dfrac{dy}{dx}+3y\right)$

18. $x^{n-1}y^{n-1}\left(x\dfrac{dy}{dx}+y\right)$

$2x+y+(x+2y)\dfrac{dy}{dx}$.

20. $4x^3+2xy^2+(2x^2y+4y^3)\dfrac{dy}{dx}$.

21. $3ax^2+2bxy+cy^2+(bx^2+2cxy+3dy^2)\dfrac{dy}{dx}$.

22. $2a(ay+b)\dfrac{dy}{dx}$

23. $3a(ay+b)^2\dfrac{dy}{dx}$.

24. $na(ay+b)^{n-1}\dfrac{dy}{dx}$.

Examples X, p. 78.

$-29/(5x-3)^2$.

2. $-3/(2-x)^2$.

$(a^2-b^2)/(bx+a)^2$.

4. $-16x/(x^2-4)^2$.

$4(x^2-4)/(x^2+2x+4)^2$.

6. $-2nx^{n-1}/(x^n-1)^2$.

7. $8(1-x^2)/(x^2+1)^2$.

8. $(1+x)/\{2\sqrt{x}(1-x)^2\}$.

9. $\frac{1}{2}(2+\sqrt{x})/(1+\sqrt{x})^2$.

10. $1/\{\sqrt{x}(1-\sqrt{x})^2\}$.

11. $2b(c-ax^2)/(ax^2-bx+c)^2$.

12. $4(x-1)/(x^2+1)^2$.

13. $(3x^2-2x+2)/(3x-1)^2$.

14. $6(2x-1)/(x^2-x)^2$.

15. $-2(2x^2+6x-3)/(x^2-5x+6)^2$.

16. $\left(y-x\frac{dy}{dx}\right.$

17. $\left(x\frac{dy}{dx}-y\right)\Big/x^2$.

18. $2x\left(y-x\frac{dy}{dx}\right)\Big/y^3$.

19. $y^2\left(3x\frac{dy}{dx}-2y\right)\Big/x^3$.

20. $nx^{n-1}\left(y-x\frac{dy}{dx}\right)\Big/y^{n+1}$.

21. $ny^{n-1}\left(x\frac{dy}{dx}-y\right)\Big/x^{n+1}$.

Examples XI, p. 83.

1. $24(4x-5)^5$; $2/\sqrt{(4x-5)}$; $-8/(4x-5)^3$; $4n(4x-5)^{n-1}$; $-4/(4x-5)^2$; $-2/\sqrt{(4x-5)^3}$; $-4/\{n\sqrt[n]{(4x-5)^{n+1}}\}$.

2. $-42(3-7x)^5$; $-\frac{7}{2}/\sqrt{(3-7x)}$; $14/(3-7x)^3$; $-7n(3-7x)^{n-1}$; $7/(3-7x)^2$; $\frac{7}{2}/\sqrt{(3-7x)^3}$; $7/\{n\sqrt[n]{(3-7x)^{n+1}}\}$.

3. $12x(x^2-1)^5$; $x/\sqrt{(x^2-1)}$; $-4x/(x^2-1)^3$; $2nx(x^2-1)^{n-1}$; $-2x/(x^2-1)^2$; $-x/\sqrt{(x^2-1)^3}$; $-2x/\{n\sqrt[n]{(x^2-1)^{n+1}}\}$.

4. $-6(a-x)^5$; $-\frac{1}{2}/\sqrt{(a-x)}$; $2/(a-x)^3$; $-n(a-x)^{n-1}$; $1/(a-x)^2$; $\frac{1}{2}/\sqrt{(a-x)^3}$; $1/\{n\sqrt[n]{(a-x)^{n+1}}\}$.

5. $6nx^{n-1}(x^n-a^n)^5$; $\frac{1}{2}nx^{n-1}/\sqrt{(x^n-a^n)}$; $-2nx^{n-1}/(x^n-a^n)^3$; $n^2x^{n-1}(x^n-a^n)^{n-1}$; $-nx^{n-1}/(x^n-a^n)^2$; $-\frac{1}{2}nx^{n-1}/\sqrt{(x^n-a^n)^3}$; $-x^{n-1}/\sqrt[n]{(x^n-a^n)^{n+1}}$.

$6(ax^2+bx+c)^5(2ax+b)$; $\frac{1}{2}(2ax+b)/\sqrt{(ax^2+bx+c)}$; $-2(2ax+b)/(ax^2+bx+c)^3$; $n(ax^2+bx+c)^{n-1}(2ax+b)$; $-(2ax+b)/(ax^2+bx+c)^2$; $-\frac{1}{2}(2ax+b)/\sqrt{(ax^2+bx+c)^3}$; $-(2ax+b)/\{n\sqrt[n]{(ax^2+bx+c)^{n+1}}\}$

7. $2nx/(a^2-x^2)^{n+1}$.

8. $\frac{4}{3}x/\sqrt[3]{(a^2-x^2)^5}$.

9. $x/\sqrt{(a^2-x^2)^3}$.

10. $\frac{2}{5}x/\sqrt[5]{(a^2-x^2)^6}$.

11. $-x^2/(a^3-x^3)^{2/3}$.

12. $-\frac{9}{4}x^2/\sqrt[4]{(a^3-x^3)}$.

13. $6(x-1)^5/x^7$.

14. $\dfrac{1}{6x^2}\sqrt[6]{\left(\dfrac{x}{x-1}\right)}$

15. $\dfrac{nx^{n-1}(1-x^2)}{(x^2+1)^{n+1}}$.

16. $\frac{1}{2}(1-x^2)/\sqrt{\{x(1+x^2)^3\}}$.

17. $\frac{1}{2}a/\sqrt{\{x(a-x)^3\}}$.

18. $nax^{n-1}/(a-x)^{n+1}$.

19. $(3x+1)/\sqrt{(2x+1)}$.

20. $\frac{1}{2}x(4-5x)/\sqrt{(1-x)}$.

21. $(x+1)/\sqrt{(2x+1)^3}$.

22. $-(x+1)/\{x^2\sqrt{(2x+1)}\}$.

23. $\frac{1}{2}x(4-3x)/\sqrt{(1-x)^3}$.

24. $\frac{1}{2}(3x-4)/\{x^3\sqrt{(1-x)}\}$.

25. $x(2a^2-3x^2)/\sqrt{(a^2-x^2)}$.

26. $x(2a^2-x^2)/\sqrt{(a^2-x^2)^3}$.

27. $(2a^2-x^2)/\{x^3\sqrt{(x^2-a^2)}\}$.

28. $(a-x)^{n-1}\{a-(n+1)x\}$.

29. $-(a-x)^{n-1}\{a+(n-1)x\}/x^2$.

30. $\{a+(n-1)x\}/(a-x)^{n+1}$.

31. $-(a-x)(b-x)^2(3a+2b-5x)$.

32. $(a-x)(3a-2b-x)/(b-x)^4$.

33. $-(a-x)^{n-1}(b-x)^{m-1}\{ma+nb-(m+n)x\}$.

34. $(a-x)^{n-1}\{ma-nb-(m-n)x\}/(b-x)^{m+1}$.

35. $1/(3y^2+6y)$.

36. $\frac{1}{5}(3+2y)^2$.

37. $(y+a)^2/(y^2+2ay)$.

38. $-(y+a)^2/(y^2+2ay+ab)$.

41. $-\frac{4}{21}$.

42. $\frac{26}{9}$.

Examples XII, p. 85.

1. $-x^2/y^2$.
2. $-\sqrt{(y/x)}$.
3. $-(x/y)^{n-1}$.
4. $-(2x+y)/(2y+x)$.
5. $-(3x^2+2xy+y^2)/(3y^2+2xy+x^2)$.
6. $-(2xy+y^2)/(2xy+x^2)$.
7. $-my/nx$.
8. ny/mx.
9. $(ax-x-y)/(x+y-by)$.
10. $(ay-(x+y)^2)/((x+\ldots$
11. $\{a^2x-2x(x^2+y^2)\}/\{a^2y+2y(x^2+y^2)\}$.
12. $-(ax+by)/(bx+ay)$.
13. $-(3ax^2+2bxy+cy^2)/(bx^2+2cxy+3dy^2)$.
14. $-\sqrt[3]{(x/y)}$.
15. $-(x/y)^{p/q-1}$.
16. $-x^{n-1}(2x^n+y^n)/\{y^{n-1}(2y^n+x^n)\}$.
17. $-(3y^2+4xy)/(6x^2+\ldots$
18. $-(ax+hy+g)/(hx+by+f)$.
19. $-(1+y)/(1+\ldots$
20. $(a+y-2x)/(2y-x-a)$.
21. (i) $-v/p$. (ii) $-v/\gamma p$. (iii) $v^3(b-v)/(pv^3-av+2ab)$.

Examples XIII, p. 89.

1. ·3 in.
2. (i) 4π c. in. (ii) $·8\pi$ sq. in.
3. ·446 sq. in.
4. $\frac{1}{125}$ in. too large.
5. $2\frac{263}{270}$ ft.
6. $\frac{1}{80}$ yd.
7. (i) $3/d$. (ii) $2/d$.
8. (i) 8·67 sq. in. (ii) ·433 in.
9. $a(a-b\cos C)/c^2$.
10. 21·42 in.
11. $\frac{5}{8}$ in.
12. (i) 14·14. (ii) −·143.
13. 9.
14. (i) ·644. (ii) 2.
15. $+\frac{1}{2}$ per cent.; 432 secs.
16. +·155 per cent.; 939 secs.
17. $R_0(a+2b\theta)\delta\theta$.
18. $10k/(1+25k^2)$.
19. ·0003 yd.
20. (i) 56. (ii) 136·5.

Examples XIV, p. 96.

1. $5\cos 5x$; $\frac{1}{3}\cos\frac{1}{3}x$; $n\cos(nx-\alpha)$; $-a\sin ax$; $-(1/p)\sin(x/p)$; $2\sin(\frac{1}{5}\pi-2x)$.
2. $3\sec^2 3x$; $\sec^2(x+\alpha)$.
3. $-m\operatorname{cosec}^2 mx$; $2\operatorname{cosec}^2(\alpha-2x)$.
4. $m\sec mx\tan mx$; $\sec(\frac{1}{4}\pi+x)\tan(\frac{1}{4}\pi+x)$.
5. $-m\operatorname{cosec} mx\cot mx$; $\frac{1}{2}\operatorname{cosec}(\beta-\frac{1}{2}x)\cot(\beta-\frac{1}{2}x)$.
6. $3\sin^2 x\cos x$; $n\sin^{n-1}x\cos x$.
7. $-5\cos^4 x\sin x$; $-m\cos^{m-1}x\sin x$.
8. $\frac{1}{2}\cos x/\sqrt{\sin x}$.
9. $-2\operatorname{cosec}^2 x\cot x$.
10. $-\frac{1}{2}\cot x\sqrt{\operatorname{cosec} x}$.
11. $-\frac{1}{3}\sin x/\sqrt[3]{\cos^2 x}$.
12. $4\sec^4 x\tan x$.
13. $-n\cot^{n-1}x\operatorname{cosec}^2 x$.
14. $2\sin 4x$.
15. $-3a\cos^2 ax\sin ax$.
16. $3n\tan^{n-1}3x\sec^2 3x$.
17. $-\cot\frac{1}{2}x\operatorname{cosec}^2\frac{1}{2}x$.
18. $x^3(3x\cos 3x+4\sin 3x)$.
19. $x^{n-1}(n\cos x-x\sin x)$.
20. $(2x\sec^2 x+\tan x)/2\sqrt{x}$.
21. $(2x\cos 2x-3\sin 2x)/x^4$.
22. $3\cos 3x\cos 4x-4\sin 3x\sin 4x$.
23. $m\cos mx\cos nx-n\sin mx\sin nx$.
24. $\sin x(1+\sec^2 x)$.
25. $\sin^2 x(2+\sec^2 x)$.
26. $2b\cos x(a+b\sin x)$.
27. $-2\sin x/\sqrt{(3+4\cos x)}$.
28. $\cos^3 x$.
29. $\sec^4 x$.

30. $7\cos x/(4+3\sin x)^2$.
31. $2ab\sin x/(a+b\cos x)^2$.
32. $2/(1-\sin 2x)$.
33. $\sin x\cos x\,(2+\sin x)/(1+\sin x)^2$.
34. $2\cos^2 x\cos 2x-\sin^2 2x$.
35. $\frac{1}{2}\sin 4x$.
36. $(1+\sin^2 x)\sec^3 x$.
37. $\sin^{m-1}x\cos^{n-1}x\,(m\cos^2 x-n\sin^2 x)$.
38. $\sin^{m-1}ax\cos^{n-1}bx\,(ma\cos ax\cos bx-nb\sin ax\sin bx)$.
39. $x^{n-1}\tan^{m-1}ax\,(amx\sec^2 ax+n\tan ax)$.
40. $-(m\cos mx)/(n\sin ny)$.
41. $\sin 2x/\sin 2y$.
42. $\cot x\cot y$.
43. $y/(x^2+y^2+x)$.
44. ·49975.
45. ·8657.
46. 1·0006.
47. $-$1·0017.
48. ·9310.
49. $-$·9996.
50. 3·9919.
51. ·24924.
52. ·76 ft.
53. ·0094.
54. ·2 sq. ft.
55. ·1.
56. (i) ·09°. (ii) ·207.
57. $\cot A\,\delta A$.
58. (i) ·017 C. (ii) 1·7.
59. 7·82.
60. 19·42.

Examples XV, p. 97.

1. $5(x-3)^4$.
2. $-8(7-x)^7$.
3. $-\frac{3}{2}x^2/\sqrt{(1-x^3)}$.
4. $2x(1-x)(1-2x)$.
5. $-\frac{1}{2}(2x-3)/\sqrt{(x^2-3x-2)^3}$.
6. $28/(5-7x)^5$.
7. $-\frac{2}{3}x/\sqrt[3]{(x^2+1)^4}$.
8. $(4-2x^2)/\sqrt{(4-x^2)}$.
9. $\frac{1}{2}(4-3x^2)/\sqrt{(4x-x^3)}$.
10. $-4/\{x^2\sqrt{(4-x^2)}\}$.
11. $4/\sqrt{(4-x^2)^3}$.
12. $\frac{1}{2}(4+x^2)/\sqrt{\{x(4-x^2)^3\}}$.
13. $-\frac{1}{2}(4+x^2)/\sqrt{\{x^3(4-x^2)\}}$.
14. $\sin 2(x-\alpha)$.
15. $-\frac{1}{2}n\cos^{n-1}\frac{1}{2}x\sin\frac{1}{2}x$.
16. $-\tan^2 x$.
17. $x\sec^2 x+\tan x$.
18. $\cot x-x\operatorname{cosec}^2 x$.
19. $(x\sec^2 x-\tan x)/x^2$.
20. $\cos x(1-3\sin^2 x)$.
21. $\frac{1}{2}(2x\cos x+\sin x)/\sqrt{x}$.
22. $(\sin x-2x\cos x)/(2\sqrt{x}\sin^2 x)$.
23. $\frac{1}{2}(2x\cos x-\sin x)/\sqrt{x^3}$.
24. $\frac{1}{2}\cos x/\sqrt{\sin x}$.
25. $(\cos\sqrt{x})/2\sqrt{x}$.
26. $\frac{1}{2}(x\cos x+\sin x)/\sqrt{(x\sin x)}$.
27. $\frac{1}{2}\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x}$.
28. $\frac{1}{2}(x\cos x-\sin x)/\sqrt{(x^3\sin x)}$.
29. $(\sin\sqrt{x}-\frac{1}{2}\sqrt{x}\cos\sqrt{x})/\sin^2\sqrt{x}$.
30. $\frac{1}{2}(\sin x-x\cos x)/\sqrt{(x\sin^3 x)}$.
31. $(\frac{1}{2}\sqrt{x}\cos\sqrt{x}-\sin\sqrt{x})/x^2$.
32. $\frac{1}{2}(x\cos x+2\sin x)/\sqrt{\sin x}$.
33. $\frac{1}{2}(x\cos x-2\sin x)/(x^2\sqrt{\sin x})$.
34. $\frac{1}{2}(2\sin x-x\cos x)/\sqrt{\sin^3 x}$.
35. $\frac{1}{2}(\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x})/\sqrt{x}$.
36. $(\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x})/(2\sqrt{x}\sin^2\sqrt{x})$.
37. $\frac{1}{2}(\sqrt{x}\cos\sqrt{x}-\sin\sqrt{x})/\sqrt{x^3}$.
38. $(2-3x)/\sqrt{(1-x)}$.
39. $(\sec x\tan x)/a$.
40. $\{\sec(x/a)\tan(x/a)\}/a$.
41. $-\{a\sec(a/x)\tan(a/x)\}/x^2$.
42. $x^{m-1}(a-x)^{n-1}(ma-mx-nx)$.
43. $x^{m-1}(ma-mx+nx)/(a-x)^{n+1}$.
44. $(a-x)^{n-1}(mx-nx-ma)/x^{m+1}$.
45. $-1/\{n\sqrt[n]{(a-x)^{n-1}}\}$.
46. $-\frac{1}{2}nx^{n-1}/\sqrt{(a^n-x^n)}$.
47. $-\frac{1}{2}n\sqrt{(a-x)^{n-2}}$.
48. $-3\sin^2(\alpha-x)\cos(\alpha-x)$.
49. $-3\cos 3(\alpha-x)$.
50. $-3(\alpha-x)^2\cos(\alpha-x)^3$.
51. $2x(\cos 2x-x\sin 2x)$.
52. $2x\cos 2x(\cos 2x-2x\sin 2x)$.
53. $2x(\cos 2x+x\sin 2x)/\cos^2 2x$.
54. $-\cos 2x(4x\sin 2x+\cos 2x)/x^2$.
55. $-2\cos 2x(\cos 2x+2x\sin 2x)/x^3$.
56. $\cos 2x(\cos 2x-4x\sin 2x)$.
57. $-2(\cos 2x+x\sin 2x)/x^3$.
58. $2x(\cos 2x+2x\sin 2x)/\cos^3 2x$.
59. $(\cos 2x+4x\sin 2x)/\cos^3 2x$.
60. $\sin 4x/\sqrt{(1+\sin^2 2x)}$.

61. $2n \sin 2x (1 - \cos 2x)^{n-1}$.
62. $mb \sin 2x (a + b \sin^2 x)^{m-1}$.
63. $-\sin nx / \sqrt[n]{(1 + \cos nx)^{n-1}}$.
64. $-n \sin^{n-1} x \cos x / (1 + \sin^n x)^2$.
65. $m \cos (x/b) \{a + b \sin (x/b)\}^{m-1}$.
66. $4 \cos 2x / (1 - \sin 2x)^2$.
67. $4 \sec^2 x \tan x$.
68. $8 \sec^2 2x \tan 2x$.
69. $-\{x + \sqrt{(a^2 - x^2)}\} / \{a^2 \sqrt{(a^2 - x^2)}\}$.
70. $1/(1 - \sin x)$.
71. $2a^2 x / \sqrt{\{(a^2 + x^2)(a^2 - x^2)^3\}}$.
72. $8(x^2 - 2x - 2)/(x^2 + 5x - 3)^2$.
73. $-a/\sqrt{\{(a - x)(a + x)^3\}}$.
74. $2(x^2 + 6)/(x^2 + x - 6)^2$.
75. $ax/(2ax - x^2)^{3/2}$.
76. $3 \sin^2 x \cos 4x$.
77. $3 \cos^2 x \cos 4x$.
78. $3 \cos 6x$.
79. $\frac{3}{4} \sin^2 2x \cos 2x$.
80. $3 \sin^2 x \cos 2x / \cos^2 3x$.
81. $-3 \cos 2x / \sin^4 x$.
82. $3 \cos 2x / \cos^4 x$.
83. $-3 \cos^2 x \cos 2x / \sin^2 3x$.
84. $3 \sec^2 3x$.
85. $-3 \operatorname{cosec}^2 3x$.
86. $3 \tan^2 x \sec^2 x$.
87. $-3 \cot^2 x \operatorname{cosec}^2 x$.
88. $3 \cos^2 3x (1 - 4 \sin^2 3x)$.
89. $3 \sin^2 3x (4 \cos^2 3x - 1)$.
90. $\frac{9}{4} \sin^2 6x \cos 6x$.
91. $3(1 + 2 \sin^2 3x)/\cos^4 3x$.
92. $-3 \cot^2 3x (1 + 2 \sin^2 3x)$.
93. $-3(1 + 2 \cos^2 3x)/\sin^4 3x$.
94. $3 \tan^2 3x (1 + 2 \cos^2 3x)$.
95. $9 \tan^2 3x \sec^2 3x$.
96. $-9 \cot^2 3x \operatorname{cosec}^2 3x$.
97. $3 \sin^2 x \cos^2 3x (\cos 3x \cos x - 3 \sin 3x \sin x)$.
98. $3 \sin^2 x (\cos x \cos 3x + 3 \sin x \sin 3x)/\cos^4 3x$.
99. $-3 \cos^2 3x (\cos x \cos 3x + 3 \sin x \sin 3x)/\sin^4 x$.
100. $3 \cos^2 x \sin^2 3x (3 \cos x \cos 3x - \sin x \sin 3x)$.
101. $-3 \cos^2 x (\sin x \sin 3x + 3 \cos x \cos 3x)/\sin^4 3x$.
102. $3 \sin^2 3x (\sin x \sin 3x + 3 \cos x \cos 3x)/\cos^4 x$.
103. $3 \sin^2 x \sin 4x$.
104. $3 \sin^2 x \sin 2x / \sin^2 3x$.
105. $-6 \cot x \operatorname{cosec}^2 x$.
106. $-3 \cos^2 x \sin 4x$.
107. $3 \cos^2 x \sin 2x / \cos^2 3x$.
108. $-6 \tan x \sec^2 x$.
109. $\sin^2 3x (9 \sin x \cos 3x + \sin 3x \cos x)$.
110. $(\sin 3x \cos x - 9 \cos 3x \sin x)/\sin^4 3x$.
111. $\sin^2 3x (9 \sin x \cos 3x - \sin 3x \cos x)/\sin^2 x$.
112. $-\cos^2 3x (9 \cos x \sin 3x + \sin x \cos 3x)$.
113. $(9 \cos x \sin 3x - \cos 3x \sin x)/\cos^4 3x$.
114. $\cos^2 3x (\cos 3x \sin x - 9 \cos x \sin 3x)/\cos^2 x$.
115. $\cos^2 3x (\cos 3x \cos x - 9 \sin 3x \sin x)$.
116. $(\cos 3x \cos x + 9 \sin 3x \sin x)/\cos^4 3x$.
117. $-\cos^2 3x (\cos 3x \cos x + 9 \sin 3x \sin x)/\sin^2 x$.
118. $\sin^2 3x (9 \cos x \cos 3x - \sin x \sin 3x)$.
119. $-(\sin x \sin 3x + 9 \cos x \cos 3x)/\sin^4 3x$.
120. $\sin^2 3x (\sin x \sin 3x + 9 \cos x \cos 3x)/\cos^2 x$.
121. $x(2a^2 - 3x^2)/\sqrt{(a^2 - x^2)}$.
122. $(x^2 - 2a^2)/\{x^3 \sqrt{(a^2 - x^2)}\}$.
123. $x(2a^2 - x^2)/(a^2 - x^2)^{3/2}$.
124. $x^2(3a^2 - 4x^2)/\sqrt{(a^2 - x^2)}$.
125. $x^2(3a^2 - 2x^2)/(a^2 - x^2)^{3/2}$.
126. $x^2(a^2 - x^2)^{n-1}(3a^2 - 3x^2 - 2nx^2)$.
127. $nx^{n-1}(a^2 - x^2)^{n-1}(a^2 - 3x^2)$.
128. $2x(1 + x)/(1 + 2x)^2$.
129. $-2(1 + x)/(1 + 2x)^3$.
130. $-2(1 + x)(2 + x)/(1 + 2x)^4$.
131. $-(1 - x)^3(7 + x)/(1 + x)^4$.
132. $(1 - x)^2(x - 4)/(2 - x)^3$.
133. $(6x - 8)/(1 + 3x)^3$.
134. $(x - 7a)(a - x)^2/(a + x)^5$.
135. $2x(3 + x^2)/(1 - x^2)^3$.
136. $2x(1 + x^2)(3 - x^2)/(1 - x^2)^2$.

137. $2x(x^2-3a^2)/(a^2+x^2)^3$. 138. $-8a^2x(a^2-x^2)/(a^2+x^2)^3$.
139. $n(a-b)(a-x)^{n-1}/(b-x)^{n+1}$. 140. $n(2x-a-b)(a-x)^{n-1}(b-x)^{n-1}$.
141. $2nx(2x^2-a^2-b^2)(a^2-x^2)^{n-1}(b^2-x^2)^{n-1}$.
142. $(3-6x+4x^2)/\sqrt{(3-4x+2x^2)}$. 143. $(3-10x+8x^2)\sqrt{(3-4x+2x^2)}$.
144. $(2x-3)/\{x^2\sqrt{(3-4x+2x^2)}\}$. 145. $-1/\sqrt{(2x+x^2)^3}$.
146. $2x(3-5x+3x^2)/\sqrt{(3-4x+2x^2)}$.
147. $2x(3-3x+x^2)/(3-4x+2x^2)^{3/2}$. 148. $n(x\sin x)^{n-1}(x\cos x+\sin x)$.
149. $nx^{n-1}\sin nx(2x\cos nx+\sin nx)$. 150. $n(x\sin nx)^{n-1}(nx\cos nx+\sin nx)$.

Examples XVI, p. 105.

1. $85°46'$. 2. $106°42'$. 3. $82°53'$. 4. $40°54'$.
5. $8x-y=13$; $x+8y=91$. 6. $4x-5y+12=0$; $5x+4y=26$.
7. $2x+y+10=0$; $x-2y=0$. 8. $2x+3y=30$; $3x-2y=19$.
9. $11x+3y=36$; $3x-11y+2=0$. 10. $y+1=0$; $x=2$.
11. $Xx/a^2-Yy/b^2=1$. 12. $Xx+Yy+g(X+x)+f(Y+y)+c=0$.
13. $(2, -12)$, $(-2, 20)$. 14. $(\pm a, \pm\frac{1}{2}a)$.
15. $(\frac{5}{4}a, \frac{1}{2}a)$. 16. $(\pm 4, \mp 3)$.
18. $15\frac{1}{4}°$. 19. They touch at $(2, 4)$.
20. $48°12'$. 22. $X/x^{1/3}+Y/y^{1/3}=a^{2/3}$; intercept $=a$.
25. $OT=(n-1)\,TN$. 26. $mX/x+nY/y=m+n$.
28. $aXx+h(Xy+Yx)+bYy+g(X+x)+f(Y+y)+c=0$.
29. $2Xy^3-Yx(x^2+3y^2)+ax^3=0$.
30. Touches OX. Bisects $\angle XOY$. Touches OY.
32. Curve bisects $\angle XOY$. 34. $\tan^{-1}\left\{\left(x\frac{dy}{dx}-y\right)\Big/\left(x+y\frac{dy}{dx}\right)\right\}$.

Examples XVII, p. 113.

1. $6\sqrt{10}$; $2\sqrt{10}$; 18; 2.
2. $\frac{1}{2}\sqrt{(a^2+\frac{4}{3}b^2)}$; $\frac{1}{4}a\sqrt{(3a^2+4b^2)}/b$; $b/\sqrt{3}$; $\sqrt{3}\,a^2/4b$.
3. 10; $-7\frac{1}{2}$; -8; $-4\frac{1}{2}$. 4. $-\sqrt{5}$; $\frac{1}{2}\sqrt{5}$; 2; $\frac{1}{2}$. 8. $y^2/(n+1)x$.
11. $4a$. 12. a^2y^2/b^2x; b^2x/a^2. 14. $x\sec\theta+y\operatorname{cosec}\theta=a$.
15. $a\sin^2\theta$; $a\sin^3\theta\sec\theta$; $a\sin^2\theta\cos\theta$; $a\sin^4\theta\sec\theta$.
16. (i) $x-y=a(\frac{1}{2}\pi-2)$. (ii) $x\cot\frac{1}{2}\theta-y=a(\theta\cot\frac{1}{2}\theta-2)$.
17. a; a. 18. $(b\operatorname{cosec}\theta)/a$. 19. $(x\cos\theta)/a+(y\sin\theta)/b=1$.
20. $m=\cot\psi$, where ψ is inclination of tangent to OX.
21. $y\tan\frac{3}{2}\theta$; $y\cot\frac{3}{2}\theta$.
22. $y\Big/\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$; $y\frac{dy}{dx}\Big/\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$.
23. $2c^2/\sqrt{(x^2+y^2)}$; $OY\,.\,OP=2c^2$.

Examples XVIII, p. 122.

1. Min. $(3, -1)$. 2. Max. $(-1, 19)$.
3. Max. $(-2, 21)$; min. $(2, -11)$. 4. Max. $(2, 28)$; min. $(3, 27)$.
5. None. 6. Max. $(1, 18)$; min. $(5, -14)$.

7. Pt. of inflexion $(1, 0)$.
8. Max. $(1, 43)$; min. $(0, 40)$, $(5, -85)$.
9. Max. $(2, 4)$; min. $(1, 3)$ and $(3, 3)$.
10. Max. $(1, 0)$; min. $(3, -28)$; pt. of inflexion $(0, -1)$.
11. Min. $(\frac{3}{2}, -\frac{51}{16})$.
12. Max. $(1, 0)$; min. $(\frac{5}{3}, -\frac{4}{27})$.
13. Max. $(\frac{8}{5}, \frac{108}{3125})$; min. $(2, 0)$; pt. of inflexion $(1, 0)$.
14. Max. $(5, \frac{2}{3})$; min. $(1, 0)$.
15. Max. $(-2, 3)$; min. $(2, \frac{1}{3})$.
16. Max. $(0, 1)$.
17. Min. $(1, 27)$; pt. of inflexion $(4, 0)$.
18. Min. $(-4, \frac{1}{2})$; max. $(4, \frac{1}{18})$.
19. Max. $(3, \frac{1}{3})$.
20. Max. $[-\sqrt{(ab)}, (\sqrt{a}-\sqrt{b})^2]$; min. $[\sqrt{(ab)}, (\sqrt{a}+\sqrt{b})^2]$.
21. Max. $(4, \sqrt[3]{4})$.
22. Min. $(0, 0)$; max. $(\frac{3}{4}a, \frac{3}{4}\sqrt{3}\,a^2)$.
23. Min. $(\frac{2}{3}a, \frac{9}{4}a)$; max. $(2a, \frac{1}{4}a)$.
24. Min. $(a, \sqrt{2})$.
25. Max. $(\frac{1}{2}, -\frac{5}{4}\sqrt{5})$; min. $(1, -2\sqrt{2})$.
26. Max. $\sqrt{2}$, when $x = (2n+\frac{1}{4})\pi$; min. $-\sqrt{2}$, when $x = (2n+\frac{5}{4})\pi$.
27. $\begin{cases}\text{If } a > b, \text{ max. } a \text{ when } x = (n+\frac{1}{2})\pi; \text{ min. } b \text{ when } x = n\pi. \\ \text{If } a < b, \text{ max. and min. interchange.}\end{cases}$
28. Max. when $x = (n+\frac{1}{6})\pi$; min. when $x = (n-\frac{1}{6})\pi$.
29. None.
30. Max. $\frac{9}{8}$ when $x = \sin^{-1}\frac{1}{4}$; min. 0 and -2 when $x = (n+\frac{1}{2})\pi$.
31. Min. -1 when $x = (2n+\frac{1}{4})\pi$.

If a and b are $+$, min. $2\sqrt{(ab)}$ when $\tan x = +\sqrt{(a/b)}$; max. $-2\sqrt{(ab)}$ when $\tan x = -\sqrt{(a/b)}$. If a and b are both $-$, interchange results.

Max. $\frac{3}{16}\sqrt{3}$ when $x = (n+\frac{1}{3})\pi$; min. $-\frac{3}{16}\sqrt{3}$ when $x = (n-\frac{1}{3})\pi$; pts. of inflexion $(n\pi, 0)$.

34. Max. when $x = \frac{1}{2}(\alpha+\beta)+(n+\frac{1}{4})\pi$; min. when $x = \frac{1}{2}(\alpha+\beta)+(n+\frac{3}{4})\pi$.
35. Max. $\frac{1}{4}\sqrt{2}$ when $x = (2n+\frac{1}{4})\pi$; min. $-\frac{1}{4}\sqrt{2}$ when $x = (2n+\frac{5}{4})\pi$.
36. Min. $-3\sqrt{3}$ when $x = (2n+\frac{1}{3})\pi$; max. $3\sqrt{3}$ when $x = (2n-\frac{1}{3})\pi$.
39. $x = \frac{1}{2}a$.
40. $\sqrt{A/C}$.
42. $34\,C$.
43. $x = \frac{1}{2}(a-b)$.
44. $x = -\frac{3}{2}$.
45. $x = \pm a/\sqrt{2}$.
46. 1.
47. Max. when $x = 1$.
48. $\{\frac{1}{2}(\gamma+1)\}^{\gamma/(1-\gamma)}$; ·067.
49. $\sqrt{(nR/r)}$.
50. $-100\,l\alpha/(lx-x^2)$; $x = \frac{1}{2}l$.

Examples XIX, p. 128.

1. 20, 20.
2. 125, 25.
3. $\frac{2}{5}a$, $\frac{3}{5}a$.
4. Max. sum $= -2$; min. $+2$.
5. Max., $\frac{4}{32}$; min. $-\frac{4}{32}$.
6. $\frac{1}{2}$ and $-\frac{1}{2}$.
7. When it is a square. (i) Min. perimeter, 20 ft., (ii) min. diagonal, $5\sqrt{2}$ ft.
9. Max. perimeter $= 2\sqrt{2}\,a$.
10. Square of area $\frac{1}{2}(a+b)^2$.
11. Height $= \sqrt{2} \times$ radius $= \frac{2}{3}\sqrt{3} \times$ radius of sphere.
12. $r = \frac{1}{2}h = 5$ ft.
13. $r = h = 3·73$ ft.
14. $h = 2r = \sqrt{2} \times$ radius of sphere.
15. Max. area $= \frac{1}{2}$ area of triangle. Perimeter continually increases from $2a$ to $2b$ as corner moves along hypotenuse.
16. (i) Max. vol. $= \frac{4}{9}$ vol. of cone. (ii) When $h = \frac{1}{2}$ height of cone. (iii) When $h/r = \cot\alpha - 2$, where α is semi-vertical angle of cone. No max. if $\alpha > 26° 34'$.
17. Height $= 4 \times$ radius of sphere.
18. Max. area $= \frac{1}{2}$ area of triangle.
19. When equilateral.

20. Height $= \sqrt{2} \times$ radius of base $= \frac{4}{3}$ radius of sphere.

23. 2 ft. long, 4 ft. girth.

24. When $h = 1{\cdot}05\ldots r$.

25. Max. vol. $= \frac{4}{27}$ vol. of given cone.

26. $2\,ab$.

27. $(\sqrt{a}+\sqrt{b})^2$.

28. $(a^{2/3}+b^{2/3})^{3/2}$.

29. $4\,ab$.

30. Max. area $= r^2$, when angle of sector $= 2$ radians.

31. When isosceles.

32. 1 ft.

33. 6 in.

34. 20.

35. 34·64 ft.

36. 22·06 in.

37. $\frac{3}{2}a$.

38. 1 mile from nearest point of road; about 1·3 minutes.

39. 10·6 ft.

40. $\frac{2}{3}$ way from brighter light.

41. (latus rectum)$^2/6\sqrt{3}$.

43. $9\sqrt{3}$ in.

44. $32\sqrt{3}$ sq. in.

46. πa^3, if $4a$ be latus rectum of parabola.

47. Equilateral triangle of area $\frac{3}{4}\sqrt{3}\,r^2$.

48. When the line is parallel to AB.

49. 3·749 ft.

50. 24.

51. (i) $9\sqrt{3}$. (ii) $9\sqrt{3}$.

52. $2/\sqrt{5}$.

54. After $2\frac{4}{19}$ minutes; min. distance, 1·192 miles.

55. The point half-way between the feet of the perpendiculars from the given points to the line.

56. 60°.

57. Breadth, 6·928 in.

58. ·414 m.

59. $\sin^{-1}(m/2M)$.

60. $\frac{3}{8}h$.

61. 41 ft. 8 in.

62. When side of square $=$ ·14 l.

63. $\sqrt{(2aW/w)}$.

64. $(av \sim bu)\sin\theta/\sqrt{(u^2+v^2-2\,uv\cos\theta)}$.

66. $\frac{1}{2}ab$.

67. $\frac{1}{18}\sqrt{3} \times$ vol. of sphere.

68. When height $= 4$ diameter.

69. $6\sqrt{3}\,a$.

71. 53° 8′ W. of N. or S.

72. 955·2 sq. yds.

Examples XX, p. 134.

1. $10x^9$; $90x^8$; $720x^7$; $\dfrac{10!}{(10-n)!}x^{10-n}$ if $n<10$, $10!$ if $n=10$, 0 if $n>10$.

2. $-\dfrac{b}{x^2}$; $\dfrac{2b}{x^3}$; $-\dfrac{6b}{x^4}$; $\dfrac{(-1)^n n!\,b}{x^{n+1}}$.

3. $-\dfrac{3}{x^4}$; $\dfrac{12}{x^5}$; $-\dfrac{60}{x^6}$; $\dfrac{(-1)^n(n+2)!}{2\,x^{n+3}}$.

4. $-\dfrac{1}{2\,x^{3/2}}$; $\dfrac{3}{4\,x^{5/2}}$; $\dfrac{15}{8\,x^{7/2}}$; $(-1)^n\dfrac{1.3.5\ldots(2n-1)}{2^n x^{n+\frac{1}{2}}}$.

5. $\dfrac{1}{2\,x^{1/2}}$; $-\dfrac{}{4\,x^{3/2}}$; $\dfrac{}{8\,x^{5/2}}$; $(-1)^{n-1}\dfrac{1.3.5\ldots(2n-3)}{2^n x^{n-\frac{1}{2}}}$.

6. $10a(ax+b)^9$; $90a^2(ax+b)^8$; $720a^3(ax+b)^7$; $10!\ a^n(ax+b)^{10-n}/(10-n)!$ if $n<10$, $10!\ a^{10}$ if $n=10$, 0 if $n>10$.

7. $-\dfrac{2}{(2x+1)^2}$; $\dfrac{8}{(2x+1)^3}$; $-\dfrac{48}{(2x+1)^4}$; $(-1)^n\dfrac{2^n.n!}{(2x+1)^{n+1}}$.

8. $\dfrac{1}{(1-x)^2}$; $\dfrac{2}{(1-x)^3}$; $\dfrac{6}{(1-x)^4}$; $\dfrac{n!}{(1-x)^{n+1}}$.

9. $\cos(x+\alpha)$; $-\sin(x+\alpha)$; $-\cos(x+\alpha)$; $\sin(x+\alpha+\frac{1}{2}n\pi)$.

10. $-\sin x$; $-\cos x$; $\sin x$; $\cos(x+\frac{1}{2}n\pi)$.

11. $\sin 2x$; $2\cos 2x$; $-4\sin 2x$; $-2^{n-1}\cos(2x+\frac{1}{2}n\pi)$.

12. $-2\sin 4x$; $-8\cos 4x$; $32\sin 4x$; $2^{2n-1}\cos(4x+\frac{1}{2}n\pi)$.

13. $x\cos x+\sin x$; $-x\sin x+2\cos x$; $-x\cos x-3\sin x$.

14. $-x^2\sin x+2x\cos x$; $-x^2\cos x-4x\sin x+2\cos x$; $x^2\sin x-6x\cos x-6\sin x$.

15. $\sec^2 x$; $2\sec^2 x \tan x$; $2\sec^2 x\,(1+3\tan^2 x)$.
16. $3x^2(x\cos 3x+\sin 3x)$; $3x(2-3x^2)\sin 3x+18x^2\cos 3x$; $3(2-27x^2)\sin 3x-27x(x^2-2)\cos 3x$.
17. $x(2+x)/(1+x)^2$; $2/(1+x)^3$; $-6/(1+x)^4$.
18. $nx^{n-1}(\cos nx - x\sin nx)$; $(n-1-nx^2)\,nx^{n-2}\cos nx-2n^2x^{n-1}\sin nx$; $n^2x^{n-2}\{nx^2-3(n-1)\}\sin nx - nx^{n-3}\{3n^2x^2-(n-1)(n-2)\}\cos nx$.
19. $\sec x\tan x$; $\sec x\,(1+2\tan^2 x)$; $\sec x\tan x\,(5+6\tan^2 x)$.
20. $x/\sqrt{(a^2+x^2)}$; $a^2/(a^2+x^2)^{3/2}$; $-3a^2x/(a^2+x^2)^{5/2}$.

Examples XXI, p. 139.

1. Down. 2. Up. 3. Up; down. 4. Down; down.
8. Up when x is $+$; down when $-$. 9. Up when $x<\frac{4}{3}$; down when $>\frac{4}{3}$.
10. Down when $0<x<4$; up elsewhere.
11. Up when $-\infty<x<-1$, and when $0<x<1$; down elsewhere.
12. $x=(n+\frac{1}{2})\pi$. 13. $x=n\pi$. 14. $x=-b/3a$.
15. Infl. (0, 0); min. $(\frac{2}{3}\sqrt{3}, -\frac{8}{9}\sqrt{3})$; max. $(-\frac{2}{3}\sqrt{3}, \frac{8}{9}\sqrt{3})$.
16. Infl. $(\pm\frac{1}{3}\sqrt{3}, \frac{1}{4})$; min. (0, 0).
17. Infl. $\{(n+\frac{1}{4})\pi, 0\}$; max. $\{(2n+\frac{3}{4})\pi, \sqrt{2}\}$; min. $\{(2n-\frac{1}{4})\pi, -\sqrt{2}\}$.
18. Infl. (0, 0), $(\pm 6, \pm\frac{9}{2})$; no max. or min.
19. Infl. $(\pm\sqrt{\frac{2}{3}}, \frac{20}{9})$; min. (0, 0); max. $(\pm\sqrt{2}, 4)$.
20. Infl. $(-2, -2)$; min. $(-1, -\frac{9}{4})$. 21. Infl. $(\pm 1, 1)$; max. $(0, \frac{4}{3})$.
22. Infl. (0, 0); max. $(\pm\sqrt{2}, 2)$; min. $(\pm\sqrt{2}, -2)$.
23. Infl. $(b/c, a)$; no max. or min.
24. $y=x^4$, none; $y=x^5$, infl. at origin.
25. $n-2$. 27. $(\frac{3}{4}a, \pm a/\sqrt{3})$. 29. $\{(\frac{1}{2}n+\frac{1}{4})\pi, \frac{1}{2}(a+b)\}$.
30. Intersects OX at $(\pm 1, 0)$ $(\pm 3, 0)$; OY at (0, 9). Max. (0, 9); min. $(\pm\sqrt{5}, 16)$; Concave up if $|x|>\sqrt{\frac{5}{3}}$; down if $<\sqrt{\frac{5}{3}}$. Points of inflexion $(\pm\sqrt{\frac{5}{3}}, -\frac{44}{9})$. Tangents at points of inflexion, $\pm 40\sqrt{15}\,x+9y=156$.
31. Touches OX at $(\pm 1, 0)$; cuts OY at (0, 1). Max. (0, 1); min. $(\pm 1, 0)$; Concave up if $|x|>\sqrt{\frac{1}{3}}$; down if $<\sqrt{\frac{1}{3}}$. Points of inflexion $(\pm\sqrt{\frac{1}{3}}, \frac{4}{9})$. Tangents at points of inflexion, $\pm 8\sqrt{3}\,x+9y=12$.
32. Touches OX at (0, 0), cuts at $(\pm 2, 0)$; cuts OY at (0, 0). Max. $(\pm\sqrt{2}, 4)$; min. (0, 0). Concave up if $|x|<\sqrt{\frac{2}{3}}$; down if $>\sqrt{\frac{2}{3}}$. Points of inflexion $(\pm\sqrt{\frac{2}{3}}, \frac{20}{9})$. Tangents at points of inflexion, $\pm 16\sqrt{6}\,x-9y=12$.

Examples XXII, p. 144.

1. (i) $v=20-4t$; $a=-4$. (ii) $v=12$; $a=-4$. (iii) $v=20$; $a=-4$.
2. (i) $v=3t^2-4t$; $a=6t-4$. (ii) $v=4$; $a=8$. (iii) $v=0$; $a=-4$.
3. (i) $v=-\frac{10}{3}\pi\sin\frac{1}{3}\pi t$; $a=-\frac{10}{9}\pi^2\cos\frac{1}{3}\pi t$.
(ii) $v=-5\pi/\sqrt{3}$; $a=\frac{5}{9}\pi^2$. (iii) $v=0$; $a=-\frac{10}{9}\pi^2$.
4. (i) $v=6-8/(t+1)^2$; $a=16/(t+1)^3$. (ii) $v=5\frac{1}{9}$; $a=\frac{16}{27}$.
(iii) $v=-2$; $a=16$. 5. 0; 6. 6. 8; -32.
7. s -20; -11; 16; 325; 2320. $v=4$; 16; 40; 184; 664.
6; 18; 30; 66; 126.

8. $s = 10$; $21{\cdot}21$; 20; $-21{\cdot}21$; 20.
$v = 15{\cdot}71$; $5{\cdot}56$; $-7{\cdot}85$; $-5{\cdot}56$; $-7{\cdot}85$.
$a = -6{\cdot}17$; $-13{\cdot}08$; $-12{\cdot}34$; $13{\cdot}08$; $-12{\cdot}34$.

11. After 4 secs. when $s = 144$.

12. After $n+\frac{1}{2}$ secs. [n any integer], when $s = \pm a$.

Examples XXIII, p. 145.

3. The force varies inversely as the square of the distance from the origin.

4. Mass $\times \mu a$.

6. 9 poundals.

Examples XXIV, p. 149.

50 at 53° 8′ below the horizontal.

2. Each $= c$.

2·4 miles per hour.

4. 34·64 m. p. h.

5. 9 ft. per sec.

6. 9 ft. per sec.

7. (i) $5\frac{1}{3}$ ft.-secs. (ii) When the foot is 24·04 ft. from wall. (iii) When the foot is 30·4 ft. from wall.

8. 20·83 ft. per min.

9. $1\frac{9}{16}$ ft. per sec.

10. When P is 7·43 ft. from 0.

11. 2 ft. per min.

12. 21·25 m. p. h.

13. 15·08 m. p. h.

14. (i) Decreasing at 4·61 m. p. h. (ii) Increasing at 4·715 m. p. h.

15. (i) Decreasing at 6·94 m. p. h. (ii) Increasing at 6·215 m. p. h.

16. Receding from C at 1·12 m. p. h.; approaching A at 2·22 m. p. h. receding from B at 1·405 m. p. h.

18. (i) 12·57. (ii) 15·71. (iii) 25·13 miles per min.

19. (i) $7\frac{1}{9}$, (ii) $11\frac{1}{9}$ ft. per sec.

20. ·0442 in. per sec.

21. 840,000 c. ft.

22. 12·46 ft. per sec.

23. 4·82 in.; ·298 in. per min.

24. 5·014 in.; ·133 in. per sec.

25. $2k/ab$ in. per sec.

26. 10 ft.-secs.; 14·14 ft.-secs.

27. 86·74 ft.-secs. at an angle 86° 42′ with and below the horizontal.

29. $-13\frac{1}{3}$ ft.-secs.; $16\frac{2}{3}$ ft.-secs. at an angle 36° 52′ with the major axis.

30. (i) 2·828, (ii) 2 in. per sec.

31. Each 7·07 ft.-secs.

32. $\pm ua^2y/\sqrt{(b^4x^2+a^4y^2)}$; $\mp ub^2x/\sqrt{(b^4x^2+a^4y^2)}$.
1·486 and 3·714 m. p. h.

Examples XXV, p. 155.

1. au/r^2 where $r = AP$.

2. (i) $\omega r/(a-r)$. (ii) $-\omega r^2/(a^2+r^2)$.

3. $-68{\cdot}52$; $41{\cdot}11$.

4. The same (in terms of θ and ϕ) as in Art. 69.

Examples XXVI, p. 160.

1. $\frac{1}{7}x^7$, $3x^7$, $-1/5x^5$, $-2/x^5$, $\frac{6}{7}x^{7/6}$, $\frac{6}{5}x^{5/6}$.

2. x^3-x^2+x.

3. $-1/x$, $-1/9x^9$, $2\sqrt{x}$, $\frac{3}{2}x^{2/3}$.

4. $\dfrac{n}{n+1}x^{1+1/n}$, $-\dfrac{1}{(n-1)x^{n-1}}$, $\dfrac{n}{n-1}x^{1-1/n}$.

5. $\frac{1}{4}x^4+2x^3+5x^2-5x$; $x^7-2x^5+3x^3-x$.

6. $-1/2x^2-6/x-5x$; $-7/5x^5+10/3x^3-9/x-x$.

7. $\frac{1}{6}x^6 - x^4 + \frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x$; $\frac{1}{7}x^7 - \frac{4}{5}x^5 + 2x^3 - 8x$.
8. $\frac{1}{5}ax^5 + \frac{1}{4}bx^4 + \frac{1}{3}cx^3 + \frac{1}{2}dx + ex$; $-a/3x^3 - b/2x^2 - c/x + dx$.
9. $-(2+x+x^2)/x^3$; $2\sqrt{x}\,(1+x+x^2)$.
10. $(2x^4+3x^2-3)/6x$; $3x^{\frac{2}{3}}\left(\frac{1}{2}+\frac{1}{8}x^2+\frac{1}{7}x^4\right)$.
11. $x-3x^2+3x^3$; $x+x^3+\frac{3}{5}x^5+\frac{1}{7}x^7$.
12. $-(3a^2x^2+3abx+b^2)/3x^3$; $(16x^4+72x^2-27)/3x$.
13. $-\dfrac{a}{(n-3)x^{n-3}} - \dfrac{b}{(n-2)\,x^{n-2}} - \dfrac{c}{(n-1)x^{n-1}}$; $\dfrac{x^{2n+1}}{2n+1} + \dfrac{x^{n+1}a^n}{n+1} + a^{2n}x$;

$\dfrac{}{m-p+1} + \dfrac{}{n-p+1}$.

14. $\frac{2}{5}\sqrt{x}\,(ax^2+5b)$.
15. $y = 2x^4 - x^2 - 20$.
16. $y = \frac{5}{2} - \cos x$.
17. $y = 3x - 2x^2 + C$.
18. $y = 2x + 3/x + C$.
19. $6y = 2x^3 - 3x^2 - 21$.
20. $y^2 - 4x - 2y + 1 = 0$. $y = 2\sqrt{x}+1$
21. $y = 1 + \sin x$.
22. $3y = x^3 + 3x^2 + 3x$.
23. $xy + c^2 = Cx$.
24. $y = \frac{1}{2}x^2/a + C$, where a is the length of NG.
25. $2x^3 - 2x^2 + 3x + 7$.
26. $(34x^2-16)/3x^2$.

Examples XXVII, p. 164.

1. $\frac{1}{3}x^3$, $\frac{1}{3}(7+x)^3$, $-\frac{1}{3}(5-x)^3$, $\frac{1}{9}(3x-4)^3$, $\frac{1}{3}(px+q)^3/p$.
2. $\dfrac{x^{n+1}}{n+1}$, $\dfrac{(x-a)^{n+1}}{n+1}$, $\dfrac{(9x+4)^{n+1}}{9(n+1)}$, $\dfrac{(3-2x)^{n+1}}{-2(n+1)}$, $\dfrac{(ax+b)^{n+1}}{a(n+1)}$, $-\dfrac{(p-qx)^{n+1}}{q(n+1)}$.
3. $-\cos x$, $-\frac{1}{4}\cos 4x$, $-(1/m)\cos mx$, $-3\cos\frac{1}{3}x$, $-(1/p)\cos(px+\alpha)$, $\frac{1}{2}\cos(\alpha-2x)$, $\cos(\frac{1}{4}\pi - x)$.
4. $\frac{2}{3}\sqrt{x^3}$, $\frac{2}{3}\sqrt{(1+x)^3}$, $-\frac{1}{6}\sqrt{(3-4x)^3}$, $\frac{2}{3}\sqrt{(px+q)^3/p}$, $\frac{2}{3}a\sqrt{(1+x/a)^3}$, $\frac{2}{9}$ $\frac{2}{3}\sqrt{(3x^3)}$, $\frac{2}{3}\sqrt{(mx^3)}$
5. $-\dfrac{1}{x}$, $\dfrac{1}{5(2-5x)}$, $-\dfrac{1}{7(7x+2)}$, $\dfrac{1}{a-x}$, $-\dfrac{1}{m(mx-n)}$.
6. $\tan x$, $\tan(x+\alpha)$, $(1/m)\tan mx$, $\frac{1}{2}\tan(\alpha+2x)$, $m\tan(x/m)$, $(1/n)\tan(nx+m)$.
7. $-\dfrac{1}{(n-1)x^{n-1}}$, $\dfrac{1}{4(n-1)(4x-5)^{n-1}}$, $\dfrac{1}{2(n-1)(1-2x)^{n-1}}$, $\dfrac{1}{(n-1)(c-x)^{n-1}}$, $-\dfrac{1}{b(n-1)(bx-a)^{n-1}}$.
8. $\dfrac{n}{n-1}\sqrt[n]{x^{n-1}}$, $\dfrac{n}{n-1}\sqrt[n]{(x+3)^{n-1}}$, $\dfrac{n}{2(n-1)}\sqrt[n]{(2x-5)^{n-1}}$, $-\dfrac{n}{n-1}\sqrt[n]{(a-x)^{n-1}}$, $\dfrac{1}{n-1}\sqrt[n]{(nx+c)^{n-1}}$, $\dfrac{n}{n-1}\sqrt[n]{\dfrac{x^{n-1}}{2}}$, $\dfrac{n}{n-1}\sqrt[n]{\dfrac{x^{n-1}}{m}}$.
9. $-\dfrac{1}{3x^3}$, $-\dfrac{1}{3(x-3)^3}$, $\dfrac{1}{3(3-x)^3}$, $\dfrac{1}{21(3-7x)^3}$, $-\dfrac{1}{3p(px+q)^3}$.
10. $\frac{1}{42}(7y-4)^6$.
11. $-(2/5b)(a-bt)^{5/2}$.
12. $\dfrac{n}{q(n+1)}\sqrt[n]{(p+qz)^{n+1}}$.
13. $-\frac{1}{2}\sqrt[3]{(5-3z)^2}$.
14. $2/\sqrt{(1-u)}$.
15. $\frac{1}{6}/(a-3\theta)^2$.

16. $-(a-t)^{p+1}/(p+1)$.

17. $1/\{4(n-1)(7-4u)^{n-1}\}$.

18. $-\sqrt[n]{(b-ny)^{n-1}}/(n-1)$.

19. $\frac{1}{3}\sin 3\theta,\ 3\sin\frac{1}{3}\theta,\ -\sin(\alpha-\theta),\ (1/n)\sin(n\theta+\alpha)$.

20. $-2/\sqrt{z},\ 2/\sqrt{(1-y)},\ -\frac{2}{3}/\sqrt{(3u-5)},\ -2/\{a\sqrt{(av+b)}\}$.

21. $9y = (3x-4)^3+82$.

Examples XXVIII, p. 167*.

The parabolas $y = \frac{1}{2}kx^2 + C$.

2. The conics $y^2 = kx^2 + C$.

$y = kx^{n+1}/(n+1) + C$.

4. The parabolas $y^2 = 2kx + C$.

The circles $x^2+y^2 = 2kx + C$. In any one of these circles, the sum of the abscissa and the subnormal is equal to the distance from the origin to the centre of the circle.

6. $ky^2 - x^2 = C$.

7. $y^3 = kx^3 + C$.

8. $2y^3 = 3kx^2 + C$.

9. $y^2(kx+C) = -1/2$.

10. $ky^2 = 2x + C$.

11. 65 ft.-secs.; 132 ft.

12. $u + 6f/\pi$; $3u$.

13. $k\sqrt{(a^2-s^2)}$, if the acceleration be $-k^2s$.

14. After 3 secs.; $58\frac{1}{2}$ ft.

15. $17\frac{1}{3}$ ft.-secs.; $6\frac{4}{15}$ ft.

16. 32 ft.

17. 12 ft.-secs.; 216 ft.

18. 7·746 ft.-secs.

19. 2·236 ft.-secs.; 3·873 ft.

20. $(16, 42\frac{2}{3})$.

21. $8+2t$; 48 ft.

22. $243x^2 = y(y-81)^2$.

Examples XXIX, p 174.

1. $20\frac{1}{4}$; $312\frac{3}{7}\pi$.
2. 34; $678\frac{1}{56}\pi$.
3. 12; 54π.
4. 192; 864π.
5. $20\frac{5}{6}$; $104\frac{1}{6}\pi$.
6. $166\frac{2}{3}$; $3333\frac{1}{3}\pi$.
7. 2; $\frac{1}{2}\pi^2$.
8. $\frac{2}{5}\sqrt{3}\,a^2$; $\frac{3}{16}\pi a^3$.
9. 12; 52π.

10. $\frac{3}{4}$; $\frac{9}{35}\pi$.
12. $\frac{1}{6}a^2$.
13. $\frac{512}{3}\pi$.
16. $\frac{1024}{3}\pi$.

18. $10\frac{2}{3}$.
19. $\frac{1}{3}$.
20. $\frac{4}{15}$.
21. $\frac{32}{105}\pi a^3$.

22. 156π.
24. $\frac{1}{12}\pi a^3$.
25. $\frac{4}{27}\pi a^2$.
27. $\frac{243}{2}\pi$.

29 $4\pi a^3$.
30. $\frac{13}{81}\pi a^3$.
31. $\frac{1}{6}$.
32. $(p_1v_1 - p_2v_2)/(\gamma-1)$.

Examples XXX, p. 178.

1. ·0316.
2. ·05.
3. ·2468.
4. $·0262a$.
5. 19.
6. 1·034.
7. $84\frac{3}{16}$.
8. 10·75.
9. 108.
10. $42\frac{1}{2}$.
11. (i) $277\frac{1}{3}\pi$. (ii) $202\frac{2}{3}\pi$.
12. $69\frac{1}{3}\pi$.
13. $118\frac{63}{64}\pi$.
14. $4\frac{155}{256}\pi$.
15. 3π.
16. 6·928.
17. $6a$.
18. 4·69 inches.

Examples XXXI, p. 184.

1. Divergent.
2. Semi-convergent.
3. Convergent.
4. Convergent.
5. Convergent if $|x| < 1$; semi-convergent if $x = -1$; divergent otherwise.

* It is immaterial whether the constant occurs in the form C simply, or $2C$, $\frac{1}{2}C$, C^2, &c.

6. Convergent if $|x| <$ or $= 1$.
7. Divergent.
8. Convergent if $|x| <$ or $= 1$.
9. Convergent if $|x| < 1$.
10. Divergent.
11. Convergent.
12. Convergent.
13. Convergent if $|x| < 5$.
14. Semi-convergent.

Examples XXXII, p. 193.

1. e.
2. 1.
3. 0; 0;
4. ·0183; 1·3956; ·6065.
6. $2(1 + x^2/2! + x^4/4! + \ldots)$.
7. 6·389.
8. 4·482.
13. (i) 1·5431, 1·1750, ·7616. (ii) 1·0314, ·2526, ·2449.
14. 2·9957, 2·7726, 1·4427.
21. 1; 1; $-\frac{1}{2}$.
22. ·3466, ·8813, $\pm$1·3169.
25. ·6931; 2·3026; 2·7081.
26. 1·649; ·513; ·135.
27. ·6367; 3·7622; ·9051; 1·138; 1·763; 2·164.

Examples XXXIII, p. 199.

$4e^{4x}$; $-e^{3-x}$; be^{a+bx}; $2xe^{x^2}$; $e^{x/a}/a$; $-qe^{p-qx}$.
$e^{\sin x}\cos x$; $-e^{\cos x}\sin x$; $ae^{\sin ax}\cos ax$; $-ae^{a\cos x}\sin x$; $e^{\tan x}\sec^2 x$.
$3x^2e^{3x}(x+1)$; $x^{n-1}e^{ax}(ax+n)$; $ae^{ax}(\sin ax + \cos ax)$;
$-3e^{-3x}(\cos 3x + \sin 3x)$; $e^{ax}(a\cos bx - b\sin bx)$; $e^{ax}\sin x(2\cos x + a\sin x)$.
$2e^{2x}(x-1)/x^3$; $\frac{1}{2}e^{ax}(2ax-1)/x^{3/2}$; $e^x(\cot x - \operatorname{cosec}^2 x)$;
$(2ax + b - ax^2 - bx - c)/e^x$.

5. $2/(2x-1)$; $-1/(2-x)$; $2x/(x^2-1)$; $2bx/(a+bx^2)$.
6. $7/(5+7x)$; $-q/(p-qx)$; $(2x-3)/(x^2-3x-1)$; $-3x^2/(1-x^3)$.
7. $-\cot x$; $2\operatorname{cosec} 2x$; $b\cos x/(a + b\sin x)$; $4\sin x/(3 - 4\cos x)$; $-\sin 2x/(1+\cos^2 x)$.
8. $x^{n-1}(1 + n\log x)$; $2x\log(2-x) - x^2/(2-x)$; $\log(1-x^2) - 2x^2/(1-x^2)$.
9. $(1-\log x)/x^2$; $(1-n\log x)/x^{n+1}$; $\dfrac{a}{x^2(ax+b)} - \dfrac{2}{x^3}\log(ax+b)$; $\dfrac{1-\frac{1}{2}\log x}{x^{3/2}}$.
10. $\dfrac{n}{x} + \dfrac{1}{x+2}$; $\dfrac{2x}{n(1+x^2)}$; $\dfrac{a^2}{x(x^2+a^2)}$; $\dfrac{1}{x} - \dfrac{1}{1-x} + \dfrac{2}{3-x}$.
11. $\frac{1}{2}\cot x$; $\dfrac{1-2x}{2x(1-x)}$; $\dfrac{2x-4}{(x-1)(2x-3)}$; $\dfrac{2\cos x - 1}{\sin x(2-\cos x)}$.
12. $\dfrac{1}{\sqrt{(x^2-1)}}$; $\dfrac{1}{2\sqrt{(x^2-1)}}$; $\dfrac{b}{2\sqrt{(b^2x^2-a^2)}}$.
13. $a^n e^{ax}$.
14. $(-1)^{n-1}(n-1)!/x^n$.
15. $(-1)^n e^{-x}$.
16. $(-b)^n e^{a-bx}$.
17. $-(n-1)!/(1-x)^n$.
18. $(-1)^{n-1}b^n(n-1)!/(a+bx)^n$.
20. $\frac{1}{2}e^{2x}$; $-e^{5-x}$; $ae^{x/a}$; e^{px+q}/p; $2e^{x/2}$; $ne^{x/n}$; $-e^{-x}$; $-ae^{-x/a}$.
21. $\frac{1}{5}\log(5x+3)$; $-\frac{1}{2}\log(7-2x)$; $\log(x-a)$; $-\{\log(p-qx)\}/q$; $\{\log(bx+c)\}/b$; $\frac{1}{3}\log(8+3x)$.
22. $-\frac{1}{5}\log(8-5x)$; $-\log(1-x)$; $\log(4x-5)$; $-\log(a-bx)$; $(a/b)\log(bx+c)$; $-\frac{1}{3}\log(5-2x)$.

Examples XXXIV, p. 204.

1. $\frac{1}{\sqrt{(9-x^2)}}$; $\frac{-a}{x\sqrt{(x^2-a^2)}}$; $\frac{1}{2\sqrt{(x-x^2)}}$; $\frac{1}{x\sqrt{(2x-1)}}$.

2. $\frac{-2x}{\sqrt{(1-x^4)}}$; $\frac{-m}{\sqrt{(1-m^2x^2)}}$; $\frac{1}{2x\sqrt{(x-1)}}$; -1.

3. $\frac{-1}{1+(a-x)^2}$; -1; $\frac{2a^2x}{a^4+x^4}$; $\frac{1}{2\sqrt{x}(1+x)}$.

4. $\frac{-1}{1+x^2}$; $\frac{-a}{a^2+x^2}$; $\frac{1}{1+x^2}$.

5. $\frac{1}{x\sqrt{(x^2-1)}}$; $\frac{-1}{x\sqrt{(x^2-1)}}$; $\frac{-1}{\sqrt{(a^2-x^2)}}$; $\frac{-1}{\sqrt{(1-\ldots}}$

6. $\frac{3}{4}\cosh\frac{3}{4}x$; $2x\cosh(x^2)$; $-\{\cosh(1/x)\}/x^2$; $\sinh 2x$.

7. $a\sinh(ax+b)$; $3\cosh^2 x\sinh x$; $\cosh^3 x$.

8. $\text{sech}^2 x$; $-\text{cosech}^2 x$; $(1/a)\,\text{sech}^2(x/a)$; $(a/x^2)\,\text{cosech}^2(a/x)$.

9. $\frac{1}{\sqrt{(9+x^2)}}$; $\frac{2x}{\sqrt{(x^4-a^4)}}$.

10. $\frac{1}{1-x^2}(x^2<1)$; $\frac{-1}{x^2-1}(x^2>1)$.

11. $1+2x\tan^{-1}x$; $\cos^{-1}x-x/\sqrt{(1-x^2)}$.

12. $1-(x\sin^{-1}x)/\sqrt{(1-x^2)}$.

13. $-1/\sqrt{(1-x^2)}$.

14. $1/\sqrt{(1-x^2)}$.

15. $2/(1+x^2)$.

16. $2/(1+x^2)$.

17. $\sin^{-1}\frac{1}{3}x$; $\sin^{-1}(x/\sqrt{5})$; $\sin^{-1}(x+1)$; $\frac{1}{2}\sin^{-1}\frac{2}{3}x$; $(1/b)\sin^{-1}(bx/a)$.

18. $\tan^{-1}x$; $\frac{1}{10}\tan^{-1}\frac{1}{10}x$; $\sqrt{\frac{1}{7}}\tan^{-1}(x\sqrt{\frac{1}{7}})$; $\frac{1}{6}\tan^{-1}\frac{3}{2}x$; $\sqrt{\frac{1}{10}}\tan^{-1}(x\sqrt{\frac{2}{5}})$; $\{\tan^{-1}(ax/b)\}/ab$; $\{1/\sqrt{(ab)}\}\tan^{-1}\{x\sqrt{(b/a)}\}$.

19. $\sinh^{-1}\frac{1}{4}x$; $\cosh^{-1}\frac{1}{3}x$; $\sinh^{-1}(x/\sqrt{5})$; $\frac{1}{2}\cosh^{-1}2x$; $\frac{1}{5}\sinh^{-1}\frac{5}{3}x$; $(1/b)\cosh^{-1}(bx/a)$.

20. $\frac{1}{3}\cosh 3x$; $\frac{1}{2}\sinh 2x$; $a\cosh(x/a)$; $a\sinh(x/a)$.

Examples XXXV, p. 208.

2. $c\coth(x/c)$; $\frac{1}{2}c\sinh(2x/c)$.

6. $y=x-1$; $x+y=1$.

7. $s=a\sinh(x/a)$.

8. $81°47'$.

9. When the number is e.

10. Min. $=\sqrt{(a^2-b^2)}$, if $a>b$; no max. or min. if $a<b$.

11. $2\sqrt{(ab)}$.

12. Min. (e, e); point of inflexion $(e^2, \frac{1}{2}e^2)$.

13. Max. $=\cdot368$; point of inflexion $(2, \cdot27)$.

14. $\cdot607$.

15. When $x=\pm1\cdot317$.

16. $(-1)^n\times\cdot7071\,e^{-(n+\frac{1}{4})\pi}$.

17. $(-1)^n\times\cdot8192\,e^{-\cdot7n\pi}$.

18. $(\pm\cdot707, \cdot607)$.

19. Max. $(\cdot707, \cdot859)$; min. $(-\cdot707, -\cdot859)$; points of inflexion $(0, 0)$, $(\pm1\cdot225, \pm\cdot273)$.

22. $\cdot0001263$.

23. $\cdot0000729$.

24. $19\cdot09$.

25. 632.

26. $18\cdot326$.

27. $a^2\sinh(b/a)$.

28. $26\cdot31$.

29. $\cdot361$.

30. $1-e^{-a}$; 1.

31. (i) $1\cdot7624\,a^2$. (ii) $2a^2\log\{b/a+\sqrt{(b^2/a^2+1)}\}$.

32. $\frac{1}{4}\pi^2a^3$; $\pi a^3\tan^{-1}(b/a)$. $\frac{1}{2}\pi^2a^3$.

33. $\frac{1}{3}\pi a^2$.

35. $3\cdot097$ ft.-secs.

36. $80\cdot37$ ft.

37. Initially, $s=1$, $v=-{\cdot}25$, $a=-2{\cdot}405$. After 1 sec., $s=0$, $v=-1{\cdot}223$, $a={\cdot}611$. After 2 secs., $s=-{\cdot}607$, $v={\cdot}152$, $a=1{\cdot}46$. After 4 secs., $s={\cdot}368$, $v=-{\cdot}092$, $a=-{\cdot}885$. After 10 secs., $s=-{\cdot}082$, $v={\cdot}021$, $a={\cdot}197$.

38. The third curve touches $s=e^{-t/10}$ where $t=(n+\frac{1}{4})\pi$, and touches $s=-e^{-t/10}$ where $t=(n+\frac{3}{4})\pi$. It cuts the axis of t where $t=-\frac{1}{2}n\pi$, at an angle $\tan^{-1}(2e^{-n\pi/20})$.

40. ${\cdot}0675$; ${\cdot}0495$.

41. If acceleration $=-n^2s$, then $v=n\sqrt{(a^2-s^2)}$, and $s=a\cos nt$.

Examples XXXVI, p. 213.

1. $-3\cot(\alpha-3x)$.
2. $3\sin x\cos^2 x/(1-\cos^3 x)$.
3. $\sec x$.
4. $\frac{1}{2}/\sqrt{(x^2-1)}$.
5. $-\sin 4x/\sqrt{(2-\sin^2 2x)}$.
6. $-3na\sin ax\cos^2 ax(1+\cos^3 ax)^{n-1}$.
7. $6\tan 3x\sec^2 3x/(1-\tan^2 3x)^2$.
8. $[\log(1+\sqrt{x})]/(\sqrt{x}+x)$.
9. $\dfrac{\sec\frac{1}{2}x\tan\frac{1}{2}x}{2n(1+\sec\frac{1}{2}x)^{1-1/n}}$.
10. $\dfrac{-a\sin 2ax}{1+\cos^2 ax}$.
11. $e^{1+\sin^2 x}\sin 2x$.
12. $\frac{1}{2}\cos x/\sqrt{(\sin x-\sin^2 x)}$.
13. $\dfrac{e^{x/2}}{2(1+e^{x/2})}$.
14. $\dfrac{-\sqrt{3}}{2\sqrt{(5x-2-3x^2)}}$.
15. $\dfrac{2xa^2}{(x^4+a^4)^{3/2}}$.
16. $\dfrac{-\sin x}{2+2\cos x+\cos^2 x}$.
17. $\dfrac{-a}{x\sqrt{(x^2-a^2)}}$.
18. $-\dfrac{\tan(1+\sqrt{x})}{2\sqrt{x}}$.
19. $3^x\log 3$.
20. $10^{2x-1}\times 2\log 10$.
21. $a^{bx-c}\times b\log a$.
22. $-(2^{1/x}\log 2)/x^2$.
23. $-(na^{1/x^n}\log a)/x^{n+1}$.
24. $2\log 5\times\cos x\,.\,5^{1+2\sin x}$.
25. $\dfrac{(6-6x-x^2)\sqrt{x}}{2\sqrt{\{(x-1)(x-2)^7\}}}$.
26. $\dfrac{\sqrt[n]{(a-3x)}}{(a-x)^3}\left[\dfrac{3}{a-x}-\dfrac{3}{n(a-3x)}\right]$.
27. $(2x-x^2-\frac{2}{3})/[x(1-x)(2-x)]^{4/3}$.
28. $\sin^m x\cos^n x(m\cot x-n\tan x-4/x)/x^4$.
29. $\dfrac{x^3\sqrt{(3-2x)}}{(1+x)(2-x)}\left[\dfrac{3}{x}-\dfrac{1}{3-2x}-\dfrac{1}{1+x}+\dfrac{1}{2-x}\right]$.
30. $\dfrac{(a-x)^2(b-x)^3}{(c-2x)^4}\left[\dfrac{8}{c-2x}-\dfrac{2}{a-x}-\dfrac{3}{b-x}\right]$.
31. $e^{-x}\sin^m x\cos^n x(m\cot x-n\tan x-1)$.
32. $(a+x)^3\sin x\cos^3 2x[\cot x-6\tan 2x+3/(a+x)]$.
33. $\frac{1}{2}\sqrt{[a^x\sin(x+\alpha)\cos(x-\beta)]}[\log a+\cot(x+\alpha)-\tan(x-\beta)]$.
34. $1/(x\log x)$.
35. $x^x(1+\log x)$.
36. $(\log x)^x[\log(\log x)+1/\log x]$.
37. $x^{\sin x}[\log x\cos x+(\sin x)/x]$.
38. $1/[x\sqrt{(3x^2-1-2x^4)}]$.
39. $2/(1+x^2)$.
40. $2/(1+x^2)$.
41. $1-x^2-3x\sqrt{(1-x^2)}\sin^{-1}x$.
42. $\cos\alpha/(1+2x\sin\alpha+x^2)$.
43. $1/[2x\sqrt{(x-1)}]$.
44. $1/x^2+1/[x^2\sqrt{(1-x^2)}]$.
45. $2x-2x^3/\sqrt{(x^4-1)}$.
46. $(1/a)\tanh(x/a)$.
47. $-(a/x^2)\coth(a/x)$.
48. $a\sinh 2ax/(1+\cosh^2 ax)$.
49. $\frac{1}{3}(a^2+4ax-3x^2)/[(a^2+x^2)(a-x)^4]^{1/3}$.

50. $3a^2x^2/(a^2+x^2)^{5/2}$.
51. $-n(1+1/\log x)^{n-1}/(x\log^2 x)$.
52. $2x(1+x^4)^{n-1}(1+2nx^2\tan^{-1}x^2)$.
53. $\frac{1}{2}n\sin 4x\,(a^2+\sin^2 x\cos^2 x)^{n-1}$.
54. $-\frac{2}{3}$.
55. $nb\cot(bx+c)$.
56. $-1/\sqrt{(x^2-a^2)}$.
57. $\frac{1}{2}\sqrt{(a^2-b^2)}/(a+b\cos x)$.
58. $\sec x$.
59. $-\sqrt{(b^2-a^2)}/(b+a\cos x)$.
60. $2\sqrt{(a^2+x^2)}$.
61. $2\sqrt{(a^2-x^2)}$.
62. $(a^2+b^2)e^{ax}\sin bx$.
63. $-\frac{1}{2}/(x^2+1)$.
64. $1/\sqrt{(1-x^2)}$.
65. $\frac{1}{2}\sec x$.
66. $\frac{1}{2}$.
67. $2ax^2/(x^4-a^4)$.
68. $x/(1+x^2)$.
69. $-\log_x a/(x\log_e x)$.
70. $2a^3/(a^4-x^4)$.
71. $\operatorname{cosech} x$.
72. 2.
73. $2/(\tan^2 x-1)$.
74. (i) $ab^{\theta/(\gamma+\theta)}\times\gamma\log b/(\gamma+\theta)^2$. (ii) $p\,(b\alpha^\theta\log\alpha-c\beta^\theta\log\beta)$.
75. If $a^2<b^2$, $|b+a\cos x|>|a+b\cos x|$, and therefore the inverse cosine is imaginary.

Examples XXXVII, p. 218.

1. $2a^2(a^2-4)/(2y-ax)^3$.
3. $-(n-1)\,a^n x^{n-2}/y^{2n-1}$.
4. $-2a^3(4x^3+a^3)/y^7$.
5. $4a^2/(a-2y)^3$.
6. $-64e^{-2x}\sin(2x+\alpha)$.
7. $(-a)^n 2^{n/2}\,.\,e^{-at}\cos(at-\frac{1}{4}n\pi)$.
9. $e^x(x^2+20x+90)$.
10. $a^{n-3}e^{ax}[(x^3+a^3)a^3+3na^2x^2+3n(n-1)ax+n(n-1)(n-2)]$.
11. $-12/x^4$.
12. $(-1)^{n-1}2(n-3)!/x^{n-2}$.
13. $32(x^2-5)\cos 2x+160x\sin 2x$.
14. $(-1)^n e^{-x}[x^3-3nx^2+3n(n-1)x-n(n-1)(n-2)]$.
16. $(1-x^2)D^{n+2}y-(2n+1)xD^{n+1}y-n^2D^ny=0$.
18. $(x^2-a^2)D^{n+2}y+(2n+1)xD^{n+1}y+n^2D^ny=0$.
19. $[x^2+2nx(1+x)+\frac{1}{2}n(n-1)(1+x)^2]\times n!$
22. $(1-x^2)D^{n+2}y-(2n+1)xD^{n+1}y-n^2D^ny=0$.
23. $x^2D^ny+2nxD^{n-1}y+n(n-1)D^{n-2}y$.
24. $(\dot{x}\ddot{y}-\dot{y}\ddot{x})/\dot{x}^3$.

Examples XXXVIII, p. 222.

1. Between $-\infty$ and 0, 0 and 4, 4 and ∞.
2. Between $-\infty$ and -3, -3 and 0, 0 and 3, 3 and ∞.
3. Between $-\infty$ and 2, 2 and 6, 6 and ∞.
4. Between $-\infty$ and -5, -5 and 0, 0 and 2, 2 and ∞.
5. Between $-\infty$ and -2, -2 and 3, 3 and ∞.
6. No real roots.
7. $x=1$, 1, and -4.
8. $x=-\frac{1}{2}$, $-\frac{1}{2}$, and 5.
9. $x=2$, 2, 2, and -2.
10. $x=\pm 1$, ± 1, and 7.
11. $x=-\frac{3}{2}$, $-\frac{3}{2}$, and $\frac{2}{3}$.
12. $x=3$, 3, and $\pm\sqrt{(-1)}$.
13. (i) $g^2=ac$. (ii) $f^2=bc$.
16. $y=0$ when $x=1$ or $\frac{4}{3}$; $dy/dx=0$ when $x=\frac{7}{6}$.
17. $y=0$ when $x=1$ or 3; $dy/dx=0$ when $x=\frac{7}{3}$.
18. $y=\frac{7}{2}$ when $x=1$ or 2; $dy/dx=0$ when $x=1{\cdot}414$.
19. $y=0$ when $x=2$ or 4; $dy/dx=0$ when $x=2{\cdot}828$.
20. $y=0$ when $x=(n+\frac{1}{4})\pi$; $dy/dx=0$ when $x=(n+\frac{3}{4})\pi$.
21. $f(x)=0$ when $x=0$ or 4, but is discontinuous when $x=1$. Theorem does not apply.

22. $f(x) = 0$ when $x = n\pi$, but $f(x)$ and $f'(x)$ are both discontinuous when $x = (n+\frac{1}{2})\pi$.

23. $f(x) = 0$ when $x = 0$ or 16, but $f'(x)$ is discontinuous when $x = 8$.

24. $4p^3 + 27q^2 = 0$.

Examples XXXIX, p. 228.

$\underline{\sqrt{(x^2+xh)}-x}$

2. $\frac{1}{h}\log\frac{e^h-1}{h}$.

3. $-\frac{x}{h}+\frac{1}{h}\cos^{-1}\frac{\sin(x+h)-\sin x}{h}$.

4. $\frac{1}{\log(1+h/x)}-\frac{x}{h}$.

6. $f(x)$ discontinuous when $x = 1$.

7. $f(x)$ and $f'(x)$ discontinuous when $x = \frac{1}{2}\pi$.

8. $f'(x)$ discontinuous when $x = 8$.

9. $\frac{1}{3}$.

10. $\sqrt{(x^2/h^2+\frac{2}{3}x/h+\frac{1}{6})}-x/h$.

11. $\sqrt[3]{(x^3+x^2h)}/h-x/h$.

12. $(1/h)\log[2(e^h-1-h)/h^2]$.

22. $f'(x)$ and $f''(x)$ discontinuous when $x = 4$.

23. $f''(x)$ discontinuous when $x = 1$.

24. $f'(x)$ and $f''(x)$ discontinuous when $x = \frac{1}{2}\pi$.

25. (i) -1. (ii) $\cos\alpha$. (iii) a/b.

26. (i) $-\frac{1}{2}$. (ii) $\frac{1}{6}$. (iii) 1.

27. (i) 0. (ii) $\frac{2}{5}$. (iii) 0.

Examples XL, p. 231.

1. $x-3\log(x+3)$.

2. $\frac{1}{2}x^2-3x+9\log(x+3)$.

3. $\frac{1}{3}x^3-\frac{3}{2}x^2+9x-27\log(x+3)$.

4. $\frac{1}{2}x+\frac{3}{4}\log(2x-3)$.

5. $\frac{1}{4}x^2+\frac{3}{4}x+\frac{9}{8}\log(2x-3)$.

6. $-\frac{1}{2}x-\frac{1}{4}\log(1-2x)$.

7. $-\frac{1}{4}x^2-\frac{1}{4}x-\frac{1}{8}\log(1-2x)$.

8. $\frac{1}{2}x-\frac{1}{4}\log(1-2x)$.

9. $2x+11\log(x-4)$.

10. $-x+\log(2x-1)$.

11. $-ax-\frac{1}{2}x^2-a^2\log(a-x)$.

12. $\frac{1}{6}x^3+\frac{1}{8}x^2+\frac{1}{8}x+\frac{1}{16}\log(2x-1)$.

13. $\frac{x}{a}-\frac{b}{a^2}\log(ax+b)$.

14. $\frac{x^2}{2p}+\frac{qx}{p^2}+\frac{q^2}{p^3}\log(px-q)$.

15. $-\frac{1}{6}x^3-\frac{1}{8}cx^2-\frac{1}{8}c^2x-\frac{1}{16}c^3\log(c-2x)$.

16. $\frac{ax}{c}+\frac{bc-ad}{c^2}\log(cx+d)$.

Examples XLI, p. 233.

1. $\frac{1}{2}\log(x^2-1)$.

2. $\log(x^2-5x+6)$.

3. $\frac{9}{11}\log(x+6)+\frac{13}{11}\log(x-5)$.

4. $x+\log[(x-2)/(x+2)]$.

5. $x+\frac{16}{3}\log(x-4)-\frac{1}{3}\log(x-1)$.

6. $\frac{2}{5}\log(x-1)-\frac{1}{15}\log(3x+2)$.

7. $\log(x-1)-2/(x-1)$.

8. $5\log(x-2)-12/(x-2)$.

9. $\frac{7}{3}\log[(3+x)/(3-x)]-x$.

10. $x+\frac{5}{3}\log(x-2)-\frac{2}{3}\log(x+1)$.

11. $\frac{15}{8}\log(x-3)-\frac{13}{24}\log(3x-1)$.

12. $x-6\log(x+1)-9/(x+1)$.

13. $\frac{1}{2}x^2+\frac{1}{2}\log(x^2-1)$.

14. $\frac{1}{4}\log(2x-1)-\frac{1}{4}/(2x-1)$.

15. $\frac{1}{2}x^2-x+\frac{125}{9}\log(x+5)+\frac{64}{9}\log(x-4)$.

16. $\frac{1}{a-b}\log\frac{x-a}{\quad}$.

17. $\frac{1}{3}x^3+5x+\frac{5\sqrt{5}}{2}\log\frac{x-\sqrt{5}}{x+\sqrt{5}}$.

18. $-\frac{1}{2}x+\frac{1}{5}\log x-\frac{29}{20}\log(5-2x)$.

Examples XLII, p. 234.

1. $\frac{1}{3}\tan^{-1}\frac{1}{3}(x+1)$.

2. $\frac{1}{3}\tan^{-1}\frac{1}{3}(2x+1)$.

3. $\dfrac{1}{\sqrt{8}}\tan^{-1}\dfrac{x-2}{\sqrt{8}}$.

4. $\dfrac{1}{2\sqrt{2}}\log\dfrac{x-1-\sqrt{2}}{x-1+\sqrt{2}}$.

5. $\dfrac{1}{4\sqrt{7}}\log\dfrac{3x+4-2\sqrt{7}}{3x+4+2\sqrt{7}}$.

6. $\dfrac{1}{2\sqrt{5}}\log\dfrac{\sqrt{5}+1+x}{\sqrt{5}-1-x}$.

7. $\dfrac{1}{2\sqrt{34}}\log\dfrac{\sqrt{34}+2+3x}{\sqrt{34}-2-3x}$.

8. $\dfrac{1}{8\sqrt{2}}\log\dfrac{2x-1-2\sqrt{2}}{2x-1+2\sqrt{2}}$.

9. $\dfrac{1}{2\sqrt{35}}\log\dfrac{x\sqrt{5}-\sqrt{7}}{x\sqrt{5}+\sqrt{7}}$.

10. $x-2\tan^{-1}(x+1)$.

11. $x+\dfrac{1}{2\sqrt{5}}\log\dfrac{x+2-\sqrt{5}}{x+2+\sqrt{5}}$.

12. $\frac{1}{3}x^3-7x+7\sqrt{7}\tan^{-1}\dfrac{x}{\sqrt{7}}$.

13. $x+\dfrac{3}{\sqrt{5}}\log\dfrac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}$.

14. $\dfrac{1}{\sqrt{73}}\log\dfrac{\sqrt{73}+4x-3}{\sqrt{73}-4x+3}$.

15. $\dfrac{1}{2\sqrt{14a}}\,\dfrac{\sqrt{14a}+x-2a}{\sqrt{14a}-x+2a}$.

16. (i) $\dfrac{1}{\sqrt{(b^2-4ac)}}\log\dfrac{2ax+b-\sqrt{(b^2-4ac)}}{2ax+b+\sqrt{(b^2-4ac)}}$. (ii) $\dfrac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\dfrac{2ax+b}{\sqrt{(4ac-b^2)}}$.

Examples XLIII, p. 236.

1. $\log(x^2+3x-4)$.
2. $\frac{1}{3}\log(x^3-1)$.
3. $-\frac{1}{4}\log(a^4-x^4)$.
4. $\frac{1}{2}\log(x^2+2x+7)$.
5. $-\log\cos x$.
6. $(\log\sin ax)/a$.
7. $\frac{1}{6}\log(1+3\sin^2 x)$.
8. $\frac{1}{2}\log(x^2+a^2)$.
9. $-(1/b)\log(a+b\cos x)$.
10. $-\frac{1}{2}\log(1-e^{2x})$.
11. $\frac{1}{2}\log(ax^2+2bx+c)$.
12. $\frac{1}{4}\log(3+4\tan x)$.
13. $\log\cosh x$.
14. $\log(\log x)$.
15. $-[\log(1-x^n)]/n$.
16. $-\log(\sin x+\cos x)$.
17. $\log(1+xe^x)$.
18. $[\log(a+b\sin^2 x)]/b$.

Examples XLIV, p. 237.

1. $\frac{1}{2}\log(x^2+9)+\frac{1}{3}\tan^{-1}\frac{1}{3}x$.
2. $2\log(x^2-5)-\frac{3}{10}\sqrt{5}\log[(x-\sqrt{5})/(x+\sqrt{5})]$.
3. $\frac{1}{2}x^2-\frac{1}{2}a^2\log(x^2+a^2)$.
4. $\frac{1}{14}\sqrt{7}\log[(\sqrt{7}+x)/(\sqrt{7}-x)]+\frac{1}{2}\log(7-x^2)$.
5. $3\log(x^2+4x+13)-3\tan^{-1}\frac{1}{3}(x+2)$.
6. $2\log(x^2-2x-1)-\dfrac{\sqrt{2}}{4}\log\dfrac{x-1-\sqrt{2}}{x-1+\sqrt{2}}$.
7. $2\log(2x^2+2x+1)-7\tan^{-1}(2x+1)$.
8. $-\frac{1}{3}\log(3x^2+6x-1)+\dfrac{5\sqrt{3}}{12}\log\dfrac{x+1-2/\sqrt{3}}{x+1+2/\sqrt{3}}$.
9. $\frac{5}{2}\log(x^2-3x+5)+\dfrac{13}{\sqrt{11}}\tan^{-1}\dfrac{2x-3}{\sqrt{11}}$.

10. $x+\log(x^2-2x+5)-2\tan^{-1}\frac{1}{2}(x-1)$.
11. $\frac{1}{2}x^2+6x+13\log(x^2-6x+10)+18\tan^{-1}(x-3)$.
12. $\frac{1}{8}\log(4x^2-10x-3)+\frac{}{8\sqrt{37}}\log\frac{4x-5-\sqrt{37}}{4x-5+\sqrt{37}}$.
13. $x-\log(x^2+x+1)+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$.
14. $x-\frac{1}{2}\log(x^2-2x+3)-\sqrt{2}\tan^{-1}\{(x-1)/\sqrt{2}\}$.
15. $x-8\log(x+3)+3\log(x+2)$.
16. $\frac{1}{2}\log(x^2+ax+a^2)-\sqrt{\frac{1}{3}}\tan^{-1}[(2x+a)/\sqrt{3}a]$.
17. $\frac{1}{2}x^2-\frac{1}{2}\log(x^2+1)+\tan^{-1}x$.
18. $\frac{1}{2}\log(x^2+2ax-a^2)-\frac{1}{2}\sqrt{2}\log[(x+a-\sqrt{2}a)/(x+a+\sqrt{2}a)]$.

Examples XLV, p. 239.

1. $\log(x-1)-\log x+1/x$.
2. $\log x-\frac{1}{2}\log(x^2+1)$.
3. $\frac{1}{2}\log\{x(x-2)/(x-1)^2\}$.
4. $\frac{1}{5}\log(x-1)+\frac{2}{5}\log(x^2+4)+\frac{2}{5}\tan^{-1}\frac{1}{2}x$.
5. $\frac{1}{6}\log(x-1)+\frac{1}{2}\log(x+1)-\frac{1}{6}\log(2x+1)$.
6. $x+\frac{5}{4}\log(x-1)-\frac{1}{4}\log(x+1)-\frac{1}{2}/(x-1)$.
7. $-1/x-\tan^{-1}x$.
8. $\frac{1}{4}\log(x-1)-\frac{1}{4}\log(x+1)-\frac{1}{2}\tan^{-1}x$.
9. $-\frac{1}{2}/(x^2-1)$.
10. $\frac{1}{4}\log\{(x-1)/(x+1)\}-\frac{1}{2}x/(x^2-1)$.
11. $\log(x+1)-\frac{1}{2}\log(x^2+2x+2)$.
12. $\frac{1}{3}\log(x-1)-\frac{1}{6}\log(x^2+x+1)-\sqrt{\frac{1}{3}}\tan^{-1}\sqrt{\frac{1}{3}}(2x+1)$.
13. $\frac{1}{6}\log\{(x^2-1)/(x^2+2)\}$.
14. $x-\frac{1}{3}\log(1+x)+\frac{1}{6}\log(1-x+x^2)-\sqrt{\frac{1}{3}}\tan^{-1}\sqrt{\frac{1}{3}}(2x-1)$.
15. $\log\{x/(1-x)\}-1/x-\frac{1}{2}/x^2$.
16. $\log(x-2)-\frac{8}{9}\log(x-1)-\frac{1}{9}\log(x+2)+\frac{1}{3}/(x-1)$.
17. $\frac{1}{4}\log\{(1+x)/(1-x)\}-\frac{1}{2}\tan^{-1}x$.
18. $-\frac{1}{6}\log(x+2)+\frac{1}{12}\log(x^2-2x+4)+\frac{1}{6}\sqrt{3}\tan^{-1}\sqrt{\frac{1}{3}}(x-1)$.
19. $\tan^{-1}x-\sqrt{\frac{1}{2}}\tan^{-1}x\sqrt{\frac{1}{2}}$.
20. $\log\{(1+x+x^2)/x^2\}-1/x$.
21. $\frac{1}{16}\log\{(x^2+2x+2)/(x^2-2x+2)\}+\frac{1}{8}\tan^{-1}(x+1)+\frac{1}{8}\tan^{-1}(x-1)$.
22. $\frac{11}{12}\log(x-2)-\frac{7}{6}\log(x-1)+\frac{5}{12}\log(x+2)-\frac{1}{6}\log(x+1)$.
23. $\frac{1}{4}\log\{(1-x+x^2)/(1+x+x^2)\}+\frac{1}{6}\sqrt{3}\tan^{-1}\{x\sqrt{3}/(1-x^2)\}$.
24. $\frac{1}{2}\log(x-1)-\log x+\frac{1}{4}\log(x^2+1)+1/x+\frac{1}{2}\tan^{-1}x$.
25. $\frac{1}{2}x^2+\frac{1}{3}a^2\log(x-a)-\frac{1}{6}a^2\log(x^2+ax+a^2)+\frac{1}{3}\sqrt{3}a^2\tan^{-1}\{(2x+a)/\sqrt{3}a\}$.
26. $\frac{1}{2}x-\frac{27}{44}\sqrt{2}\tan^{-1}\frac{1}{3}\sqrt{2}x+\frac{1}{22}\log\{(x-1)/(x+1)\}$.
27. $\frac{1}{6}\log\{(x^3-1)/(x^3+1)\}$.

Examples XLVI, p. 241.

1. $\sinh^{-1}\frac{1}{3}(x+1)$ or $\log[x+1+\sqrt{(x^2+2x+10)}]$.
2. $\cosh^{-1}\frac{1}{6}(x+5)$ or $\log[x+5+\sqrt{(x^2+10x-11)}]$.
3. $\sin^{-1}\frac{1}{4}(x+3)$.
4. $\sin^{-1}\frac{1}{2}(x-2)$.
5. $\cosh^{-1}(2x+1)$.
6. $\cosh^{-1}(2x-7)$.
7. $\cosh^{-1}\frac{1}{7}(2x+3)$.
8. $\frac{1}{6}\sqrt{2}\sinh^{-1}\frac{1}{5}(6x-7)$.
9. $\sqrt{\frac{1}{2}}\sin^{-1}\frac{1}{3}(4x-3)$.
10. $\sqrt{\frac{1}{2}}\cosh^{-1}\frac{1}{3}(4x-7)$.
11. $\sin^{-1}\{(2x-3)/\sqrt{41}\}$.
12. $\frac{1}{2}\cosh^{-1}\{(9x-2a)/2a\}$.

Examples XLVII, p. 242.

1. $\sqrt{(x^2+5)}$.
2. $\sqrt{(x^2-1)}+\cosh^{-1}x$.
3. $-2\sqrt{(4-x^2)}-\sin^{-1}\frac{1}{2}x$.
4. $\sqrt{(x+x^2)}-\frac{1}{2}\cosh^{-1}(2x+1)$.
5. $-\sqrt{(4-3x-x^2)}-\frac{3}{2}\sin^{-1}\frac{1}{5}(2x+3)$.
6. $2\sqrt{(x^2+5x+6)}-2\cosh^{-1}(2x+5)$.
7. $\sin^{-1}x+\sqrt{(1-x^2)}$.
8. $\frac{1}{2}\sqrt{(2x^2+x-3)}+\frac{3}{8}\sqrt{2}\cosh^{-1}\frac{1}{5}(4x+1)$.
9. $\sqrt{(x^2+2x)}+\cosh^{-1}(x+1)$.
10. $\sqrt{(3x^2+4x+7)}-2\sqrt{3}\sinh^{-1}\{(3x+2)/\sqrt{17}\}$.
11. $\sqrt{(6+x-x^2)}+\frac{5}{2}\sin^{-1}\frac{1}{5}(2x-1)$.
12. $\frac{1}{5}\sqrt{(5x^2-4x)}+\frac{2}{25}\sqrt{5}\cosh^{-1}\frac{1}{2}(5x-2)$.

Examples XLVIII, p. 245.

1. $\frac{1}{3}\sin^3 x$.
2. $-\frac{1}{4}\cos^4 x$.
3. $\frac{1}{3}(a^2+x^2)^{3/2}$.
4. $\sec x$.
5. $-\frac{1}{3}\operatorname{cosec}^3 x$.
6. $\sqrt{(x^2-a^2)}$.
7. $-\frac{1}{4}/(x^4-1)$.
8. $1/\{2(n-1)(a^2-x^2)^{n-1}\}$.
9. $-\frac{2}{3}\sqrt{(a^3-x^3)}$.
10. $\frac{1}{2}(a^2+x^2)^{n+1}/(n+1)$.
11. $-\frac{1}{3}(a^3-x^3)^{n+1}/(n+1)$.
12. $-(a-bx^n)^3/(3bn)$.
13. $-\frac{1}{3}\log(a^3-x^3)$.
14. $\dfrac{1}{2a^2}\tan^{-1}\dfrac{x^2}{a^2}$.
15. $\dfrac{1}{6a^3}\log\dfrac{x^3-a^3}{x^3+a^3}$.
16. $-\frac{1}{2}\sqrt{(a^4-x^4)}$.
17. $\frac{1}{2}\sin^{-1}(x^2/a^2)$.
18. $\frac{1}{4}\sin^{-1}(x^4/a^4)$.
19. $\frac{1}{15}(x^3-2)^5$.
20. $\tan^{-1}(\sin x)$.
21. $\log(1+e^x)$.
22. $\frac{1}{2}\log\{(1+e^x)/(1-e^x)\}$.
23. $\frac{1}{4}\log\{(1-2\cos x)/(1+2\cos x)\}$.
24. $-(1-\sin x)^{n+1}/(n+1)$.
25. $\frac{1}{2}(\log x)^2$.
26. $(\log x)^{n+1}/(n+1)$.
27. $\frac{1}{2}\tan^2 x$.
28. $\tan^{n+1}x/(n+1)$.
29. $\log(1+\tan x)$.
30. $\frac{1}{2}\log\{(1+\tan x)/(1-\tan x)\}$.
31. $\frac{1}{3}(1+\log x)^3$.
32. $-(1-e^x)^{n+1}/(n+1)$.
33. $2\sin\sqrt{x}$.
34. $\frac{1}{2}(\sin^{-1}x)^2$.
35. $\frac{1}{2}\cosh^{-1}\frac{1}{4}x^2$.
36. $\sin^{-1}(\frac{1}{2}\sqrt{2}\sin x)$.
37. $-1/\{b(a-b\cos x)\}$.
38. $-1/(1+\log x)$.
39. $\frac{1}{2}\sin^{-1}\frac{1}{7}(2x^2+5)$.
40. $\frac{1}{3}\tan^{-1}(x^3+2)$.

Examples XLIX, p. 248.

1. $-\frac{2}{3}(x+2)\sqrt{(1-x)}$.
2. $-\frac{2}{15}(8a^2+4ax+3x^2)\sqrt{(a-x)}$.
3. $2\sqrt{(1+x)}-\log x+\log\{2+x-2\sqrt{(1+x)}\}$.
4. $2\sqrt{(x+2)}-4\tan^{-1}\frac{1}{2}\sqrt{(x+2)}$.
5. $2\log(1-\sqrt{x})+2\sqrt{x}$.
6. $x-2\sqrt{x}+2\log(1+\sqrt{x})$.
7. $\log\{x+\sqrt{(1-x)}\}-\dfrac{1}{\sqrt{5}}\log\dfrac{\sqrt{5}+2\sqrt{(1-x)}-1}{\sqrt{5}-2\sqrt{(1-x)}+1}$.
8. $\frac{2}{15}(3x-4)(x+2)^{3/2}$.
9. $\frac{2}{105}(ax+b)^{3/2}(15a^2x^2-12abx+8b^2)/a^3$.

10. $2\sqrt{x}-x+\frac{2}{3}x\sqrt{x}-2\log(1+\sqrt{x})$.   11. $\frac{2}{3}\sqrt{3}\tan^{-1}\sqrt{(\frac{1}{3}x)}$.

12. $\dfrac{1}{a}\log\dfrac{\sqrt{(a+x)}-\sqrt{a}}{\sqrt{(a+x)}+\sqrt{a}}$.   13. $\dfrac{x}{a^2\sqrt{(a^2+x^2)}}$.

14. $-\frac{1}{4}x/\sqrt{(x^2-4)}$.   15. $-\operatorname{cosech}^{-1}x$.

16. $-\sinh^{-1}\{(2+x)/\sqrt{3}x\}$.

18. $-\frac{1}{2}\sqrt{2}\sinh^{-1}\{(1-x)/(1+x)\}$.   19. $-\sqrt{\frac{1}{7}}\cosh^{-1}\{(14-5x)/9x\}$.

20. $(x+2)/\sqrt{(x^2+4x+5)}$.

21. $\frac{1}{2}\log x-\log\{\sqrt{(1+x)}-1\}-\sqrt{(1+x)}/x$.

22. $\sqrt{x}-\sqrt{\frac{3}{2}}\tan^{-1}\sqrt{(\frac{2}{3}x)}$.   23. $-(1/a)\cosh^{-1}(a/x)$.

24. $3\log\{(1+\sqrt{x})/(1-\sqrt{x})\}-6\sqrt{x}$.   25. $\frac{1}{6}\log\{x^2/(2x^2+3)\}$.

26. $(1/3a^3)\log\{x^3/(x^3+a^3)\}$.   27. $(1/n)\log\{x^n/(1-x^n)\}$.

28. $\frac{1}{12}\log\{x^4/(3-2x^4)\}$.   29. $\frac{1}{2}\log\{x^2/(1+x^2)\}+\frac{1}{2}/(1+x^2)$.

30. $\frac{3}{2}\log\{\sqrt[3]{(1+x)}-1\}-\frac{1}{2}\log x-\sqrt{3}\tan^{-1}\sqrt{\frac{1}{3}}\{2\sqrt[3]{(1+x)}+1\}$.

Examples L, p. 252.

1. $-\frac{1}{2}\log\cos 2x$.   2. $(\log\sin mx)/m$.

3. $2\log\tan\frac{1}{4}(\pi+x)$.   4. $\frac{1}{3}\log\tan\frac{3}{2}x$.

5. $a\log\tan(\frac{1}{2}x/a)$.   6. $3\tan\frac{1}{3}x-x$.

7. $-(\cot nx)/n$.   8. $\frac{1}{3}\cos^3 x-\cos x$.

9. $\sin x-\frac{1}{3}\sin^3 x$.   10. $\frac{1}{7}\cos^7 x-\frac{1}{5}\cos^5 x$.

11. $(\sin^{n+1}x)/(n+1)-(\sin^{n+3}x)/(n+3)$.   12. $\tan x+\frac{2}{3}\tan^3 x+\frac{1}{5}\tan^5 x$

13. $\frac{1}{3}\sec^3 x-\sec x$.   14. $\frac{1}{3}\sin^3 x-2\sin x-\operatorname{cosec} x$.

15. $\frac{1}{3}\tan^3 x$.   16. $\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$.

17. $\frac{1}{3}\tan^3 x-\tan x+x$.   18. $-\cot x-\frac{1}{3}\cot^3 x$.

19. $\frac{1}{5}\tan^5 x-\frac{1}{3}\tan^3 x+\tan x-x$.   20. $\log\tan x$.

21. $\frac{1}{2}\tan^2 x+\log\cos x$.   22. $\log\tan x+\frac{1}{2}\tan^2 x$.

23. $-2\cot 2x$.   24. $\operatorname{cosec}^2 x-\frac{1}{4}\operatorname{cosec}^4 x+\log\sin x$.

25. $\frac{1}{8}x-\frac{1}{32}\sin 4x$.   26. $\frac{1}{9}\tan^9 x$.

27. $\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$.   28. $-\frac{1}{3}\cot^3 x-\frac{1}{5}\cot^5 x$.

29. $\tan\frac{1}{2}x$.   30. $-\cot\frac{1}{2}x$.

31. $\tan x-\sec x$.   32. $\tan x+\sec x$.

33. $\log\tan(\frac{1}{4}\pi+\frac{1}{2}x)-\operatorname{cosec} x$.   34. $-\frac{1}{6}\cos 3x-\frac{1}{10}\cos 5x$.

35. $\frac{1}{10}\sin 5x+\frac{1}{2}\sin x$.   36. $-\dfrac{\cos(m+n)x}{2(m+n)}-\dfrac{\cos(m-n)x}{2(m-n)}$.

37. $\dfrac{\sin(p-q)x}{2(p-q)}-\dfrac{\sin(p+q)x}{2(p+q)}$.   38. $\frac{1}{6}\sin 3x-\frac{1}{20}\sin 5x-\frac{1}{4}\sin x$.

39. $-\dfrac{\cos mx}{2m}-\dfrac{\cos(m+2)x}{4(m+2)}-\dfrac{\cos(m-2)x}{4(m-2)}$

40. $\frac{1}{4}\log\tan(\frac{1}{4}\pi+x)+\frac{1}{2}x$.   41. $-\frac{1}{2}\log\cos x$.

42. $\dfrac{1}{3}\log\dfrac{3+\tan\frac{1}{2}x}{3-\tan\frac{1}{2}x}$.   43. $\dfrac{1}{84}\log\dfrac{13\tan\frac{1}{2}x+1}{\tan\frac{1}{2}x+13}$.

44. $\frac{1}{3}\tan^{-1}(\frac{1}{3}\tan x)$.   45. $\frac{1}{5}\tan^{-1}(\frac{1}{5}\tan x)$.

46. $\frac{1}{6}\tan^{-1}(\frac{3}{2}\tan x)$.   47. $\frac{1}{2}x+\frac{1}{2}\log(\cos x+\sin x)$.

48. $\frac{1}{2}\tan^2 x+4\log\tan x-\frac{1}{6}\cot^6 x-\cot^4 x-3\cot^2 x$.

49. $\frac{2}{3}\sqrt{3}\tan^{-1}\{(2\tan\frac{1}{2}x+1)/\sqrt{3}\}$.   50. $\frac{1}{2}\tan^{-1}(2\tan\frac{1}{2}x)$

Examples LI, p. 255.

1. $\frac{1}{2}x\sqrt{(9-x^2)}+\frac{9}{2}\sin^{-1}\frac{1}{3}x$.
2. $\frac{1}{2}x\sqrt{(x^2-a^2)}-\frac{1}{2}a^2\cosh^{-1}(x/a)$.
3. $\frac{1}{2}x\sqrt{(1+x^2)}+\frac{1}{2}\sinh^{-1}x$.
4. $\frac{1}{2}x\sqrt{(x^2-4)}-2\cosh^{-1}\frac{1}{2}x$.
5. $-\sqrt{(25-x^2)}/x-\sin^{-1}\frac{1}{5}x$.
6. $-\sqrt{(1-x^2)}/x$.
7. $\sinh^{-1}x-\sqrt{(1+x^2)}/x$.
8. $\frac{1}{2}a^2\sin^{-1}(x/a)-\frac{1}{2}x\sqrt{(a^2-x^2)}$.
9. $\frac{1}{2}x\sqrt{(9+x^2)}-\frac{9}{2}\sinh^{-1}\frac{1}{3}x$.
10. $\frac{1}{2}x\sqrt{(x^2-a^2)}+\frac{1}{2}a^2\cosh^{-1}(x/a)$.
11. $\frac{3}{2}a^2\sin^{-1}(x/a)-\frac{1}{2}x\sqrt{(a^2-x^2)}$.
12. $-\sqrt{(a^2+x^2)}/a^2x$.
13. $-\frac{1}{3}(x^2+2)\sqrt{(1-x^2)}$.
14. $\frac{1}{3}(x^2+2a^2)\sqrt{(a^2+x^2)}$.
15. $x/\{a^2\sqrt{(a^2-x^2)}\}$.
16. $x/\sqrt{(1-x^2)}-\sin^{-1}x$.
17. $\sin^{-1}\sqrt{(x-1)}-\sqrt{(3x-2-x^2)}$.
18. $3\sin^{-1}\sqrt{\{\frac{1}{3}(x-2)\}}+\sqrt{(7x-10-x^2)}$.
19. $4\sin^{-1}\frac{1}{2}\sqrt{(x-3)}-\frac{1}{2}(5-x)\sqrt{(10x-21-x^2)}$.
20. $\frac{25}{4}\sin^{-1}\sqrt{\{\frac{1}{5}(x+1)\}}-\frac{1}{4}(3-2x)\sqrt{(4+3x-x^2)}$.
21. $\sin^{-1}\frac{1}{3}(2x-5)$.
22. $\sin^{-1}\{(2x-\alpha-\beta)/(\beta-\alpha)\}$.
23. $\frac{1}{4}(\beta-\alpha)^2\sin^{-1}\sqrt{\{(x-\alpha)/(\beta-\alpha)\}}-\frac{1}{4}(\alpha+\beta-2x)\sqrt{\{(x-\alpha)(\beta-x)\}}$.
24. $4a^2\sin^{-1}\sqrt{\{\frac{1}{4}(x-2a)/a\}}-\frac{1}{2}(4a-x)\sqrt{(8ax-12a^2-x^2)}$.
25. $(\beta-\alpha)\sin^{-1}\sqrt{\{(x-\alpha)/(\beta-\alpha)\}}-\sqrt{\{(x-\alpha)(\beta-x)\}}$.
26. $8\sin^{-1}\sqrt{\{\frac{1}{8}(x+4)\}}+\sqrt{(16-x^2)}$.
27. $(\alpha-\beta)\sin^{-1}\sqrt{\{(x-\beta)/(\alpha-\beta)\}}+\sqrt{\{(x-\beta)(\alpha-x)\}}$.
28. $\frac{1}{2}\tan^{-1}x+\frac{1}{2}x/(1+x^2)$.
29. $\frac{1}{2}\tan^{-1}(x+2)+\frac{1}{2}(x+2)/(x^2+4x+5)$.
30. $\frac{1}{729}\tan^{-1}\frac{1}{9}(2x-3)+\frac{1}{162}(2x-3)/(2x^2-6x+45)$.
31. $\frac{1}{4}(x^2-2)/(x^2+2x+2)-\frac{1}{2}\tan^{-1}(x+1)$.
32. $\frac{1}{2}\tan^{-1}x-\frac{1}{2}x/(x^2+1)$.

Examples LII, p. 258.

1. $\frac{1}{5}x^5(\log x-\frac{1}{5})$.
2. $\frac{2}{3}x^{3/2}(\log x-\frac{2}{3})$.
3. $(x^{m+1}\log x)/(m+1)-x^{m+1}/(m+1)^2$.
4. $-\frac{1}{2}(\log x+\frac{1}{2})/x^2$.
5. $x\sin x+\cos x$.
6. $-(x\cos mx)/m+(\sin mx)/m^2$.
7. $e^x(x-1)$.
8. $-e^{-ax}(ax+1)/a^2$.
9. $\frac{1}{4}(x^4-1)\tan^{-1}x+\frac{1}{4}x-\frac{1}{12}x^3$.
10. $\frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\log(x^2+1)$.
11. $x\sin^{-1}x+\sqrt{(1-x^2)}$.
12. $x\log x-x$.
13. $x\tan x+\log\cos x$.
14. $(\log\sin mx)/m^2-(x\cot mx)/m$.
15. $\frac{1}{4}(2x^2-1)\sin^{-1}x+\frac{1}{4}x\sqrt{(1-x^2)}$.
16. $x\cosh x-\sinh x$.
17. $ax\sinh(x/a)-a^2\cosh(x/a)$.
18. $x\sinh^{-1}x-\sqrt{(1+x^2)}$.
19. $x\cosh^{-1}x-\sqrt{(x^2-1)}$.
20. $-x^2\cos x+2x\sin x+2\cos x$.
21. $2x^2\sin\frac{1}{2}x+8x\cos\frac{1}{2}x-16\sin\frac{1}{2}x$.
22. $(x^3-3x^2+6x-6)e^x$.
23. $-(x^2+2x+2)e^{-x}$.
24. $-\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x$.

Examples LIII, p. 259.

1. $\frac{1}{2}x\sqrt{(x^2-a^2)}-\frac{1}{2}a^2\cosh^{-1}(x/a)$.
2. $\frac{1}{2}x\sqrt{(a^2-x^2)}+\frac{1}{2}a^2\sin^{-1}(x/a)$.
3. $\frac{1}{2}x\sqrt{(32+2x^2)}+8\sqrt{2}\sinh^{-1}\frac{1}{4}x$.
4. $\frac{1}{2}x\sqrt{(12-3x^2)}+2\sqrt{3}\sin^{-1}\frac{1}{2}x$.
5. $\frac{1}{2}(x+1)\sqrt{(x^2+4x+5)}+2\sinh^{-1}\frac{1}{2}(x+1)$.
6. $\frac{1}{4}(2x+5)\sqrt{(6-5x-x^2)}+\frac{49}{8}\sin^{-1}\frac{1}{7}(2x+5)$.
7. $\frac{1}{6}(3x+2)\sqrt{(3x^2+4x-7)}-\frac{25}{18}\sqrt{3}\cosh^{-1}\frac{1}{5}(3x+2)$.
8. $\frac{1}{12}(6x+5)\sqrt{(8-5x-3x^2)}+\frac{121}{72}\sqrt{3}\sin^{-1}\frac{1}{11}(6x+5)$.

9. $\frac{1}{6}(3x-1)\sqrt{(3x^2-2x)}-\frac{1}{18}\sqrt{3}\cosh^{-1}(3x-1)$.
10. $\frac{1}{16}(8x-5)\sqrt{(5x-4x^2)}+\frac{25}{64}\sin^{-1}\frac{1}{5}(8x-5)$.
11. $\frac{1}{13}e^{3x}(2\sin 2x+3\cos 2x)$.
12. $\frac{1}{29}e^{2x}(2\sin 5x-5\cos 5x)$.
13. $\frac{4}{5}e^{-x}(\frac{1}{2}\sin\frac{1}{2}x-\cos\frac{1}{2}x)$.
14. $-\frac{1}{2}(\sin ax+\cos ax)e^{-ax}/a$.
15. $\frac{1}{10}e^{x}(5+2\sin 2x+\cos 2x)$.
16. $\frac{1}{8}e^{2x}(2-\sin 2x-\cos 2x)$.
17. $\frac{1}{2}(\sinh x\sin x-\cosh x\cos x)$.
18. $\frac{1}{2}(\cosh x\cos x+\sinh x\sin x)$.
19. $\frac{1}{2}(\cosh x\sin x-\sinh x\cos x)$.
20. $-Le^{-Rt/L}(R\sin pt+pL\cos pt)/(R^2+p^2L^2)$.
21. $Le^{-Rt/L}\{pL\sin(pt+\epsilon)-R\cos(pt+\epsilon)\}/(R^2+p^2L^2)$.

Examples LIV, p. 264.

1. $e^{ax}(x^3a^3-3x^2a^2+6xa-6)/a^4$.
2. $-e^{-x}(x^4+4x^3+12x^2+24x+24)$.
3. $\frac{1}{6}x(3-2x^2)\cos 2x+\frac{1}{4}(2x^2-1)\sin 2x$.
4. $(x^3-6x)\sin x+(3x^2-6)\cos x$.
5. $-(x^4-12x^2+24)\cos x+(4x^3-24x)\sin x$.
6. $\frac{1}{27}x^3\{9(\log x)^2-6\log x+2\}$.
7. $\frac{1}{32}x^4\{8(\log x)^2-4\log x+1\}$.
8. $(x^2+2)\sinh x-2x\cosh x$.
9. $(x^3+6x)\cosh x-(3x^2+6)\sinh x$.
10. $\frac{1}{3}\tan^3\theta-\tan\theta+\theta$.
11. $-\frac{1}{5}\cot^5\theta+\frac{1}{3}\cot^3\theta-\cot\theta-\theta$.
12. $\frac{1}{7}\tan^7\theta-\frac{1}{5}\tan^5\theta+\frac{1}{3}\tan^3\theta-\tan\theta+1$.
13. $\frac{1}{4}\tan^4\theta$.
14. $\frac{1}{2}\tan^2\theta+\log\tan\theta$.
15. $\frac{1}{3}\tan^3\theta+3\tan\theta-3\cot\theta-\frac{1}{3}\cot^3\theta$.
16. $2\sqrt{\tan\theta}$.
17. $\frac{5}{16}\theta-\frac{5}{16}\cos\theta\sin\theta-\frac{5}{24}\cos\theta\sin^3\theta-\frac{1}{6}\cos\theta\sin^5\theta$.
18. $\frac{1}{6}\sin^3\theta\cos^3\theta+\frac{1}{8}\sin^3\theta\cos\theta-\frac{1}{16}\sin\theta\cos\theta+\frac{1}{16}\theta$.
19. $\frac{1}{4}\sin\theta\cos^3\theta+\frac{3}{8}\sin\theta\cos\theta+\frac{3}{8}\theta$.
20. $\frac{1}{2}\tan\theta\sec\theta+\frac{1}{2}\log(\tan\theta+\sec\theta)$.
21. $-\cot\theta-\frac{1}{3}\cot^3\theta$.
22. $\log(\sec\theta+\tan\theta)-\operatorname{cosec}\theta$.

23. $$I_{m,n}=\frac{\sin^{m-1}\theta\cos^{n+1}\theta}{m+n}+\frac{m-1}{m+n}I_{m-2,n}.$$

24. $\int\sin^6\theta\cos^2\theta\,d\theta=-\frac{1}{8}\sin^5\theta\cos^3\theta+\frac{5}{8}\int\sin^4\theta\cos^2\theta\,d\theta$.

25. $$I_{m,n}=\frac{\sin^{m-1}\theta\cos^{n+1}\theta}{n+1}+\frac{m-1}{n+1}I_{m-2,n+2}.$$

26. $$\int\frac{\sin^6\theta}{\cos^4\theta}d\theta=\frac{1}{3}\frac{\sin^5\theta}{\cos^3\theta}-\frac{5}{3}\int\frac{\sin^4\theta}{\cos^2\theta}d\theta.$$

27. $$I_{m,n}=\frac{\sin^{m+1}\theta\cos^{n-1}\theta}{m+1}+\frac{n-1}{m+1}I_{m+2,n-2}$$

28. $$\int\frac{\cos^5\theta}{\sin^3\theta}d\theta=-\frac{1}{2}\frac{\cos^4\theta}{\sin^2\theta}-2\int\frac{\cos^3\theta}{\sin\theta}d\theta.$$

29. $$=-\frac{\sin^{m+1}\theta\cos^{n+1}\theta}{n+1}+\frac{m+n+2}{n+1}I_{m,n+2}.$$

30. $$\int\frac{d\theta}{\sin\theta\cos^3\theta}=\frac{1}{2\cos^2\theta}+\int\frac{d\theta}{\sin\theta\cos\theta}.$$

31. $$I_{m,n}=\frac{\sin^{m+1}\theta\cos^{n+1}\theta}{m+1}+\frac{m+n+2}{m+1}I_{m+2,n}.$$

32. $$\int\frac{d\theta}{\sin^4\theta\cos^2\theta}=-\frac{}{3\sin^3\theta\cos\theta}+\frac{4}{3}\int\frac{d\theta}{\sin^2\theta\cos^2\theta}.$$

Miscellaneous Examples, LV, p. 265.

1. $-\frac{1}{4}\log(1-4x)$.
2. $\frac{1}{4}\log\{(1+2x)/(1-2x)\}$.
3. $-\frac{1}{2}\sqrt{(1-4x)}$.
4. $-\frac{1}{4}(1-4x)^{n+1}/(n+1)$.
5. $-\frac{1}{12}(1-4x^2)^{3/2}$.
6. $\frac{1}{2}x\sqrt{(1-4x^2)}+\frac{1}{4}\sin^{-1}2x$.
7. $-\frac{1}{4}\sqrt{(1-4x^2)}$.
8. $\frac{1}{2}\sin^{-1}2x$.
9. $1/\{4(1-4x)\}$.
10. $-\frac{1}{8}(1-4x^2)^{n+1}/(n+1)$.
11. $-\frac{1}{4}n(1-4x)^{(n+1)/n}/(n+1)$.
12. $-\frac{1}{18}(1-4x^3)^{3/2}$.
13. $-\frac{1}{8}x\sqrt{(1-4x^2)}+\frac{1}{16}\sin^{-1}2x$.
14. $-\frac{1}{12}(2x+1)\sqrt{(1-4x)}$.
15. $-\frac{1}{8}x^2-\frac{1}{16}x-\frac{1}{64}\log(1-4x)$.
16. $-\frac{1}{8}\sqrt{(1-4x^4)}$.
17. $\frac{1}{4}\sin^{-1}2x^2$.
18. $\frac{1}{16}\{(1-4x)^{n+2}/(n+2)-(1-4x)^{n+1}/(n+1)\}$.
19. $\frac{1}{16}\{1/(1-4x)+\log(1-4x)\}$.
20. $\frac{1}{16}\{x+\frac{1}{2}\log(1-4x)+\frac{1}{4}/(1-4x)\}$.
21. $\frac{1}{16}\log\{(1+2x)/(1-2x)\}-\frac{1}{4}x$.
22. $\frac{1}{64}\{\sin^{-1}2x-2x(1-8x^2)\sqrt{(1-4x^2)}\}$.
23. $\frac{1}{3}x^3-2x^4+\frac{16}{5}x^5$.
24. $-\frac{1}{420}(1+6x+30x^2)(1-4x)^{3/2}$.
25. $x/\sqrt{(1-4x^2)}$.
26. $\frac{1}{4}(1-4x^2)^{-\frac{1}{2}}$.
27. $1/\{8(n-1)(1-4x^2)^{n-1}\}$.
28. $\frac{1}{2}(1-4x)^{-1/2}$.
29. $-\frac{1}{60}(1+2x+6x^2)\sqrt{(1-4x)}$.
30. $\frac{1}{32}\{\log(1-4x^2)+1/(1-4x^2)\}$.
31. $\log\{x/(1-4x)\}$.
32. $\log\{x/\sqrt{(1-4x^2)}\}$.
33. $\log\{x/(1-4x)\}+1/(1-4x)$.
34. $4\log\{x/(1-4x)\}-1/x$.
35. $\log\{(1+2x)/(1-2x)\}-1/x$.
36. $\frac{1}{2}\log\{x^2(1-2x)/(1+2x)^3\}$.
37. $x+\log x-\frac{3}{4}\log(1-4x)$.
38. $-\frac{1}{24}(1+2x^2)\sqrt{(1-4x^2)}$.
39. $1/\{8(1-4x^2)\}$.
40. $-\frac{1}{60}(1+6x)(1-4x)^{3/2}$.
41. $\frac{1}{2}x-\frac{1}{4}(\sin 2ax)/a$.
42. $2\sin\frac{1}{2}x-\frac{2}{3}\sin^3\frac{1}{2}x$.
43. $-\frac{1}{8}\cos 4x$.
44. $-\frac{1}{5}\cos^5 x$.
45. $\frac{1}{4}\sin^4 x$.
46. $\frac{1}{4}\sin^4 x-\frac{1}{6}\sin^6 x$.
47. $\frac{1}{8}x-\frac{1}{32}\sin 4x$.
48. $\frac{1}{2}\cos x-\frac{1}{6}\cos 3x$.
49. $\frac{1}{2}\sin x+\frac{1}{6}\sin 3x$.
50. $\frac{1}{2}\sin x-\frac{1}{6}\sin 3x$.
51. $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x$.
52. $\frac{1}{2}\tan 2x-x$.
53. $-2\cot\frac{1}{2}x-x$.
54. $\frac{1}{2}\tan x\sec x-\frac{1}{2}\log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$.
55. $\frac{1}{3}\tan^3 x-\tan x+x$.
56. $\frac{1}{2}\tan^2 x$.
57. $\frac{1}{3}\sec^3 x$.
58. $-\operatorname{cosec} x$.
59. $\frac{1}{2}\log\tan x$.
60. $\frac{1}{2}\log\tan(\frac{1}{4}\pi+x)$.
61. $(\sin nx)/n^2-(x\cos nx)/n$.
62. $2x\sin\frac{1}{2}x+4\cos\frac{1}{2}x$.
63. $(x\tan mx)/m+(\log\cos mx)/m^2$.
64. $x\tan x+\log\cos x-\frac{1}{2}x^2$
65. $(2-x^2)\cos x+2x\sin x$.
66. $-\frac{1}{4}(2x+1)e^{-2x}$.
67. $(x^2-2x+2)e^x$.
68. $\frac{1}{2}e^{x^2}$.
69. $(a-b+bx)e^x$.
70. $\frac{1}{7}x^7(\log x-\frac{1}{7})$.
71. $\frac{1}{2}(x^2-1)\log(1+x)-\frac{1}{4}x^2+\frac{1}{2}x$.
72. $\frac{1}{2}(\log x)^2$.
73. $-\{(n-1)\log x+1\}/\{(n-1)^2x^{n-1}\}$.
74. $-(a-x)\log(a-x)-x$.
75. $\frac{1}{2}(1+x^2)\log(1+x^2)-\frac{1}{2}x^2$.
76. $-\frac{1}{26}(\sin 5x+5\cos 5x)e^{-x}$.
77. $\frac{1}{6}(\sin 3x+\cos 3x)e^{3x}$.
78. $\frac{1}{10}(\sin 2x-2\cos 2x)e^x$.
79. $-\frac{1}{10}(2\sin 2x-\cos 2x+5)e^{-x}$.
80. $\frac{1}{4}(x^4-1)\tan^{-1}x-\frac{1}{12}x^3+\frac{1}{4}x$.
81. $\frac{1}{4}(2x^2-1)\cos^{-1}x-\frac{1}{4}x\sqrt{(1-x^2)}$.
82. $x\sec^{-1}x-\cosh^{-1}x$.
83. $\frac{1}{2}x^2\operatorname{cosec}^{-1}x+\frac{1}{2}\sqrt{(x^2-1)}$.
84. $(\log\cosh ax)/a$.
85. $-\log(\sin x+\cos x)$.
86. $\log\{x/\sqrt{(x^2+1)}\}$.

87. $\frac{1}{6}\log\{x^3/(x^3+2)\}$. 88. $\frac{1}{2}\log(x^2+1)$.
89. $\sqrt{(x^2+1)}$. 90. $\log[\{\sqrt{(x^2+1)}-1\}/x]$ or $-\operatorname{cosech}^{-1}x$.
91. $-\sqrt{(x^2+1)}/x$. 92. $x-\tan^{-1}x$.
93. $\frac{1}{2}x\sqrt{(1+x^2)}-\frac{1}{2}\sinh^{-1}x$. 94. $\sinh^{-1}x-\sqrt{(1+x^2)}/x$.
95. $\frac{1}{3}(x^2+1)^{3/2}$. 96. $\frac{3}{8}\sinh^{-1}x+\frac{1}{8}x(5+2x^2)\sqrt{(1+x^2)}$.
97. $\frac{1}{2}x/(x^2+1)+\frac{1}{2}\tan^{-1}x$. 98. $x/\sqrt{(x^2+1)}$.
99. $-1/\sqrt{(1+x^2)}$. 100. $\frac{1}{5}(x^2+1)^{5/2}$.
101. $-\frac{2}{15}(2+3x)(1-x)^{3/2}$. 102. $\frac{2}{15}(2+3x)(x-1)^{3/2}$.
103. $-\frac{2}{3}(x+2)\sqrt{(1-x)}$. 104. $\frac{2}{3}(x+5)\sqrt{(x-1)}$.
105. $\sin^{-1}\sqrt{x}-\sqrt{\{x(1-x)\}}$.
106. $\frac{1}{4}(2x-1)\sqrt{\{x(x-1)\}}-\frac{1}{8}\cosh^{-1}(2x-1)$.
107. $\frac{1}{4}(2x-1)\sqrt{\{x(1-x)\}}+\frac{1}{8}\sin^{-1}(2x-1)$.
108. $\sin^{-1}\sqrt{x}+\sqrt{\{x(1-x)\}}$. 109. $\sin^{-1}(2x-1)$.
110. $\frac{1}{4}(2x+1)\sqrt{\{x(x+1)\}}-\frac{1}{8}\cosh^{-1}(2x+1)$.
111. $-\log(1+\cos x)$. 112. $\tan\frac{1}{2}x$.
113. $x-\tan\frac{1}{2}x$. 114. $x-\sin x$.
115. $\tan\frac{1}{2}x+\sin x-x$. 116. $1/(1+\cos x)$.
117. $\frac{1}{2}\tan\frac{1}{2}x+\frac{1}{6}\tan^3\frac{1}{2}x$. 118. $\frac{1}{2}\tan\frac{1}{2}x-\frac{1}{6}\tan^3\frac{1}{2}x$.
119. $\frac{1}{6}\tan^3\frac{1}{2}x-\frac{3}{2}\tan\frac{1}{2}x+x$. 120. $2\tan\frac{1}{2}x-x$.
121. $\log(e^x+1)$. 122. $x-\log(e^x+1)$.
123. $e^x-\log(e^x+1)$. 124. $2\tan^{-1}e^x$.
125. $\log\tanh\frac{1}{2}x$. 126. $x\sinh x-\cosh x$.
127. $2x\cosh\frac{1}{2}x-4\sinh\frac{1}{2}x$. 128. $\frac{1}{2}x+\frac{1}{4}\sinh 2x$.
129. $\frac{1}{4}\sinh 2x-\frac{1}{2}x$. 130. $x\sinh^{-1}x-\sqrt{(1+x^2)}$.
131. $\frac{1}{4}(2x^2-1)\cosh^{-1}x-\frac{1}{4}x\sqrt{(x^2-1)}$.
132. $\frac{1}{2}(\sin ax\cosh ax-\cos ax\sinh ax)/a$.
133. $\frac{1}{10}(5+\cos 2x)\cosh x+\frac{1}{5}\sin 2x\sinh x$.
134. $\frac{1}{13}(3\sin 2x\sinh 3x-2\cos 2x\cosh 3x)$.
135. $(n\cos mx\sinh nx+m\sin mx\cosh nx)/(m^2+n^2)$.
136. $\frac{1}{10}\tan^{-1}\frac{1}{10}(x+3)$. 137. $\frac{1}{2}\log(x^2+6x+109)$.
138. $\frac{1}{2}\log(x^2+6x+109)-\frac{3}{5}\tan^{-1}\frac{1}{10}(x+3)$.
139. $x-3\log(x^2+6x+109)-\frac{91}{10}\tan^{-1}\frac{1}{10}(x+3)$.
140. $\sinh^{-1}\frac{1}{10}(x+3)$. 141. $\sqrt{(x^2+6x+109)}$.
142. $\sqrt{(x^2+6x+109)}-3\sinh^{-1}\frac{1}{10}(x+3)$.
143. $\frac{1}{2}(x+3)\sqrt{(x^2+6x+109)}+50\sinh^{-1}\frac{1}{10}(x+3)$.
144. $\frac{1}{6}(2x^2+3x+191)\sqrt{(x^2+6x+109)}-150\sinh^{-1}\frac{1}{10}(x+3)$.
145. $\frac{1}{109}[\log\{x/\sqrt{(x^2+6x+109)}\}-\frac{3}{10}\tanh^{-1}\frac{1}{10}(x+3)]$.
146. $-\sqrt{\frac{1}{109}}\sinh^{-1}\{(3x+109)/10x\}$. 147. $\log x-\frac{1}{3}\log(1+x^3)$.
148. $\frac{1}{6}\log\{(1+x)^2/(1-x+x^2)\}+\sqrt{\frac{1}{3}}\tan^{-1}\{(2x-1)/\sqrt{3}\}$.
149. $\frac{1}{6}\log\{(1-x+x^2)/(1+x)^2\}+\sqrt{\frac{1}{3}}\tan^{-1}\{(2x-1)/\sqrt{3}\}$.
150. $-\frac{1}{4}(1+2x^2)/(1+x^2)^2$.

Examples LVI, p. 272.

1. $204{\cdot}6$; $\frac{1}{6}$; 2; $293\frac{1}{3}$; $\frac{16}{15}$.
2. 13; $-\frac{5}{72}$; $a^{n+1}(2^{n+1}-1)/(n+1)$; $\cdot 828$; $\frac{28}{15}a^{3/2}$.
3. $\log 4$; $\log 2$; $\log 4$. 4. 1; 0; 0. 5. $\frac{1}{4}\pi$; $\frac{1}{2}\pi$; 1.

6. $\frac{1}{4}\pi$; $\frac{1}{6}\pi$; $\cdot 8812$; $\cdot 446$.
7. $\frac{1}{4}\pi$; $\frac{1}{2}a^2\{\log(1+\sqrt{2})+\sqrt{2}\}$; $1\cdot 074$.
8. $\sqrt{2}-1$; $\frac{1}{2}\log\frac{3}{2}$; $\tan^{-1}2-\frac{1}{4}\pi$.
9. $\frac{1}{4}\pi$; $\frac{2}{3}$; $\log 2$.
10. $2\log 2-\frac{3}{4}$; $\frac{64}{3}\log 4-7$; $b\log b-a\log a+a-b$.
11. 1; 1.
12. $\frac{1}{2}\pi-1$; $\frac{1}{4}\pi-\frac{1}{2}\log 2$.
13. $\frac{1}{5}(e^{2\pi}+1)$; $\frac{1}{2}(e^{-\pi/2}+1)$.
14. $e-2$; $\sinh 1$.
15. $\frac{1}{3}a^3$; $a(\sqrt{2}-1)$.
16. π; π^2-4.
17. 0; $\frac{2}{3}$.
18. $\cdot 446$; $2\cdot 287$.
19. $a(1-\frac{1}{2}\pi)$; $\frac{19}{3}-8\log\frac{3}{2}$.
20. $\frac{1}{2}\log 3$; $\frac{1}{2}\log 3$.
21. $\frac{2}{3}\tan^{-1}\frac{1}{3}$; $\frac{1}{3}\log 2$.
22. $\frac{1}{4}\pi-\frac{2}{3}$; $\frac{3}{16}\pi$.
23. $\frac{2}{15}$; 0.
24. 0.
25. $\frac{1}{2}+\log\frac{3}{4}$; $\frac{1}{2}+\log\frac{3}{4}$.
26. $\frac{1}{16}\pi^2+\frac{1}{4}$; $\frac{3}{16}\pi^2+\frac{1}{4}$.
27. 0; 0.
28. $\frac{4}{3}$.
29. $\frac{1}{4}\pi-\frac{1}{2}$; 1.
30. $\pi/(ab)$.
31. $\frac{1}{2}\pi-1$.
32. $\frac{1}{2}\alpha/\sin\alpha$.
33. $6-4\log 2$.
34. $\tan^{-1}e-\frac{1}{4}\pi$.
35. $\frac{1}{2}(e^{-\pi}+1)$.
36. $\frac{1}{3}(\log 2+\pi/\sqrt{3})$.
37. $\frac{1}{10}(\pi+\log\frac{2}{9})$.
38. $\cdot 446$.
39. $\cdot 562$; $1\cdot 7624$.
40. $\pi/\sqrt{3}-\frac{1}{18}\pi^2-\log 2$.
41. $\frac{1}{8}\pi-\frac{1}{2}\tan^{-1}\frac{1}{2}$; $\log\frac{256}{243}$.
42. $\frac{1}{3}\pi$.
43. $\frac{1}{2}a^2$.
44. 1.
45. 1.
46. e^b-e^a.
47. 0.
48. $\frac{1}{4}(b^4-a^4)$.

Examples LVII, p. 277.

In each of Ex. 1–12, I denotes the integral of the given function from 0 to $\frac{1}{2}\pi$.

1. $2I$.
2. $2I$.
3. $4I$.
4. $4I$.
5. $8I$.
6. $8I$.
7. $-2I$.
8. $4I$.
9. $-3I$.
10. $3I$.
11. $2I$.
12. $3I$.
13. $\frac{4}{35}$.
14. $a^{n+2}/\{(n+1)(n+2)\}$.
15. $\frac{128}{105}\sqrt{2}$.
16. 0.
17. 0.
18. 0.
19. $\frac{1}{429}$.
20. $\frac{44}{105}\sqrt{(2a^7)}$.
21. 0.
22. 0.
23. 0.
24. 2.
25. 0.
26. 0.
27. $\cdot 5$ and $\cdot 5236$.
28. $\cdot 1132$ and $\cdot 1192$.
29. $1\cdot 571$ and $1\cdot 679$.
30. $1\cdot 785$ and 2.
34. $\cdot 1163$.

Examples LVIII, p. 281.

In cases where no answer is given, the integral does not exist.

1. $\frac{1}{2}$.
2. $3/\sqrt[3]{2}$.
4. 3.
6. 0.
8. $4\frac{1}{2}$.
9. $\frac{1}{2}\pi$.
10. $\pi/(ab)$.
13. $\frac{1}{2}$.
14. π.
15. 1.
16. a.
18. $\frac{1}{12}\pi$.
20. $\log 2$.
21. $1-\frac{1}{4}\pi$.
23. π.
24. π.
25. $n!$.

Examples LIX, p. 285.

1. $\frac{3}{256}\pi$.
2. $\frac{1}{40}$.
3. $\frac{8}{105}$.
4. $\frac{63}{512}\pi$.
5. $\frac{32}{315}$.
6. $\frac{32}{35}$.
7. $\frac{5}{64}\pi$.
8. $\frac{2}{7}$.
9. $\frac{3}{2}\pi$.
10. $\frac{8}{693}$.
11. $\frac{1}{2}\pi a^2$.
12. 48π.
13. $\frac{2}{15}a^5$.
14. $\frac{7}{512}\pi a^{10}$.
15. $\frac{8192}{3465}$.
16. $1/(12a^4$
17. $\frac{135}{64}\pi$.
18. 0.
19. $\frac{1}{140}a^7$.
20. $\frac{1}{16}\pi$.
21. 3π.
22. $\frac{32}{35}$.
23. $\frac{1}{8}\pi a^2$.
24. $\frac{3}{2}\pi a^2$.
25. $\frac{5}{128}\pi$.
26. $\frac{81}{40}$.
27. $1/(24a^6)$.
28. $1/(6a^3)$
29. π.
30. $\frac{1}{2}(\beta-\alpha)\pi$.
31. $\frac{9}{8}\pi$.
32. $\frac{25}{8}\pi$.
33. $a\pi$.
34. $a\pi$.
35. π.
36. $\frac{65}{8}\pi$.

Examples LX, p. 291.

1. $10\frac{2}{3}$.
2. $\frac{81}{4}\pi$.
3. $3\pi a^2$.
4. $\frac{3}{8}\pi ab$.
5. $\frac{25}{16}\sqrt{2}\pi$.
7. $\frac{4}{3}ab$.
8. $\frac{16}{15}a^2$.
9. $\frac{128}{81}$.
10. $106\frac{2}{3}$.
11. (i) $\frac{1}{2}(4-\pi)a^2$. (ii) $\frac{1}{2}(4+\pi)a^2$.
12. $\cdot 67a^2$, $14\cdot 37a^2$, $\cdot 67a^2$.
13. $\frac{256}{15}a^2$; $\frac{2}{5}\sqrt{3}$.
15. $\{2\sqrt{3}-\log(2+\sqrt{3})\}ab$; $ab\log(2+\sqrt{3})$.
16. 2π.
17. 34π.
18. $\frac{1}{2}\pi$.
19. $4ab\tan^{-1}(b/a)$.
20. π.
21. $\frac{15}{128}\pi a^2$.
22. $4\pi a^2$.
23. $6\pi a^2$.
24. $\frac{3}{8}\pi a^2$.

Examples LXI, p. 298.

1. $1\cdot 111$.
2. $14\cdot 902$.
3. $\cdot 6$.
4. $1\cdot 37$.
8. $3\cdot 57$.
9. $\cdot 256$.
10. (i) $2na/\pi$, (ii) $\frac{1}{4}na\pi$; (i) $2n^2a/\pi$, (ii) $\frac{1}{2}n^2a$.
12. (i) $\frac{1}{4}\pi a$. (ii) $2a/\pi$.
13. $47\cdot 75$.
14. $1\cdot 274$.
15. 50.
16. $\frac{1}{3}a^2$.
17. $1\cdot 273a$.
18. $\frac{1}{6}a^2$.
19. $\frac{8}{3}a$.
20. $\frac{1}{4}\pi b$.
21. 0 for a complete revolution, $2r/\pi$ for half a revolution.
22. $3\cdot 9$ sq. in.
23. $14\cdot 42$ sq. in.
24. $16\cdot 72$; $16\cdot 64$.
25. $1\cdot 5$

Examples LXII, p. 303.

1. $\frac{1}{12}\pi a^3$.
2. $\frac{4}{15}\pi a^3$.
3. $\frac{1}{5}$ of circumscribing cylinder.
4. $\frac{8}{15}$ of circumscribing cylinder.
5. $\frac{1}{2}\pi ab^2$.
6. $\frac{2}{3}\pi a^3$.
7. $\frac{1}{2}\pi^2a^3$.
8. $\frac{1}{12}\sqrt{2}(10-3\pi)\pi r^3$.
9. $\frac{1}{4}\pi^2a^3$.
10. $\frac{32}{105}\pi a^3$.
11. $\frac{1}{2}\pi^2a^3$.
12. $\frac{1}{6}(9\pi^2-16)\pi a^3$.
13. $2\cdot 813\pi c^3$.
14. $\frac{96}{5}\pi a^3$.
15. $\frac{2}{3}\pi a^3$.
16. $2\pi^2r^3$.
17. $\frac{1}{4}\pi(\pi^2-8)ab^2$.
19. $\frac{1}{15}\pi a^3$.
20. $\pi a\{b(2a+b)\log(b/a)+2ab+\frac{1}{2}a^2-\frac{5}{2}b^2\}$.
21. $2\pi^2br^2$.
23. $17\cdot 69$ c. ft.
24. 2186 c. in.

25. $\frac{1}{12}\pi(14+3\pi)a^3$.
26. $\pi[2ab^2-\frac{2}{3}b^4/a-2ea^2b\sin^{-1}(b/a)]$.
27. 183·8.
28. $\frac{1}{3}(2-\sqrt{3})\pi r^3$.
29. $\frac{2}{3}\pi bh^2$.
30. $\frac{2}{3}\sqrt{2}\pi r^3$.

Examples LXIII, p. 307.

2. 11·304.
3. $\frac{13}{3}a$.
5. ·82.
6. $6a$.
8. $2\int_0^{\frac{1}{2}\pi}\sqrt{(a^2+b^2\cos^2\theta)}\,d\theta$.
9. $4a/\sqrt{3}$.
10. 46·66.
11. 48·87.
13. 1·317.
14. $4(a^2+ab+b^2)/(a+b)$.
16. $8a\cos\frac{1}{2}\alpha$.
17. $\log(2\cosh a)$.
18. $\frac{1}{2}a\alpha^2$.
20. 9·76.

Examples LXIV, p. 309.

1. 620.
2. 452·5 sq. in.
3. $\frac{32}{3}\pi a^2$.
4. $\frac{8}{3}(3\pi-4)\pi a^2$.
5. $262{\cdot}2a^2$.
6. $4\pi^2 ar$.
7. $\pi(\pi-2\sqrt{2})a^2$.
8. $\pi a^2(4-\pi)/\sqrt{2}$.
9. $\frac{12}{5}\pi a^2$.
10. $2\pi c^2(1-1/e)$.
11. $2\pi\{ab(\sin^{-1}e)/e+b^2\}$.
12. $\pi(b^2/e)\log\{(1+e)/(1-e)\}+2\pi a^2$.
13. $\pi a^2\{3\sqrt{2}-\log(\sqrt{2}+1)\}$.
14. $\pi\{2+\sqrt{2}\log(\sqrt{2}+1)\}a^2$.
15. $\pi c^2(2+\sinh 2)$.
16. $\frac{1}{3}\pi a^2$.
17. $\pi\{\sqrt{2}+\log(\sqrt{2}+1)\}$.
18. $2\pi a^2$.
19. 4288 sq. in.
20. 1096 sq. in.

Examples LXV, p. 312.

13. $r=2a\sin\theta\tan\theta$.
14. $r^2\cos 2\theta=a^2$.
15. $r(1+\cos\theta)=2a$.

Examples LXVI, p. 317.

3. 120°.
4. $\sqrt{(2ar)}$.
8. 107° 39′.
21. $r/a=1+\sin(\theta+C)$.
23. $\frac{3}{2}\sqrt{3}a$.
24. $\frac{1}{4}a$.
25. $a/\sqrt{2}$.
28. 60° and 120° with initial line.
29. When $\cos\theta=\{-a\pm\sqrt{(a^2+8b^2)}\}/4b$.
30. $b^2/p^2=\pm 2a/r+1$.
31. $a^2b^2/p^2=b^2+a^2-r^2$.
32. $a^2/4b$.
36. 0°.

Examples LXVII, p. 321.

1. $\frac{3}{2}\pi a^2$.
2. 1.
4. 14·14.
5. 35·525; 2·174.
6. πa^2.
7. a^2.
8. $\frac{1}{12}\pi a^2$.
9. $\frac{1}{16}\pi a^2$.
10. $\frac{1}{12}(4\pi-3\sqrt{3})a^2$.
11. $(\frac{3}{4}\pi\pm 2)a^2$.
12. $\frac{8}{3}a^2$.
13. $\sqrt{2}(r_1\sim r_2)$.
14. $3{\cdot}925a$.
15. $\frac{3}{2}\pi a$.
16. $4{\cdot}59a$.
17. $2a\int^{\frac{1}{4}\pi}\frac{d\theta}{\sqrt{(1-2\sin^2\theta)}}$.
18. πa^2.
19. $8a$.
21. $\frac{4}{15}\pi a^3$.
22. 233·7.
23. 335·1.
24. πa^2.
25. $\frac{1}{6}\pi a^3$.
26. $\frac{8}{3}\pi(\sqrt{8}-1)a^2$.
27. $2\pi a^3$.
28. $\frac{2}{5}\sqrt{2}\pi(e^{2\pi}-1)$.

Examples LXVIII, p. 324.

1. $x = b(4\cos\theta - \cos 4\theta)$, $y = b(4\sin\theta - \sin 4\theta)$; $x = b(2\cos\theta + \cos 2\theta)$, $y = b(\sin\theta - \sin 2\theta)$.
2. $\tan(\theta + \frac{1}{2}\phi)$.
3. $ds/d\theta = 2(a+b)\sin(\frac{1}{2}a\theta/b)$; $8(a+b)b/a$.
4. $ds/d\theta = 2(a-b)\sin(\frac{1}{2}a\theta/b)$; $8(a-b)b/a$.
5. $x(\sqrt{2}+1) + y = (4+3\sqrt{2})b$.
6. $x + \sqrt{3}y = 2b$.
7. $6\pi b^2$.
9. $r^2 = a^2 - 8p^2$.
10. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

Examples LXIX, p. 332.

1. (i) $\frac{4}{3}\sqrt{2}\,r/\pi$ from centre, on middle radius. (ii) $\frac{4}{3}a/\pi$, $\frac{4}{3}b/\pi$.
2. $\frac{3}{4}h$ from vertex.
3. $\left(\dfrac{b-c}{\log(b/c)}, \dfrac{a^2(b-c)}{2bc\log(b/c)}\right)$
4. $\bar{y} = \frac{1}{8}\pi b$.
5. $\bar{x} = 5\frac{131}{812}$.
6. $(\frac{3}{4}a, \frac{3}{10}a)$.
7. $(\frac{2}{5}, \frac{4}{7})$.
8. $\bar{x} = \frac{2}{3}a$.
9. $\bar{x} = \frac{3}{8}a$.
10. $\bar{y} = \frac{3}{8}b$.
11. $\cdot 92\,a$ from centre.
12. $\bar{x} = \frac{5}{6}a$.
13. $\bar{x} = \bar{y} = \frac{256}{315}a/\pi$.
14. $\cdot 876\,r$ from vertex.
15. $\bar{x} = \frac{27}{40}r$.
16. $5\frac{16}{37}$ in. from larger end.
17. $\cdot 87\,r$ from vertex.
18. $(3\frac{1}{2}, \frac{9}{10})$.
19. $\bar{x} = \frac{2}{3}(a^3 - b^3)/(abe + a^2\sin^{-1}e)$.
20. $(\frac{5}{6}, 3\frac{19}{30})$.
21. $\bar{x} = 3\frac{1}{34}$.
22. $\bar{x} = \frac{3}{2}$.
23. $\bar{y} = \frac{4}{3}a$.
24. $\bar{y} = \frac{5}{6}a$.
25. $\bar{y} = \frac{1}{4}\{a\,\mathrm{cosech}\,(a/c) + c\cosh(a/c)\}$.
26. $\pi\sqrt{3}a^2$; $\frac{1}{4}\pi a^3$.
27. $4\sqrt{2}\pi a^2$; $\sqrt{2}\pi a^3$.
28. $4\pi^2 r^2$; $2\pi^2 r^3$.
29. $2\pi^2 a^2 b/e$.
30. 76 lb. wt.
31. 984·7 sq. in.; 1583 c. in.
33. 265·4 lb. wt.
34. (3·95, ·96).
35. $\bar{x} = 3{\cdot}9$.

Examples LXX, p. 336.

1. $\frac{3}{4}h$.
2. $\frac{1}{2}(6h^2 + 8ah + 3a^2)/(3h + 2a)$, if a be height of triangle.
3. $4\frac{1}{3}$ ft.
4. $\frac{3}{16}\pi r$.
5. $\frac{5}{4}a$.
6. $h(a+3b)/(2a+4b)$, where a is the side in the surface and b the parallel side.
7. $\frac{4}{7}a$.
9. Depth below surface increased by $h(h+2a-b)/(h+a)$, where a and b denote original depths of C. G. and C. P. respectively.

Examples LXXI, p. 340.

1. $\frac{1}{3}Ma^2$.
2. $\frac{1}{9}Mb^2$.
3. $\frac{5}{2}Mr^2$.
4. $\frac{1}{4}Mr^2$.
5. $\frac{1}{2}Mh^2$.
6. $\frac{1}{6}Mh^2$.
7. $\frac{1}{6}Mb^2$ [$b = \frac{1}{2}$ base].
8. $\frac{1}{18}Mh^2$.
9. $\frac{1}{2}Mr^2$.
10. $\frac{3}{10}Mr^2$.
11. $\frac{2}{5}Mb^2$.
12. $Ma^2(e^2 + \frac{1}{4})$.
13. $\frac{4}{3}Mab$.
14. $\frac{1}{2}Mr^2$.
15. $\frac{2}{3}Mr^2$.
16. $\frac{3}{7}Mb^2$.
17. $\frac{4}{5}Mab$.
18. $\frac{8}{35}Mb^2$.
19. $1{\cdot}25\,M$.
20. $1{\cdot}95\,M$.
21. $\frac{1}{2}Mr^2\{2 + \cos 2\alpha - \frac{3}{2}(\sin 2\alpha)/\alpha\}$.
22. $\frac{25}{36}Ma^2$.
23. $\frac{63}{40}Ma^2$.
24. $\frac{7}{64}Ma^2$.

Examples LXXII, p. 345.

1. $\frac{1}{4}M(r^2+r'^2)$. 2. $\frac{2}{3}Ma^2$. 3. $\frac{1}{4}M(a^2+b^2)$. 4. $\frac{1}{6}Ma^2$.
5. $\frac{1}{6}Ma^2$. 6. (i) $\frac{5}{4}Ma^2$. (ii) $Ma^2(e^2+\frac{1}{4})$. (iii) $Ma^2(1/e^2+\frac{1}{4})$.
7. $\frac{1}{12}Ma^2$. 8. $\frac{3}{2}Ma^2$. 9. $\frac{7}{5}Mr^2$. 10. $M\{\frac{1}{3}a^2+(a-b)^2\}$.
11. $\frac{1}{12}Ma^2$. 12. $M(\frac{1}{12}a^2+b^2)$. 13. $\frac{3}{2}Ma^2b^2/(a^2+b^2)$.
14. $\frac{1}{3}Mb^2h^2/(b^2+h^2)$. 15. (i) $\frac{3}{80}M(h^2+4r^2)$. (ii) $\frac{1}{20}M(3r^2+2h^2)$.
16. $M(\frac{1}{3}h^2+\frac{1}{4}r^2)$. 17. (i) $\frac{1}{12}M(h^2+3r^2)$. (ii) $M(\frac{1}{3}h^2+\frac{5}{4}r^2)$.
18. $M(c^2+\frac{1}{4}b^2)$. 19. $M(c^2+\frac{3}{4}r^2)$. 20. $2Mr^2\{1-(\sin\alpha)/\alpha\}$.
21. $\frac{1}{2}Ma^2$. 22. $\frac{3}{5}Mr^2$.
23. $\frac{1}{6}Mc^2$, if c be the length of the hypotenuse.
24. $\frac{1}{6}Mb(4a+3b)$. 25. $\frac{1}{5}M(b^2+6a^2)$.

Examples LXXIII, p. 348.

1. M/r. 2. $(M/h)\sinh^{-1}(h/r)$.
3. $(M/r^2h)\{h\sqrt{(r^2+h^2)}+r^2\sinh^{-1}(h/r)-h^2\}$.
4. $2M/l$, where l is the slant height. 5. $3M(l-h)/r^2$.
6. $2\pi m(r^2-r'^2)$, where m is the density, at an internal point; M/c at an external point. 7. M/r.
8. $2\pi m(r^2-r'^2-\frac{1}{3}c^2+\frac{1}{3}c'^2)$, where c and c' are the distances of the point from the centres, at an internal point; $M/c-M'/c'$, at an external point; $\frac{2}{3}\pi m(3r^2-c^2-2r'^3/c')$, at a point between the spheres.
9. $2\pi m(R-R')$, where R and R' are the distances of the point from the edges of the ring. 10. $2M/r$.

Examples LXXIV, p. 352.

1. $(2m\sin\alpha)/p$. 2. $2\pi m(1-\cos\alpha)$. 3 and 4. $(2m\sin\frac{1}{2}APB)/p$.
5. $(M/h)(1/R_1-1/R_2)$, where R_1 and R_2 are the distances of the point from the circumferences of the ends.
6. $2\pi m(h+R_1-R_2)$. 7. $2\pi mh(1-\cos\alpha)$.
8. (i) 0. (ii) $\frac{4}{3}\pi m(x^3-r'^3)/x^2$. (iii) M/x^2.
9. (i) $\frac{4}{3}\pi md$, where d is the distance between the centres A and B.
(ii) Resultant of M'/BP^2 along BP and M/AP^2 along PA.
10. $M/PA.PB$.
11. $2\pi m(\cos\beta-\cos\alpha)$, where α and β are the angles subtended at the point by the radii of the ring.
12. (i) 31·376. (ii) 30·63. 13. 515×10^5 ft.-lb.

Examples LXXV, p. 360.

1. (i) £32 5*s.* 11*d.* (ii) £32 6*s.* 3*d.* (iii) £32 7*s.* 4*d.*
2. (i) 29·1°. (ii) 40·55 min. 3. 10·5°; $20\frac{1}{2}$ min.
4. $T/T_0=2{\cdot}85$. 5. 3048 lb. wt. 6. 20·5.
7. 5870. 8. ·44. 9. 19·5.
10. 36·95; 14·98. 11. 78·5. 12. (i) ·2054. (ii) 6·1.
13. (i) ·0059. (ii) 5·44. 14. (i) ·00098. (ii) 1·24 secs.

15. $1{\cdot}74 \times 10^{14}$.
16. (i) $\cdot 268$. ii) $\cdot 0028$.
17. $4{\cdot}97$.
18. $-3{\cdot}09$.
19. $-{\cdot}02$.
20. $\frac{3}{4}V$; $2{\cdot}32$ secs.
21. $44{\cdot}26$ per cent.; $12{\cdot}5$.
22. $1{\cdot}99$ secs.
23. $185{\cdot}8$ min.
24. $46{\cdot}45$ years.

Examples LXXVI, p. 368.

1. 1 ft.-lb.
2. a ft.-lb.
3. 19080.
4. 129360.
5. 92363; 186540.
6. 77625; 47520.
7. $111{\cdot}45$.
8. 1,000,000.
9. 3522×10^4. (i) $2\frac{1}{2}$ ft. per min. (ii) $1\frac{5}{8}$ ft. per min.
10. 78680; $433{\cdot}8°$.
11. 61790.
12. At $56° 19'$ to horizontal.
13. $2{\cdot}836 W$.
14. $2W$.
15. $2W$.
16. $41° 24'$.
17. 9 in. below AB.
18. $43° 54'$ to wall; unstable.
19. $16° 25'$ to horizontal; unstable.
20. $\frac{1}{2}W \cot \alpha$, if W be total weight of rods.
21. $W(2c-a)/a\sqrt{3}$.
22. $50 \cot \alpha$.
23. $294{\cdot}8$.
24. $79{\cdot}72$ ft.-lb.

Examples LXXVII, p. 375.

1. (i) $8\sqrt{3}$. (ii) 16. (iii) $-1{\cdot}67$. (iv) $\frac{1}{2}\pi$.
2. (i) 16. (ii) $8\sqrt{5}$. (iii) $-1{\cdot}86$. (iv) $\frac{1}{2}\pi$.
3. $64\frac{1}{2}$ days.
4. (i) $32\sqrt{3}$. (ii) $\frac{1}{16}\pi$. (iii) $\cdot 185$.
5. $21\frac{1}{4}$ min.; $4{\cdot}946$ miles per sec. 85 min.
6. $2{\cdot}92$ miles per sec.
7. 84 miles.
8. $1{\cdot}49$ miles per sec.
9. $25{\cdot}82$ ft.-secs.
10. $17{\cdot}17$.
11. $\sqrt{\mu}/x$.
12. (i) 4 ft. (ii) $1{\cdot}11$ secs. (iii) $4{\cdot}9$ ft.-secs. (iv) $\cdot 24$ secs. (v) $3{\cdot}95$ ft.
13. $\sqrt{(\mu x^2+u^2)}$; $\sqrt{(1/\mu)} \sinh^{-1}(x\sqrt{\mu}/u)$.
14. $\sqrt{\{\mu(x^2-a^2)\}}$; $\sqrt{(1/\mu)} \cosh^{-1}(x/a)$.
15. $\sqrt{\{2\mu(x-a)/ax\}}$; $\sqrt{(a^3/2\mu)}\ \{\cosh^{-1}\sqrt{(x/a)}+\sqrt{(x^2-ax)}/a\}$.
16. 16 ft.
17. $a^2/\sqrt{\mu}$.
18. (i) $\cdot 556$ secs. (ii) $5{\cdot}656$. (iii) $-3{\cdot}32$. (iv) $4{\cdot}306$.
19. (ii) $2{\cdot}828$. (iii) $-1{\cdot}66$. (iv) $3{\cdot}847$.
20. $\cdot 7854$; $x = \cos 8t$, if x be distance from centre at time t.
21. $\cdot 7854$; $x = \cos 8t$, if x be distance above position of equilibrium at time t.

Examples LXXVIII, p. 384.

1. (i) $v=u-ks$. (ii) $v=ue^{-kt}$. (iii) $u(1-e^{-kt})/k$. (iv) $t=\infty$. (v) $s=u/k$.
2. (i) $1000/(100x+1)$. (ii) $1000/\sqrt{(2t \times 10^5+1)}$.
3. (i) $1{\cdot}96$ secs. (ii) $69{\cdot}3$ ft. (iii) $1{\cdot}3$ ft.-secs. downwards. (iv) $63{\cdot}9$ ft.-secs.
4. (i) $2{\cdot}36$ secs. (ii) $92{\cdot}16$ ft. (iii) $11{\cdot}46$ ft.-secs.
5. (i) 100. (ii) $79{\cdot}8$. (iii) $174{\cdot}4$.
6. (i) $31{\cdot}623$. (ii) $31{\cdot}62$. (iii) $104{\cdot}9$ ft. (iv) $30{\cdot}37$ ft.-secs.

7. $\cdot488$ lb. wt.

8. (i) $1\cdot16$ secs. (ii) $22\cdot3$ ft. (iii) $35\cdot78$ ft.-secs. (iv) $1\cdot203$ secs.

9. $22\cdot74$ ft.-secs. 10. $\sqrt[n]{(g/k)}$, if mkv^n be the resistance.

11. $(1/k)\log(1+ku/g)$; $u/k-(g/k^2)\log(1+ku/g)$.

12. $\{\tan^{-1}(u\sqrt{k}/\sqrt{g})\}/\sqrt{(gk)}$.

13. (i) $\cdot89$ ft.-sec. (ii) $3\cdot3$ ft. (iii) $\cdot0048$ ft.

14. $2k\sqrt{u/g}$; $\frac{2}{3}ku^{\frac{3}{2}}/g$. 15. $14\cdot3$ ft. diameter.

16. $40\cdot8$ secs.; 1739 ft. 17. $(1/k)\log\{u/(u-kh)\}$.

18. $61\cdot66$ ft.-secs.; $62\cdot3$ secs. 19. $1\cdot37$ secs.; $18\cdot8$ ft.

20. $(x+y)^2+\frac{1}{2}y^2=25$. 21. $4(4x+3y)^2+25y^2=1600$.

22. $x=a\cosh t\sqrt{\mu}$, $y=(u/\sqrt{\mu})\sinh t\sqrt{\mu}$; $x^2/a^2-\mu y^2/u^2=1$.

Examples LXXIX, p. 391.

1. $8\sqrt{(\frac{1}{3}\cos\theta)}$; $\dfrac{\sqrt{6}}{8}\displaystyle\int_0 \frac{d\phi}{\sqrt{(1-\frac{1}{2}\sin^2\phi)}}$. 2. (i) $-2\frac{1}{2}°$. (ii) $-3\cdot27°$.

3. (i) $3\cdot14$ secs. (ii) 16 ft.-secs. (iii) $13\cdot86$ ft.-secs. (iv) $-3\cdot33$ ft. (v) $\cdot66$ sec. (vi) $s=6\cdot93$. (vii) $14\cdot57$ ft.-secs. upwards. (viii) $\cdot425$ sec.

4. $\sqrt{(24\cos\theta)}$; $\dfrac{\sqrt{3}}{6}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\frac{1}{2}\sin^2\phi)}}$.

5. $8\sqrt{\cos\theta}$; $\sqrt{2}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\frac{1}{2}\sin^2\phi)}}$.

6. $\frac{8}{3}\sqrt{(6\cos\theta)}$; $\dfrac{\sqrt{3}}{8}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\frac{1}{2}\sin^2\phi)}}$

7. $32\sqrt{\left(\dfrac{\cos\theta}{3\pi r}\right)}$; $\dfrac{\sqrt{(\frac{3}{2}\pi r)}}{16}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\frac{1}{2}\sin^2\phi)}}$.

8. $4\sqrt{\left\{\dfrac{3}{a}\left(\sqrt{2}\cos\theta-1\right)\right\}}$; $\sqrt{\dfrac{a}{24\sqrt{2}}}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\sin^2\frac{1}{8}\pi\sin^2\phi)}}$.

9. (i) $2\pi\sqrt{\dfrac{2a}{3g}}$. (ii) $2\pi\sqrt{\left\{\dfrac{2a^4+3a^2b^2+2b^4}{3g(a^2+b^2)^{3/2}}\right\}}$. (iii) $2\pi\sqrt{\left\{\dfrac{2}{3g}\sqrt{(a^2+b^2)}\right\}}$.

(iv) $2\pi\sqrt{\left\{\dfrac{5ab}{6g\sqrt{(a^2+b^2)}}\right\}}$.

13. $\cdot736\sqrt{a}$. 14. $h=$ radius of gyration about C. G.

15. 2 ft. 16. $\dot{\phi}^2=2G(\cos\phi-\cos\alpha)/J$; $2\pi\sqrt{(J/G)}$.

Examples LXXX, p. 395.

1. $8\cdot9$ ft. 2. (i) $353\cdot3$. (ii) $356\cdot8$. (iii) $354\cdot9$ lb. wt.

5. $\frac{1}{4}$ inch. 6. 101 ft.; $6\cdot28$ ft. 7. 101 ft.; $6\frac{1}{4}$ ft. 10. $\cdot04$ ft.

Examples LXXXI, p. 401.

1. 5·59. $x^2+y^2+8x-7y=3$. 2. 42·16. $(x+38)^2+(y-\frac{52}{3})^2=\frac{16000}{9}$.
3. $-4·63$. 4. -5. 5. $(x^2+y^2)^{\frac{3}{2}}/2c^2$. 6. $-a^2(1-e^4)^{\frac{3}{2}}/b$.
7. $-4\sqrt{2}\,a$. $x^2+y^2-10ax+4ay=3a^2$. $(9a, -6a)$.
8. c. 9. $-(2x^2-a^2)^{\frac{3}{2}}/a^2$. 10. $\frac{125}{192}a$.
11. $-3\sqrt[3]{(axy)}$. 12. -2. 13. $4\frac{1}{2}$. $x^2+y^2=9x$.
14. $-\frac{1}{4}\sqrt{2}$. 15. $4a\cos\frac{1}{2}\theta$. 16. $-\frac{1}{4}a$.
17. $\frac{125}{64}$. 18. $-a\operatorname{cosec}(x/a)$. 19. $3a\sin\theta\cos\theta$.
20. $c\sec^2\psi$. 21. y^2/a. At the vertex. 25. $e=\frac{1}{2}\sqrt{2}$.
27. $4a\cos 3\theta$.
29. (i) Max. at origin. (ii) Max. when x (iii) No max. or min.
32. $-a\cot\theta$.

Examples LXXXII, p. 406.

1. $EIy=\frac{1}{2}Wx^2(l-\frac{1}{3}x)$; $\frac{1}{3}Wl^3/EI$. 2. See result of Art. 200, Ex. (iii).
3. See result of Art. 200, Ex. (ii).
4. $EIy=\frac{1}{24}Wx^2(18l^2-8lx+x^2)/l$; $\frac{11}{24}Wl^3/EI$.
5. As $3\pi:16$. 8. $\frac{19}{2048}wl^4/EI$. 9. $\frac{3}{2048}wl^4/EI$.
10. If x be measured from clamped end, $EIy=\frac{1}{48}wx^2(l-x)(3l-2x)$. Where $x=·58\,l$.
11. $\frac{1}{144}Wl^3/EI$. 12. 1·39 in.

Examples LXXXIII, p. 413.

1. $2\sqrt{(r^3/a)}$. 2. (i) $\frac{1}{3}a$. (ii) $\frac{1}{3}a\sqrt{2}$.
3. (i) $\frac{4}{3}a$. (ii) $2a/\sqrt{3}$. (iii) $\frac{2}{3}a$. 4. $(2ar-r^2)^{3/2}/ab$.
5. $3\sqrt{3}\,a$. 6. $(a^2+r^2)^{3/2}/(2a^2+r^2)$.
7. $r(a^2+r^2)^{3/2}/a^3$. 8. $a^n/\{(n+1)r^{n-1}\}$. 9. $\frac{3}{4}\sqrt[3]{(ar^2)}$.
10. $\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}\div\left\{r^2-r\frac{d^2r}{d\theta^2}+2\left(\frac{dr}{d\theta}\right)^2\right\}$.
14. $f\propto 1/r^5$. 15. $(h\operatorname{cosec}\alpha)/r$. 16. $f\propto 1/r^4$.
17. $f\propto 1/r^7$. 18. The lemniscate $r^3=a^2p$.
20. $h^2=\mu$ (semi-latus rectum); $v^2=\mu(1/a-2/r)$.

Examples LXXXIV, p. 420.

1. A concentric circle. 2. Two concentric circles. 3. A circle.
4. $4xy=c^2$. 5. $\sqrt{x}+\sqrt{y}=\sqrt{a}$. 6. $x^{2/3}+y^{2/3}=c^{2/3}$.
7. $2xy=\pm c^2$, if πc^2 be the constant area. 8. $y^2=4a(x+a)$.
9. $y=\pm x$. 10. $y^2=\frac{4}{27}x^3$. 11. $y^3=\frac{27}{4}a^2x$.
12. $x^2+y^2=a^2$. 13. $x^2-y^2=a^2$. 14. $4xy=-a^2$.
15. $x^{2/3}+y^{2/3}=c^{2/3}$. 16. $y=\pm x$. 17. $x^4+y^4=a^4$.
18. The cardioid $r=a(1+\cos\theta)$, if a be the radius of the fixed circle.
19. The lemniscate $(x^2+y^2)^2=a^2(x^2-y^2)$, if a be a semi-axis of the hyperbola.
20. $x^2/(a^2+b^2)+y^2/b^2=1$. 21. A cycloid.

22. The parabola $x^2 = 4h(h-y)$. 23. $x \pm y \pm a = 0$, if a be sum of semi-axes.
24. $(x^2+y^2-a^2-b^2)^2 = 4r^2\{(x-a)^2+(y-b)^2\}$, if (a, b) be the fixed point, and $x^2+y^2 = r^2$ the fixed circle. 25. $x^2+y^2 = \frac{1}{2}y$.
26. $(x^2+y^2)^2 = 16xy$. 27. An epicycloid (Art. 169) in which $a = 2b$.
28. A cardioid. 29. An equal cycloid.
30. $(ax)^{2/3}+(by)^{2/3} = (a^2-b^2)^{2/3}$. 31. $(x+y)^{2/3}+(x-y)^{2/3} = 2a^{2/3}$.
32. $(x+y)^{2/3}-(x-y)^{2/3} = (4c)^{2/3}$. 33. $(ax)^{2/3}-(by)^{2/3} = (a^2+b^2)^{2/3}$.
34. x^2+y^2 35. A parabola, focus S, touching the given line.
36. An ellipse with the two fixed points as foci.

Examples LXXXV, p. 426.

1. $x\dfrac{dy}{dx}+y = 0$. 2. $y = x\dfrac{dy}{dx}+a \Big/ \dfrac{dy}{dx}$.

3. (i) $\dfrac{dy}{dx} = -\cot\alpha$. (ii) $y-x\dfrac{dy}{dx} = p\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}$ (iii) $\dfrac{d^2y}{dx^2} = 0$.

4. (i) $\dfrac{dy}{dx} = by$. (ii) $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$. 6. $y = x\dfrac{dy}{dx} \pm a\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}$.

7. $\dfrac{d^3y}{dx^3} = 0$. 9. $\dfrac{d^2y}{dx^2} = m^2y$.

10. $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2 = y\dfrac{dy}{dx}$. 12. $\dfrac{d^2y}{dt^2}+k\dfrac{dy}{dt} = (n^2-\frac{1}{4}k^2)\,y$.

13. $x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx} = 0$. 20. Same as 10.

21. $\left(y-x\dfrac{dy}{dx}\right)^2 = 2xy\left\{1+\left(\dfrac{dy}{dx}\right)^2\right.$

22. $x\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^3$. 24. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y = 0$.

Examples LXXXVI, p. 429.

1. $y^2 = 2ax+C$. 2. $y = Ce^{x/a}$. 3. $y = Ce^{kx}$.
4. $y^2 = Cx$. 5. $y^2 = 3x^2+C$. 6. $xy = C$.
7. $x^my^n = C$. 8. $y^2 = \pm x^2+C$.
9. (i) $r = a/(\theta-C)$. (ii) $r = a(\theta-C)$. 10. $r = Ce^{\theta\cot\alpha}$.
11. $y+b = C(x+a)$. 12. $2x^2(y-C)+4x+3 = 0$.
13. $1+y^2 = Cx^2$. 14. $ay+b = Ce^{-ax}$. 15. $\frac{1}{2}ax^2+bx+y = C$.
16. $y+b = Ce^{x^2}$. 17. $a+abx+b^2y = Ce^{bx}$.
18. $(x+2)^2(y+2)^2 = Ce^{x+y}$. 19. $(1+x^2)(1+y^2) = Cx^2$.
20. $y = Cxe^x$. 21. $x^2+y^2 = Ce^{2x}$.
22. $\sin y = C\sin x$. 23. $x+C = \tan\frac{1}{2}(x+y)$.
24. $y+1 = Ce^{\frac{1}{2}x(x-2)}$. 25. $2/\sqrt{(4-3x)}$. 26. $\sqrt{\{\frac{1}{3}(x-1)\}}$.

Examples LXXXVII, p. 435.

1. $x^4+2x^2y^2 = C$. 2. $y-2x = Cx^2y$. 3. $2xy+x^2 = C$.
4. $x^2+xy-y^2 = C$. 5. $x^2+2Cy = C^2$. 6. $x^2+y^2 = Cy$.
7. $xy(x-y) = C$. 8. $x^4+4xy^3-y^4 = C$.

9. $y^2+2xy-x^2-2x+2y = C$.
10. $(y-x+3)^4 = C(y+2x-3)$.
11. $\log(x+y-1) = x-y+C$.
12. $x^2-6xy+5y^2+4x = C$.
13. $4xy = x^4+C$.
14. $5x^4y = x^5+C$.
15. $y\sin x = x+C$.
16. $2y\cos x = x+\sin x\cos x+C$.
17. $ye^x = \frac{1}{2}e^{2x}+C$.
18. $\frac{1}{2}x^2y+x+Cy = 0$.
19. $y(\sin x - C\cos x) = 1$.
20. $y^{n-1}(1+Cx^{n-1}) = 1$.
21. $x^2y^3(3+Cx) = 1$.
22. $y = (a\cos bx + b\sin bx)/(a^2+b^2) + Ce^{-ax}$.
23. $(x-y)^2(x+2y) = C$.
24. $i = \frac{20}{29}(2\sin 500t - 5\cos 500t) + Ce^{-200t}$.
25. 28 ft.-secs.

Examples LXXXVIII, p. 440.

1. $3xy = x^3+C$.
2. $2xy = y^2+C$.
3. $3x^2y+y^3 = C$.
4. $(n+1)y = x+C/x^n$.
5. $2xy^2 = x^2+C$.
6. $x^2y+1 = Cx$.
7. $y = \frac{1}{2}x^3+Cx$.
8. $y = x^2+Cx$.
9. $y = x(y+C)$.
10. $x^3-x^2y+xy-\frac{2}{3}y^3 = C$.
11. $x = 2y^2+Cy$.
12. $\log(x/y) = \frac{1}{2}(x^2+y^2)+C$.
13. $(y-x)^2 = 4ax$; $(y-x-24a)^2 = 100ax$; $\pm\frac{3}{2}$.
14. $y = 4x+C$, $y = 3x+C$.
15. $y = Ce^{\pm ax}$.
16. $y = C\pm\frac{1}{2}x^2$.
17. $y = C$, $y = \frac{1}{2}x^2+C$.
18. $x^2 = 2Cy+C^2$.
19. $(y+C)^2 = 2x+3$.
20. $y = Cx$, $y^2-x^2 = C$; $3y = 5x$, $y^2-x^2 = 16$; $\frac{5}{3}$, $\frac{3}{5}$.
21. $y = \frac{1}{2}x^2+C$, $y = Ce^{-x}-x+1$; $y = \frac{1}{2}x^2$, $y = 1-x-e^{-x}$; 0, 0.
22. $xy = C$, $y^2-x^2 = C$; $xy = 6$, $x^2-y^2 = 5$.
23. $x+y = C$, $x^2-2xy = C$; $11°19'$.
24. $xy = \frac{1}{2}A$.
25. $a\cosh(x/a+C)$.
26. $x^2+y^2 = a^2$.
27. $9y^4 = 16ax^3$.
28. $y = Cx+a\sqrt{(1+C^2)}$, $x^2+y^2 = a^2$
29. $y = Cx+C^2$, $x^2+4y = 0$.
30. $y-Cx = \pm 2\sqrt{C}$, $xy+1 = 0$.
31. $Cy+a = C^2x$, $y^2+4ax = 0$.
32. $4xy = a^2$.
33. $x^{2/3}+y^{2/3} = a^{2/3}$.
34. $\sqrt{x}+\sqrt{y} = \sqrt{a}$.
35. $y^2 = 4ax$.
36. $x^2 = 4ay$.

Examples LXXXIX, p. 444.

1. $y = x^{n+2}/(n^2+3n+2)+Cx+D$.
2. $y = x\log x+Cx+D$.
3. $y = \frac{1}{8}a^2(2x^2+\cos 2x)+Cx+D$.
4. $y = C\cosh(2x+D)$.
5. $Ce^y = a\cosh(Cx+D)$.
6. $(Cx+D)^2 = Cy^2-a^2$.
7. $y = Cx^2+D$.
8. $y(x+D) = C$.
9. $V = C\log r+D$.
10. $y = \frac{1}{2}x^2+C\log x+D$.
11. $y^2 = a^2x^2+Cx+D$.
12. $y^2 = C\sinh(2x+D)$.
13. $y = \frac{1}{2}x\sqrt{(C^2x^2-1)}-(1/2C)\cosh^{-1}Cx$.
14. $3y = (x+C)^3+D$.
15. $y = C+De^x$.
16. $x = D+Ce^{y/a}$.
17. $(Cx+D)^2 = 1+Cy^2$.
18. $y = C\log x+\frac{1}{4}x^2-x+D$.
19. $(y-D)^2 = (x-C)^3$.
20. $u^2 = C\pm\sqrt{(C^2-k/h^2)}\sin(2\theta+\alpha)$.
21. $(x-C)^2+(y-D)^2 = a^2$.
22. $y = C\cosh(x/C+D)$.
23. $Cy^2-1 = k(Cx+D)^2$.
24. $(x-C)^2 = 4D(y-D)$.

Examples XC, p. 450.

1. $y = Ae^{2x} + Be^{3x}$.
2. $y = Ae^{4x} + Be^{-4x}$.
3. $y = e^{5x}(A + Bx)$.
4. $y = e^{-3x}(A + Bx)$.
5. $y = e^{-3x}(A \cos x + B \sin x)$
6. $y = e^{-\frac{1}{2}x}(A \cos \frac{1}{2}\sqrt{3}\,x + B \sin \frac{1}{2}\sqrt{3}\,x)$.
7. $y = Ae^{5x} + Be^{-2x}$.
8. $y = Ae^{\frac{3}{2}x} + Be^{-\frac{2}{3}x}$.
9. $y = Ae^{x} + Be^{-\frac{5}{4}x}$.
10. $y = e^{-\frac{3}{2}x}(A + Bx)$.
11. $y = A + Be^{x} + Ce^{-x}$.
12. $y = A + B \cos x + C \sin x$.
13. $y = Ae^{2x} + Be^{-2x} + C \sin 2x + D \cos 2x$.
14. $y = A + Be^{x} + e^{-\frac{1}{2}x}(C \cos \frac{1}{2}\sqrt{3}\,x + D \sin \frac{1}{2}\sqrt{3}\,x)$.
15. $y = A + Bx + Ce^{-x}$.
16. $y = e^{x}(A \cos \sqrt{5}\,x + B \sin \sqrt{5}x)$.

Examples XCI, p. 459.

1. $y = Ae^{3x} + Be^{2x} + 2$.
2. $y = Ae^{3x} + Be^{2x} + \frac{1}{108}(18x^2 + 30x + 19)$.
3. $y = Ae^{3x} + Be^{2x} - \frac{3}{10}(\sin x + \cos x)$.
4. $y = Ae^{3x} + Be^{2x} + \frac{1}{12}e^{-x}$.
5. $y = Ae^{-2x} + Be^{-6x}$.
6. $y = Ae^{-2x} + Be^{-6x} + \frac{1}{3}$.
7. $y = Ae^{2x} + Be^{-2x} - \frac{5}{2}$.
8. $y = Ae^{2x} + Be^{-2x} - \frac{1}{5}\sin x$.
9. $y = Ae^{2x} + Be^{-2x} - \frac{1}{3}e^{x}$.
10. $y = \frac{1}{4}e^{2x}(x + A) + Be^{-2x}$.
11. $y = Ae^{-2x}\cos(x + \epsilon) + \frac{1}{5}a$.
12. $y = Ae^{-2x}\cos(x + \epsilon) + \frac{1}{15}(5x - 6)$.
13. $y = Ae^{-2x}\cos(x + \epsilon) + \frac{2}{65}(\cos 2x + 8 \sin 2x)$.
14. $y = Ae^{-2x}\cos(x + \epsilon) + \frac{1}{16}(\sin x - \cos x) + \frac{1}{5}$.
15. $y = A + Be^{3x} + Ce^{-2x}$.
16. $y = A + Be^{3x} + Ce^{-2x} - \frac{1}{54}x(3x^2 - 15x - 10)$.
17. $y = A + B\cos(x + \epsilon) + 2x$.
18. $y = A + B\cos(x + \epsilon) - \frac{1}{2}x\cos x$.
19. $y = e^{3x}(A + Bx)$.
20. $y = e^{3x}(A + Bx + \frac{1}{2}x^2)$.
21. $y = Ae^{5x} + e^{-2x}(B + \frac{1}{7}x) - \frac{2}{5}$.
22. $y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x} + C\cos(\sqrt{2}\,x + \epsilon)$.
23. $y = Ae^{-\frac{5}{2}x}\cos(\frac{1}{2}\sqrt{3}x + \epsilon)$.
24. $y = e^{x}(A + Bx + Cx^2) - \frac{1}{8}e^{-x}$.
25. $x = e^{-kt}(A + Bt)$.
26. $x = Ae^{-kt}\cos(nt + \epsilon)$.
27. $x = Ae^{-kt}\cos(nt + \epsilon) + \dfrac{(k^2 + n^2 - p^2)\cos pt + 2kp\sin pt}{(k^2 + n^2 - p^2)^2 + 4k^2p^2}$.
28. $x = A\cos(at + \epsilon) + \{k\sin(pt + \alpha)\}/(a^2 - p^2)$.
29. 28 ft. nearly, -14 ft.-secs., -14 ft.-secs. per sec. After 1·21 secs.
30. $x = \frac{100}{11}\sqrt{5}e^{-\cdot 16t}\cos(\cdot 88t - \epsilon)$, where $\tan\epsilon = \frac{2}{11}$. 2 secs. and -13 ft.-secs. nearly. 11·3 ft.
32. $\alpha p\sqrt{(1 + k^2/4p^2)}\,e^{-\frac{1}{2}k\tau}$, if τ be the corresponding time.
33. $p = \pi$, $k = \cdot 575$, $\epsilon = \cdot 091$, $C = \cdot 351$. ·75, ·75. 1·13°; ·071.
34. $\frac{1}{20}$ inch to left of A. $p = \frac{2}{3}\pi$, $k = \cdot 267$, $\epsilon = \cdot 0637$, $C = 6\cdot 06$. ·82 inches from O, ·77 from A.
35. $23\frac{1}{3}$ inches from the fixed point.
36. $\theta = \frac{1}{6}\sqrt{10}\,\pi e^{-\cdot 6t}\cos(\cdot 2t - \tan^{-1}3)$.
37. $\theta = \cdot 5236(5e^{-\cdot 8t} - 4e^{-t})$.
38. $\dot{q} = 4\cdot 162\,e^{-1020t} - 3\cdot 918\,e^{-48980t}$.
39. $\dot{q} = 5\,e^{-5000t}\sin 5000t$.
40. $x = Ae^{-1\cdot 6t}\cos(7\cdot 837t - \epsilon)$.
41. $x = e^{-128t}(Ae^{110\cdot 85t} + Be^{-110\cdot 85t})$.
42. (i) $\theta = A\cos(2t - \epsilon) - \frac{1}{40}k\cos 3t$. (ii) $\theta = A\cos(2t - \epsilon) + \frac{1}{32}kt\sin 2t$.

Examples XCII, p. 463.

1. $y = C(1-x^2) + Dx$.
2. $(y - Cx)^2 = D^2(1-x^2)$.
3. $y = x(C + D\log x)$.
4. $y = x(C + D\log x + x)$.
5. $y = (Ce^{kx} + De^{-kx})/x$.
6. $y = x^2(C + Dx)$.
7. $y = e^x(x\log x + Cx + D)$.
8. $y = x(C + Dx^2 - x)$.

Examples XCIII, p. 476.

Expansions 1–3, 6–9 hold for all finite values of x.

1. $1 + ax + a^2x^2/2! + a^3x^3/3! + \ldots$.
2. $mx - m^3x^3/3! + m^5x^5/5! - m^7x^7/7! + \ldots$.
3. $1 - m^2x^2/2! + m^4x^4/4! - m^6x^6/6! + \ldots$.
4. $\log a + x/a - x^2/2a^2 + x^3/3a^3 - \ldots$, if $|x| < |a|$ or $x = a$.
5. $\log a - x/a - x^2/2a^2 - x^3/3a^3 - \ldots$, if $|x| < |a|$ or $x = -a$.
6. $1 + x\log 2 + (x\log 2)^2/2! + (x\log 2)^3/3! + \ldots$.
7. $1 + mx\log a + (mx\log a)^2/2! + (mx\log a)^3/3! + \ldots$.
8. $x^2 - 8x^4/4! + 32x^6/6! - 128x^8/8! + \ldots$.
9. $x + x^3/3! + x^5/5! + x^7/7! + \ldots$.
39. From $-8°$ to $+8°$.
40. From $-22\frac{1}{2}°$ to $+22\frac{1}{2}°$.
41. From $-17\frac{1}{2}°$ to $+17\frac{1}{2}°$.
42. (i) From $-31°$ to $+31°$. (ii) From $-19\frac{1}{2}°$ to $+19\frac{1}{2}°$.
43. ·48481, ·87462, ·46947, ·88295.
45. ·2679, 1·4281.
46. ·2960, ·9200.
47. $e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha)$; $1 + x\cos\alpha + (x^2\cos 2\alpha)/2! + (x^3\cos 3\alpha)/3! + .$
52. $2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \ldots\right\}$. See Table IX.

Examples XCIV, p. 483.

1. $a\sec^2(ax+by)$; $b\sec^2(ax+by)$.
2. $2y/(x+y)^2$; $-2x/(x+y)^2$.
3. pe^{px+qy}; qe^{px+qy}.
4. $2(ax+hy+g)$; $2(hx+by+f)$.
5. $2nax(ax^2+by^2)^{n-1}$; $2nby(ax^2+by^2)^{n-1}$.
6. $1/\sqrt{(y^2-x^2)}$; $-x/\{y\sqrt{(y^2-x^2)}\}$.
7. $y^2/(x+y)^2$; $x^2/(x+y)^2$.
8. x/z; $-y/z$.
9. x^{n-1}/z^{n-1}; y^{n-1}/z^{n-1}.
10. $2xy^2/\{z(x^2+y^2)^2\}$; $-2x^2y/\{z(x^2+y^2)^2\}$.
11. $-ax/cz$; $-by/cz$.
12. $-z^2/x^2$; z^2/y^2.
13. $2x$; $2y$; $2z$.
14. $\frac{z}{(x+y)^2+z^2}$; $\frac{z}{(x+y)^2+z^2}$; $-\frac{x+y}{(x+y)^2+z^2}$.
15. $-xV^3$; $-yV^3$; $-zV^3$.
16. $2(ax+hy+gz)$; $2(hx+by+fz)$; $2(gx+fy+cz)$.
22. (i) $\frac{1}{3}\pi r^2$, (ii) $\frac{2}{3}\pi rh$ cubic inches per second.
23. $2x$; $2y$.
24. (i) ·033, (ii) $\frac{1}{8}$ inch per sec.
25. (i) 15·1, (ii) 21·36 sq. in. per sec.
26. $-c^2x/a^2z$; $-c^2y/b^2z$.
27. (i) k/v. (ii) $-p/v$. (iii) k/p.

Examples XCV, p. 494.

1. (i) $\delta y = \sin\theta\,\delta r + r\cos\theta\,\delta\theta$. (ii) $\delta r = \cos\theta\,\delta x + \sin\theta\,\delta y$.
(iii) $\delta\theta = (x\,\delta y - y\,\delta x)/r^2$.

3. $nx^{n-1}y^{n-1}(bx\cos bt - ay\sin at)$. 4. $\cot\frac{1}{2}t$.

5. $(ad+bc)\,e^{2t}/(cx+dy)^2$. 6. $4e^{2t}$.

7. $2\cos 2t/(1+\sin 2t)$. 8. $e^{-2t}\sin^2 t\,(3\cos^2 t - \sin^2 t - \sin 2t)$.

9. $2(x^4+x-1)/x^3$. 10. $3x^2-3xy$.

11. $xy^2(4y^2+xy-6x^2)/(2y-x)$. 12. a/y^2.

13. $(x\sin 2x - y)/(x^2+y^2)$.

14. $3(x-y)(x^2-xy+y^2+2ax+2ay)/(x-2y)$.

15. $(3x^2+10xy-4y^2)/(6y^2+8xy-5x^2)$. 16. $-\sin x\operatorname{cosec} y$.

17. $-\{2x(x^2+y^2)-a^2x\}/\{2y(x^2+y^2)+a^2y\}$.

18. $-\{y(nx^{n-m}+my^{n-m})\}/\{x(mx^{n-m}+ny^{n-m})\}$.

19. $-\tan(\frac{1}{4}\pi+y)\cot(\frac{1}{4}\pi+x)$.

20. $(b^2x-a^2x-2aby)/(b^2y-a^2y+2abx)$.

21. $\delta V = \frac{1}{3}\pi r^2\,\delta h + \frac{2}{3}\pi rh\,\delta r$. 22. $p\,\delta v + v\,\delta p = k\,\delta T$.

23. $yz\,\delta x + zx\,\delta y + xy\,\delta z = 0$. 24. $x\,\delta x + y\,\delta y = z\,\delta z$.

25. $r\,\delta f + f\,\delta r = 2mv\,\delta v$. 26. $F\,\delta s + s\,\delta F = mv\,\delta v$.

27. $\dfrac{\partial u}{\partial r} = \cos\theta\dfrac{\partial u}{\partial x} + \sin\theta\dfrac{\partial u}{\partial y}$; $\dfrac{1}{r}\dfrac{\partial u}{\partial\theta} = -\sin\theta\dfrac{\partial u}{\partial x} + \cos\theta\dfrac{\partial u}{\partial y}$.

28. $(X-x)f_y = (Y-y)f_x$.

29. $(-3,\ -10)$; $14x-5y=0$; $5x+14y=0$.

30. $(\frac{1}{2}a,\ \frac{1}{2}a)$; $x+y=0$; $3x-y=4a$; $3y-x=4a$.

31. $K(np\,\delta T + T\,\delta p)/pT$. 32. $(\mu-1)\{\delta p/p - \alpha\delta\theta/(1+\alpha\theta)\}$.

33. ·46 inch. 34. (i) $+$·05. (ii) $-$·78 inch.

35. (i) 8·77 sq. ft., ·54 per cent. (ii) 18·44 sq. ft., 1·12 per cent.

36. (i) $\delta p = -$·37. (ii) $\delta v = -$·06. (iii) $\delta v =$ ·16. (iv) $\delta T = 9$·39. (v) $\delta T = -3$·13.

37. ·407. 38. (i) 1·19. (ii) 1·69.

39. $c\,\delta c = (a - b\cos C)\,\delta a + (b - a\cos C)\,\delta b + ab\sin C\,\delta C$. ·604; ·6 per cent.

40. ·65 sq. ft. 41. $\delta a/a + \delta b/b$.

42. $\frac{1}{4}[a\,\delta a\,\{2s(s-a)-bc\}/S^2 + 2$ similar terms].

43. $\{2abc\,\delta a - c(b^2+a^2-c^2)\,\delta b - b(c^2+a^2-b^2)\,\delta c\}/(2b^2c^2\sin A)$.

44. $a\sec^2\alpha\,.\,\delta\alpha + \tan\alpha\,.\,\delta a$. ·413. 45. $\delta h/h + 2\,\delta r/r$.

46. $\pi\sqrt{(l/g)}\,(\delta l/l - \delta g/g)$. $\frac{1}{2}\delta l/l - \frac{1}{2}\delta g/g$.

47. $-$·5 per cent. 50. $\frac{1}{68}$. 51. ·0177. ·8 per cent.

53. $\{(W_2-W_1)\,\delta W + (W_1-W)\,\delta W_2 + (W-W_2)\,\delta W_1\}/\{(W-W_1)(W-W_2)\}$.

54. (i) 58·64 c. in. per sec. (ii) 27·82 sq. in. per sec.

55. (i) 360 c. in. per sec. (ii) 104 sq. in. per sec.

Examples XCVI, p. 503.

1. $6(ax+by)$; $6(cx+dy)$; $6(bx+cy)$; $6(bx+cy)$.
2. $2\sin y-y^2\sin x$; $2\sin x-x^2\sin y$; $2x\cos y+2y\cos x$.
3. $f_{xx}=m(m-1)x^{m-2}/y^n$; $f_{yy}=n(n+1)x^m/y^{n+2}$; $f_{xy}=f_{yx}=-mnx^{m-1}/y^{n+1}$; $f_{xxx}=m(m-1)(m-2)x^{m-3}/y^n$; $f_{yyy}=-n(n+1)(n+2)x^m/y^{n+3}$; $f_{yxx}=f_{xyx}=-mn(m-1)x^{m-2}/y^{n+1}$; $f_{xyy}=f_{yxy}=mn(n+1)x^{m-1}/y^{n+2}$.
8. $(1+3xyz+x^2y^2z^2)e^{xyz}$.
15. $\ddot{z}=f_x\ddot{x}+f_y\ddot{y}+f_{xx}\dot{x}^2+2f_{xy}\dot{x}\dot{y}+f_{yy}\dot{y}^2$.
16. $-(f_{xx}f_y{}^2-2f_{xy}f_xf_y+f_{yy}f_x{}^2)/f_y{}^3$.
17. $f_{xx}=\cos^2\theta f_{rr}+\sin 2\theta\,(f_\theta/r^2-f_{r\theta}/r)+\sin^2\theta f_r/r+\sin^2\theta f_{\theta\theta}/r^2$.

    $f_{yy}=\sin^2\theta f_{rr}-\sin 2\theta\,(f_\theta/r^2-f_{r\theta}/r)+\cos^2\theta f_r/r+\sin^2\theta f_{\theta\theta}/r^2$.
20. $hf_x+kf_y+\frac{1}{2}(h^2f_{xx}+2hkf_{xy}+k^2f_{yy})$.
22. $cx^2+2axy+by^2=C$.
23. $xy(x-y+a)=C$.
24. $\log(x^2+y^2)=2k\tan^{-1}(y/x)+C$.
25. $x^2e^{x^2+y^2}=C(x^2+y^2)$.
27. $d^3u/dt^3=\alpha^3f_{xxx}+3\alpha^2\beta f_{xxy}+3\alpha\beta^2f_{xyy}+\beta^3f_{yyy}$. In the general case, the coefficients are the same as in the Binomial Theorem.

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